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A STUDY ON THE GEOMETRY OF 2–DIMENSIONAL RE–MANIFOLD X_2

IN HO HWANG

1. Introduction

Manifolds with recurrent connections have been studied by many authors, such as Chung, Datta, E.M.Patterson, M.Prvanovitch, Singal, and Takano, etc (refer to [2] and [3]). Examples of such manifolds are those of recurrent curvature, Ricci-recurrent manifolds, and bi-recurrent manifolds.

In this paper, we introduce a new concept of g-recurrent connection $\Gamma_\lambda{}^\nu{}_\mu$ on a generalized n-dimensional Riemannian manifold X_n and study its differential geometric properties in the first. In the second, we prove a necessary condition for a g-recurrent connection to be einstein in X_n. The generalized 2-dimensional Riemannian manifold X_2 has some particular properties, probably due to the simplicity of its dimension.

The main purpose of the present paper is to display a precise tensorial representation of a connection in X_2 which is both g-recurrent and einstein, using useful and powerful recurrence relations in X_2 which do not hold in a higher dimensional manifold. This representation in terms of the unified field tensor $g_{\lambda\mu}$ has been shown to exist uniquely in X_2 and is the simplest ever found.

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2. Preliminaries

This section is a brief collection of definitions and notations which are needed in our subsequent considerations. Let X_n be a generalized n-dimensional Riemannian manifold referred to a real coordinate system x^ν, which obeys only coordinate transformations $x^\nu \to x^{\nu'}$ for which

$$\text{Det}\left(\frac{\partial x'}{\partial x}\right) \neq 0 \tag{2.1}$$

The manifold X_n is endowed with a general real nonsymmetric tensor $g_{\lambda\mu}$ which may be split into its symmetric part $h_{\lambda\mu}$ and skew-symmetric part $k_{\lambda\mu}$:

$$g_{\lambda\mu} = h_{\lambda\mu} + k_{\lambda\mu} \tag{2.2}$$

where

$$\mathfrak{g} = \text{Det}(g_{\lambda\mu}) \neq 0, \quad \mathfrak{h} = \text{Det}(h_{\lambda\mu}) \neq 0, \quad \mathfrak{k} = \text{Det}(k_{\lambda\mu}) \tag{2.3}$$

In virtue of (2.3) we may define a unique tensor $h^{\lambda\nu}$ by

$$h_{\lambda\mu} h^{\lambda\nu} = \delta_\mu^\nu \tag{2.4}$$

which together with $h_{\lambda\mu}$ will serve for raising and/or lowering indices of tensors in X_n in the usual manner. There exists also a unique tensor ${}^*g^{\lambda\nu}$ satisfying

$$g_{\lambda\mu}{}^*g^{\lambda\nu} = g_{\mu\lambda}{}^*g^{\nu\lambda} = \delta_\mu^\nu \tag{2.5}$$

The manifold X_n is connected by a general real connection $\Gamma_\lambda{}^\nu{}_\mu$ with the following transformation rule:

$$\Gamma_{\lambda'}{}^{\nu'}{}_{\mu'} = \frac{\partial x^{\nu'}}{\partial x^\alpha}\left(\frac{\partial x^\beta}{\partial x^{\lambda'}}\frac{\partial x^\gamma}{\partial x^{\mu'}}\Gamma_\beta{}^\alpha{}_\gamma + \frac{\partial^2 x^\alpha}{\partial x^{\lambda'}\partial x^{\mu'}}\right) \tag{2.6}$$

It may also be decomposed into its symmetric part $\Lambda_\lambda{}^\nu{}_\mu$ and its skew-symmetric part $S_{\lambda\mu}{}^\nu$, called the torsion tensor of $\Gamma_\lambda{}^\nu{}_\mu$:

$$\Gamma_\lambda{}^\nu{}_\mu = \Lambda_\lambda{}^\nu{}_\mu + S_{\lambda\mu}{}^\nu; \quad \Lambda_\lambda{}^\nu{}_\mu = \Gamma_{(\lambda}{}^\nu{}_{\mu)}; \quad S_{\lambda\mu}{}^\nu = \Gamma_{[\lambda}{}^\nu{}_{\mu]} \tag{2.7}$$

A connection $\Gamma_\lambda{}^\nu{}_\mu$ is said to be *einstein* if it satisfies the following system of Einstein's equations:

(2.8a) $$\partial_\omega g_{\lambda\mu} - \Gamma_\lambda{}^\alpha{}_\omega g_{\alpha\mu} - \Gamma_\omega{}^\alpha{}_\mu g_{\lambda\alpha} = 0$$

or equivalently

(2.8b) $$D_\omega g_{\lambda\mu} = 2S_{\omega\mu}{}^\alpha g_{\lambda\alpha}$$

where D_ω is the symbolic vector of the covariant derivative with respect to $\Gamma_\lambda{}^\nu{}_\mu$. The manifold X_n in this case is a generalization of the space-time X_4 , and *Einstein's n-dimensional unified field theory* is based upon this manifold X_n. A connection $\Gamma_\lambda{}^\nu{}_\mu$ is said to be *semi-symmetric* if its torsion tensor $S_{\lambda\mu}{}^\nu$ is of the form

(2.9) $$S_{\lambda\mu}{}^\nu = 2\delta_{[\lambda}{}^\nu X_{\mu]}$$

for an arbitrary non-null vector X_μ. The manifold X_n in this case is called a *semi-symmetric manifold.* Finally, our new concept of g-recurrent connection $\Gamma_\lambda{}^\nu{}_\mu$ is defined by the following system of equations:

(2.10) $$D_\omega g_{\lambda\mu} = -4X_\omega g_{\lambda\mu}$$

for a non-null vector X_μ. The manifold X_n connected by this connection is called an n-dimensional *g-recurrent manifold.* A connection $\Gamma_\lambda{}^\nu{}_\mu$ in X_n which is both g-recurrent and einstein is called a *RE-connection*, and the manifold X_n connected by a RE-connection is called a n-dimensional *RE-manifold.* We denote this manifold by REX_n.

The main purpose of the present paper is to find a precise tensorial representation of the 2-dimensional RE-connection in terms of the unified field tensor $g_{\lambda\mu}$. This work will be done by employing powerful relations including recurrence relations, which hold particularly in X_2.

3. The n-dimensional g-recurrent connections

This section is devoted to the investigations of the differential geometric properties of g-recurrent connections.

The following two theorems will be proved simultaneously:

THEOREM 3.1. *The system (2.10) may be decomposed into*

(3.1a) $$D_\omega h_{\lambda\mu} = -4X_\omega h_{\lambda\mu}$$

(3.1b) $$D_\omega k_{\lambda\mu} = -4X_\omega k_{\lambda\mu}$$

THEOREM 3.2. *The system (2.10) is equivalent to*

(3.2) $$D_\omega {}^*g^{\lambda\nu} = 4X_\omega {}^*g^{\lambda\nu}$$

Proof. The equations (3.1a) and (3.1b) follow from (2.10) and

$$D_\omega h_{\lambda\mu} = D_\omega g_{(\lambda\mu)}, \quad D_\omega k_{\lambda\mu} = D_\omega g_{[\lambda\mu]}$$

In virtue of (2.5), multiplication of ${}^*g^{\lambda\nu}$ to both sides of (2.10) gives

(3.3) $$-g_{\lambda\mu} D_\omega {}^*g^{\lambda\nu} = {}^*g^{\lambda\nu} D_\omega g_{\lambda\mu} = -4X_\omega g_{\lambda\mu} {}^*g^{\lambda\nu} = -4X_\omega \delta_\mu^\nu$$

The equations (3.2) may be obtained by multiplying ${}^*g^{\epsilon\mu}$ again to both sides of (3.3). Conversely, start with (3.2), and multiply this equations by $g_{\lambda\mu}$ to get (2.10).

REMARK 3.3. The form of equations (3.2) may be used for the study of g-recurrent connections in the Einstein's n-dimensional ${}^*g$-unified field theory (Refer to [6]).

THEOREM 3.4. *If the system (2.10) admits a solution $\Gamma_\lambda{}^\nu{}_\mu$, it must be of the form*

(3.4) $$\Gamma_\lambda{}^\nu{}_\mu = \{{}_\lambda{}^\nu{}_\mu\} + S_{\lambda\mu}{}^\nu + U^\nu{}_{\lambda\mu}$$

where $\{{}_\lambda{}^\nu{}_\mu\}$ are the Christoffel symbols with respect to $h_{\lambda\mu}$ and

(3.5a) $$U^\nu{}_{\lambda\mu} = U^\nu{}_{(\lambda\mu)} = -2S^\nu{}_{(\lambda\mu)} + 4X_{(\lambda}\delta_{\mu)}{}^\nu - 2X^\nu h_{\lambda\mu}$$

or equivalently

(3.5b) $$U_{\nu\lambda\mu} = U_{\nu(\lambda\mu)} = -2S_{\nu(\lambda\mu)} + 4X_{(\lambda}h_{\mu)\nu} - 2X_\nu h_{\lambda\mu}$$

Proof. In virtue of

$$D_\omega h_{\lambda\mu} = \partial_\omega h_{\lambda\mu} - \Gamma_\lambda{}^\alpha{}_\omega h_{\alpha\mu} - \Gamma_\mu{}^\alpha{}_\omega h_{\lambda\alpha}$$

we have

$$\begin{aligned} & h^{\nu\alpha}(D_\lambda h_{\alpha\mu} + D_\mu h_{\lambda\alpha} - D_\alpha h_{\lambda\mu}) \\ (3.6) \qquad & = \{\lambda{}^\nu{}_\mu\} - 2h^{\nu\alpha} S_{\alpha(\lambda\mu)} - \Gamma_{(\lambda}{}^\nu{}_{\mu)} \\ & = \{\lambda{}^\nu{}_\mu\} - 2S^\nu{}_{(\lambda\mu)} - \Gamma_\lambda{}^\nu{}_\mu + S_{\lambda\mu}{}^\nu \end{aligned}$$

On the other hand, the relation (3.1)a gives

$$(3.7) \qquad \frac{1}{2} h^{\nu\alpha}(D_\lambda h_{\alpha\mu} + D_\mu h_{\lambda\alpha} - D_\alpha h_{\lambda\mu}) = -4X_{(\lambda}\delta_{\mu)}{}^\nu + 2X^\nu h_{\lambda\mu}$$

Comparing (3.6) and (3.7), we finally have (3.4) in virtue of (3.5).

REMARK 3.5. In virtue of (3.4) and (3.5), we note that the investigation of the g-recurrent connection $\Gamma_\lambda{}^\nu{}_\mu$ is reduced to the study of the tensor $S_{\lambda\mu}{}^\nu$. In order to know the g-recurrent connection $\Gamma_\lambda{}^\nu{}_\mu$, it is necessary and sufficient to represent the tensor $S_{\lambda\mu}{}^\nu$ in terms of $g_{\lambda\mu}$. This is an open problem. Probably, the precise tensorial representation of $S_{\lambda\mu}{}^\nu$ in terms of $g_{\lambda\mu}$ may be obtained by starting from (3.1b).

THEOREM 3.6. *If a connection $\Gamma_\lambda{}^\nu{}_\mu$ in X_n is both g-recurrent and einstein, then it is semi-symmetric.*

Proof. The equations (2.8b) and (2.10) give

$$(3.8) \qquad S_{\omega\mu}{}^\alpha g_{\lambda\alpha} = -4X_\omega g_{\lambda\mu}$$

Multiplying ${}^*g^{\lambda\nu}$ to both sides of (3.8) and using (2.5), we have

$$S_{\lambda\mu}{}^\nu = -4\delta_{[\lambda}{}^\nu X_{\mu]}.$$

This implies that the connection $\Gamma_\lambda{}^\nu{}_\mu$ is semi-symmetric in virtue of (2.9).

4. 2-dimensional RE-connection in REX_2

This section is devoted to the derivation of a surveyable tensorial representation of 2-dimensional RE-connection. The representation obtained in this section is the simplest ever found. Our investigation is mainly based on the results obtained in **1**. It should also be remarked that *all indices in this section are restricted to take the values 1,2 only.* The following Mishra's abbreviations (refer to [7]) will be used in this section:

$$\overset{000}{T}_{\omega\mu\nu} = T_{\omega\mu\nu}, \quad \overset{pqr}{T}_{\omega\mu\nu} = T_{\alpha\beta\gamma}{}^{(p)}k_\omega{}^{\alpha(q)}k_\mu{}^{\beta(r)}k_\nu{}^\gamma \tag{4.1}$$

where $T_{\omega\mu\nu}$ is an arbitrary tensor and

$$\begin{aligned} &{}^{(0)}k_\lambda{}^\nu = \delta_\lambda{}^\nu, \quad {}^{(1)}k_\lambda{}^\nu = k_\lambda{}^\nu, \\ &{}^{(p)}k_\lambda{}^\nu = {}^{(p-1)}k_\lambda{}^\alpha k_\alpha{}^\nu, \quad (p,q,r = 1,2,\cdots) \end{aligned} \tag{4.2}$$

We also use the scalars g and k, defined by

$$g = \frac{\mathfrak{g}}{\mathfrak{h}}, \quad k = \frac{\mathfrak{k}}{\mathfrak{h}} \tag{4.3}$$

The following two lemmas were proved in **1**.

LEMMA 4.1. *If the condition*

$$g \neq 0 \tag{4.4}$$

is satisfied in X_2, the system (2.8) of Einstein's equations admits a unique solution

$$S_{\omega\mu\nu} = \frac{1}{g}\nabla_\nu k_{\omega\mu} \tag{4.5}$$

where ∇_ν is the symbolic vector of the covariant derivative with respect to $\{{\lambda}^\nu{}_\mu\}$.

REMARK 4.2. The condition (2.3) imposed to X_2 shows that the condition (4.4) is always satisfied. Therefore, we may remark that the system (2.8) of Einstein's equations always admit a unique solution of the form (4.5) in X_2.

LEMMA 4.3. *If $T_{\omega\mu\nu}$ is a tensor skew-symmetric in the first two indices, the following recurrence relations hold in X_2:*

$$\overset{(10)r}{T}{}_{\omega\mu\nu} = 0, \quad \overset{11r}{T}{}_{\omega\mu\nu} = k\overset{00r}{T}{}_{\omega\mu\nu} \tag{4.6a}$$

$$\overset{r(10)}{T}{}_{\nu[\omega\mu]} = 0, \quad \overset{r11}{T}{}_{\nu\omega\mu} = k\overset{r00}{T}{}_{\nu[\omega\mu]}, \quad (r = 0, 1, 2, \cdots) \tag{4.6b}$$

LEMMA 4.4. *If $T_{\omega\mu\nu}$ is a tensor skew-symmetric in the first two indices, then the following relations hold in X_2:*

$$\overset{pqr}{T}{}_{\omega\mu\nu} = -\overset{qpr}{T}{}_{\mu\omega\nu} \tag{4.7}$$

$$\overset{pqr}{T}{}_{\omega\mu\nu} + \overset{qrp}{T}{}_{\mu\nu\omega} + \overset{rpq}{T}{}_{\nu\omega\mu} = 0, \quad (p, q, r = 0, 1, 2, \cdots) \tag{4.8}$$

Proof. The relation (4.7) is a direct consequence of (4.1). The relation (4.8) for the special case p=q=r=0, that is $T_{[\omega\mu\nu]}=0$, follows easily since all indices take the values 1,2 only in X_2 and $T_{\omega\mu\nu}$ is skew-symmetric in the first two indices. The relation (4.8) for the general case may be proved from the above special case as in the following way:

$$\begin{aligned} 0 &= (T_{\alpha\beta\gamma} + T_{\beta\gamma\alpha} + T_{\gamma\alpha\beta})^{(p)}k_\omega{}^{\alpha(q)}k_\mu{}^{\beta(r)}k_\nu{}^\gamma \\ &= \overset{pqr}{T}{}_{\omega\mu\nu} + \overset{qrp}{T}{}_{\mu\nu\omega} + \overset{rpq}{T}{}_{\nu\omega\mu} \end{aligned}$$

LEMMA 4.5. *If the conditions (2.8) and (2.10) hold in X_2, they admit a unique 2-dimensional solution of the form*

$$gS_{\omega\mu\nu} = 4h_{\nu[\omega}k_{\mu]}{}^\alpha X_\alpha - 4k_{\nu[\omega}X_{\mu]} \tag{4.9}$$

Proof. Employing the abbreviations (2.11) and (2.12) and using the relations (3.4), we have

$$\begin{aligned} D_\nu k_{\omega\mu} &= \partial_\nu k_{\omega\mu} - \Gamma_\omega{}^\alpha{}_\nu k_{\alpha\mu} - \Gamma_\mu{}^\alpha{}_\omega k_{\omega\alpha} \\ &= \partial_\nu k_{\omega\mu} - (\{{}_\omega{}^\alpha{}_\nu\} + S_{\omega\nu}{}^\alpha + U^\alpha{}_{\omega\nu})k_{\alpha\mu} \\ &\quad - (\{{}_\mu{}^\alpha{}_\nu\} + S_{\mu\nu}{}^\alpha + U^\alpha{}_{\mu\nu})k_{\omega\alpha} \\ &= \nabla_\nu k_{\omega\mu} + (\overset{001}{S}{}_{\omega\nu\mu} - \overset{001}{S}{}_{\mu\nu\omega}) \\ &\quad + (U_{\alpha\omega\nu}k_\mu{}^\alpha - U_{\alpha\mu\nu}k_\omega{}^\alpha) \end{aligned} \tag{4.10}$$

Using (4.8) and (4.6a), the second term of the right-hand side of (4.10) may be written as

$$\text{(second term)} = \left(-\overset{010}{S}{}_{\nu\mu\omega} - \overset{100}{S}{}_{\mu\omega\nu}\right) + \left(\overset{010}{S}{}_{\nu\omega\mu} - \overset{100}{S}{}_{\omega\mu\nu}\right)$$
(4.11)
$$= 2\overset{(10)0}{S}{}_{\omega\mu\nu} + 2\overset{010}{S}{}_{\nu[\omega\mu]} = 2\overset{010}{S}{}_{\nu[\omega\mu]}$$

Using (3.5b), (4.6a) and (4.8) again, the third term of the right-hand side of (4.10) is

(4.12)
$$\begin{aligned}\text{(third term)} &= (-2S_{\alpha(\omega\nu)} + 4X_{(\omega}h_{\nu)\alpha} - 2X_\alpha h_{\omega\nu})k_\mu{}^\alpha \\ &\quad - (-2S_{\alpha(\mu\nu)} + 4X_{(\mu}h_{\nu)\alpha} - 2X_\alpha h_{\mu\nu})k_\omega{}^\alpha \\ &= -2\overset{010}{S}{}_{\nu[\omega\mu]} - 4X_\nu k_{\omega\mu} - 4h_{\nu[\omega}k_{\mu]}{}^\alpha X_\alpha + 4k_{\nu[\omega}X_{\mu]}\end{aligned}$$

We now substitute (3.1b), (4.11), and (4.12) into (4.10) to obtain

(4.13)
$$\nabla_\nu k_{\omega\mu} = 4h_{\nu[\omega}k_{\mu]}{}^\alpha X_\alpha - 4k_{\nu[\omega}X_{\mu]}$$

Now, the solution (4.9) is the result of (4.13) and (4.5).

LEMMA 4.6. *If the conditions (2.8) and (2.10) hold in X_2, the tensor $U^\nu{}_{\omega\mu}$ is given by*

(4.14)
$$\begin{aligned}U^\nu{}_{\omega\mu} &= \frac{4}{g}(k_{(\omega}{}^\nu X_{\mu)} - \delta_{(\omega}{}^\nu k_{\mu)}{}^\alpha X_\alpha - h_{\omega\mu}k_\alpha{}^\nu X^\alpha) \\ &\quad + 4X_{(\omega}\delta_{\mu)}{}^\nu - 2X^\nu h_{\omega\mu}\end{aligned}$$

Proof. Using (4.9) we have

(4.15)
$$-2gS^\nu{}_{(\omega\mu)} = 4(k_{(\omega}{}^\nu X_{\mu)} - \delta_{(\omega}{}^\nu k_{\mu)}{}^\alpha X_\alpha - h_{\omega\mu}k_\alpha{}^\nu X^\alpha)$$

The representation (4.14) follows immediately by substituting (4.15) into (3.5)a.

Now that we have obtained the tensor $S_{\omega\mu}{}^\nu$ and $U^\nu{}_{\omega\mu}$ in terms of $g_{\lambda\mu}$, it is possible for us to determine the 2-dimensional RE-connection $\Gamma_\omega{}^\nu{}_\mu$ by only substituting for S and U into (3.4). We formally state our main theorem as follows:

THEOREM 4.7. *In X_2 there always exists a unique 2-dimensional RE-connection $\Gamma_\lambda{}^\nu{}_\mu$ represented by*

$$\Gamma_\lambda{}^\nu{}_\mu = \{\lambda{}^\nu{}_\mu\} + \frac{4}{g}(k_\omega{}^\nu X_\mu - \delta_\mu{}^\nu k_\omega{}^\alpha X_\alpha - h_{\omega\mu} k_\alpha{}^\nu X^\alpha) + 4X_{(\omega}\delta_{\mu)}{}^\nu - 2X^\nu h_{\omega\mu} \tag{4.16}$$

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Department of Mathematics
University of Inchon
402-749, Inchon, Korea