A STUDY ON THE GEOMETRY OF
2-DIMENSIONAL RE-MANIFOLD $X_2$

IN HO HWANG

1. Introduction

Manifolds with recurrent connections have been studied by many authors, such as Chung, Datta, E.M.Patterson, M.Prvanovitch, Singal, and Takano, etc (refer to [2] and [3]). Examples of such manifolds are those of recurrent curvature, Ricci-recurrent manifolds, and bi-recurrent manifolds.

In this paper, we introduce a new concept of $g$-recurrent connection $\Gamma^\nu_{\lambda\mu}$ on a generalized $n$-dimensional Riemannian manifold $X_n$ and study its differential geometric properties in the first. In the second, we prove a necessary condition for a $g$-recurrent connection to be einstein in $X_n$. The generalized 2-dimensional Riemannian manifold $X_2$ has some particular properties, probably due to the simplicity of its dimension.

The main purpose of the present paper is to display a precise tensorial representation of a connection in $X_2$ which is both $g$-recurrent and einstein, using useful and powerful recurrence relations in $X_2$ which do not hold in a higher dimensional manifold. This representation in terms of the unified field tensor $g_{\lambda\mu}$ has been shown to exist uniquely in $X_2$ and is the simplest ever found.

Received February 24, 1994.
1991 AMS Subject Classification: 83E15.
Key words: $g$-recurrent manifold, RE-connection, RE-manifold.
This paper was supported by NON DIRECTED RESEARCH FUND, Korea Research Foundation, 1991.
2. Preliminaries

This section is a brief collection of definitions and notations which are needed in our subsequent considerations. Let $X_n$ be a generalized $n$-dimensional Riemannian manifold referred to a real coordinate system $x^\nu$, which obeys only coordinate transformations $x^\nu \to x'^\nu$ for which

\[ \text{Det}\left( \frac{\partial x'}{\partial x} \right) \neq 0 \]

The manifold $X_n$ is endowed with a general real nonsymmetric tensor $g_{\lambda\mu}$ which may be split into its symmetric part $h_{\lambda\mu}$ and skew-symmetric part $k_{\lambda\mu}$:

\[ g_{\lambda\mu} = h_{\lambda\mu} + k_{\lambda\mu} \]

where

\[ \mathfrak{g} = \text{Det}(g_{\lambda\mu}) \neq 0, \quad \mathfrak{h} = \text{Det}(h_{\lambda\mu}) \neq 0, \quad \mathfrak{k} = \text{Det}(k_{\lambda\mu}) \]

In virtue of (2.3) we may define a unique tensor $h^{\lambda\nu}$ by

\[ h_{\lambda\mu} h^{\lambda\nu} = \delta^\nu_\mu \]

which together with $h_{\lambda\mu}$ will serve for raising and/or lowering indices of tensors in $X_n$ in the usual manner. There exists also a unique tensor $^*g^{\lambda\nu}$ satisfying

\[ g_{\lambda\mu} ^*g^{\lambda\nu} = g_{\mu\lambda} ^*g^{\nu\lambda} = \delta^\nu_\mu \]

The manifold $X_n$ is connected by a general real connection $\Gamma^\nu_{\lambda\mu}$ with the following transformation rule:

\[ \Gamma^\nu_{\lambda\alpha} = \frac{\partial x'^\nu}{\partial x^\lambda} \left( \frac{\partial x^\beta}{\partial x^\lambda} \frac{\partial x^\gamma}{\partial x^\mu} \Gamma^\beta_{\gamma\alpha} + \frac{\partial^2 x^\alpha}{\partial x^\lambda \partial x'^\mu} \right) \]

It may also be decomposed into its symmetric part $\Lambda^\nu_{\lambda\mu}$ and its skew-symmetric part $S^\nu_{\lambda\mu}$, called the torsion tensor of $\Gamma^\nu_{\lambda\mu}$:

\[ \Gamma^\nu_{\lambda\mu} = \Lambda^\nu_{\lambda\mu} + S^\nu_{\lambda\mu}; \quad \Lambda^\nu_{\lambda\mu} = \Gamma_{(\lambda\nu\mu)}; \quad S^\nu_{\lambda\mu} = \Gamma_{[\lambda\nu\mu]} \]
A connection $\Gamma^\nu_\lambda\mu$ is said to be \textit{einstein} if it satisfies the following system of Einstein's equations:

$$\partial_\omega g_{\lambda\mu} - \Gamma^\alpha_\lambda\omega g_{\alpha\mu} - \Gamma_\omega^\alpha_\mu g_{\lambda\alpha} = 0$$

or equivalently

$$D_\omega g_{\lambda\mu} = 2S_{\omega\mu}^\alpha g_{\lambda\alpha}$$

where $D_\omega$ is the symbolic vector of the covariant derivative with respect to $\Gamma^\nu_\lambda\mu$. The manifold $X_n$ in this case is a generalization of the space-time $X_4$, and \textit{Einstein's $n$-dimensional unified field theory} is based upon this manifold $X_n$. A connection $\Gamma^\nu_\lambda\mu$ is said to be \textit{semi-symmetric} if its torsion tensor $S_{\lambda\mu}^\nu$ is of the form

$$S_{\lambda\mu}^\nu = 2\delta^\nu_[\lambda]X^\mu$$

for an arbitrary non-null vector $X^\mu$. The manifold $X_n$ in this case is called a \textit{semi-symmetric manifold}. Finally, our new concept of $g$-recurrent connection $\Gamma^\nu_\lambda\mu$ is defined by the following system of equations:

$$D_\omega g_{\lambda\mu} = -4X_\omega g_{\lambda\mu}$$

for a non-null vector $X^\mu$. The manifold $X_n$ connected by this connection is called an $n$-dimensional \textit{$g$-recurrent manifold}. A connection $\Gamma^\nu_\lambda\mu$ in $X_n$ which is both $g$-recurrent and einstein is called a \textit{RE-connection}, and the manifold $X_n$ connected by a RE-connection is called a $n$-dimensional \textit{RE-manifold}. We denote this manifold by $\text{RE}X_n$.

The main purpose of the present paper is to find a precise tensorial representation of the 2-dimensional RE-connection in terms of the unified field tensor $g_{\lambda\mu}$. This work will be done by employing powerful relations including recurrence relations, which hold particularly in $X_2$.

3. The \textit{n}-dimensional \textit{g}-recurrent connections

This section is devoted to the investigations of the differential geometric properties of \textit{g}-recurrent connections.

The following two theorems will be proved simultaneously:
THEOREM 3.1. The system (2.10) may be decomposed into

\[ \text{(3.1a)} \hspace{1cm} D_\omega h_{\lambda\mu} = -4X_\omega h_{\lambda\mu} \]

\[ \text{(3.1b)} \hspace{1cm} D_\omega k_{\lambda\mu} = -4X_\omega k_{\lambda\mu} \]

THEOREM 3.2. The system (2.10) is equivalent to

\[ \text{(3.2)} \hspace{1cm} D_\omega * g^{\nu\lambda} = 4X_\omega * g^{\nu\lambda} \]

Proof. The equations (3.1a) and (3.1b) follow from (2.10) and

\[ D_\omega h_{\lambda\mu} = D_\omega g_{(\lambda\mu)}, \quad D_\omega k_{\lambda\mu} = D_\omega g_{[\lambda\mu]} \]

In virtue of (2.5), multiplication of * \( g^{\lambda\nu} \) to both sides of (2.10) gives

\[ \text{(3.3)} \hspace{1cm} -g_{\lambda\mu} D_\omega * g^{\lambda\nu} = * g^{\lambda\nu} D_\omega g_{\lambda\mu} = -4X_\omega g_{\lambda\mu} * g^{\lambda\nu} = -4X_\omega \delta^{\nu}_\mu \]

The equations (3.2) may be obtained by multiplying * \( g^{\epsilon\mu} \) again to both sides of (3.3). Conversely, start with (3.2), and multiply this equations by \( g_{\lambda\mu} \) to get (2.10).

REMARK 3.3. The form of equations (3.2) may be used for the study of g-recurrent connections in the Einstein's n-dimensional *g-unified field theory (Refer to [6]).

THEOREM 3.4. If the system (2.10) admits a solution \( \Gamma_{\lambda\nu}^\mu \), it must be of the form

\[ \text{(3.4)} \hspace{1cm} \Gamma_{\lambda\nu}^\mu = \{ \lambda^\nu_{\mu} \} + S_{\lambda\mu}^\nu + U_{\nu\lambda\mu} \]

where \( \{ \lambda^\nu_{\mu} \} \) are the Christoffel symbols with respect to \( h_{\lambda\mu} \) and

\[ \text{(3.5a)} \hspace{1cm} U_{\nu\lambda\mu} = U_{\nu(\lambda\mu)} = -2S_{(\lambda\mu)}^\nu + 4X_{(\lambda\delta_\mu)}^\nu - 2X^\nu h_{\lambda\mu} \]

or equivalently

\[ \text{(3.5b)} \hspace{1cm} U_{\nu\lambda\mu} = U_{\nu(\lambda\mu)} = -2S_{\nu(\lambda\mu)} + 4X_{(\lambda h_\mu)_\nu} - 2X_\nu h_{\lambda\mu} \]
Proof. In virtue of
\[ D_\omega h_{\lambda\mu} = \partial_\omega h_{\lambda\mu} - \Gamma_\lambda^\alpha \omega h_{\alpha\mu} - \Gamma_\mu^\alpha \omega h_{\lambda\alpha} \]
we have
\[
h^\nu\alpha(D_\lambda h_{\alpha\mu} + D_\mu h_{\lambda\alpha} - D_\alpha h_{\lambda\mu})
= \{\lambda^\nu\}_{\mu} - 2h^\nu\alpha S_{\alpha(\lambda\mu)} - \Gamma(\lambda^\nu\mu)
= \{\lambda^\nu\}_{\mu} - 2S^\nu(\lambda\mu) - \Gamma\lambda^\nu\mu + S_{\lambda\mu}^\nu
\]  
(3.6)

On the other hand, the relation (3.1a) gives
\[
\frac{1}{2}h^\nu\alpha(D_\lambda h_{\alpha\mu} + D_\mu h_{\lambda\alpha} - D_\alpha h_{\lambda\mu}) = -4X(\lambda\delta\mu)^\nu + 2X^\nu h_{\lambda\mu}
\]  
(3.7)

Comparing (3.6) and (3.7), we finally have (3.4) in virtue of (3.5).

Remark 3.5. In virtue of (3.4) and (3.5), we note that the investigation of the g-recurrent connection \(\Gamma_\lambda^\nu\mu\) is reduced to the study of the tensor \(S_{\lambda\mu}^\nu\). In order to know the g-recurrent connection \(\Gamma_\lambda^\nu\mu\), it is necessary and sufficient to represent the tensor \(S_{\lambda\mu}^\nu\) in terms of \(g_{\lambda\mu}\). This is an open problem. Probably, the precise tensorial representation of \(S_{\lambda\mu}^\nu\) in terms of \(g_{\lambda\mu}\) may be obtained by starting from (3.1b).

Theorem 3.6. If a connection \(\Gamma_\lambda^\nu\mu\) in \(X_\alpha\) is both g-recurrent and einstein, then it is semi-symmetric.

Proof. The equations (2.8b) and (2.10) give
\[
S_{\omega\mu}^\alpha g_{\lambda\alpha} = -4X_\omega g_{\lambda\mu}
\]  
(3.8)

Multiplying \(*g^\lambda\nu\) to both sides of (3.8) and using (2.5), we have
\[
S_{\lambda\mu}^\nu = -4\delta[\lambda^\nu X_{\mu}].
\]

This implies that the connection \(\Gamma_\lambda^\nu\mu\) is semi-symmetric in virtue of (2.9).
4. 2-dimensional RE-connection in \( \text{REX}_2 \)

This section is devoted to the derivation of a surveyable tensorial representation of 2-dimensional RE-connection. The representation obtained in this section is the simplest ever found. Our investigation is mainly based on the results obtained in 1. It should also be remarked that all indices in this section are restricted to take the values 1, 2 only. The following Mishra's abbreviations (refer to [7]) will be used in this section:

\[
\begin{align*}
T_{\omega_{\mu \nu}} &= T_{\omega_{\mu \nu}}, \\
T^{pq r}_{\omega_{\mu \nu}} &= T_{\alpha \beta \gamma} (p) k_{\omega}^\alpha (q) k_{\mu}^\beta (r) k_{\nu}^\gamma
\end{align*}
\]

where \( T_{\omega_{\mu \nu}} \) is an arbitrary tensor and

\[
\begin{align*}
(0) k_\lambda^\nu &= \delta_\lambda^\nu, \\
(1) k_\lambda^\nu &= k_\lambda^\nu, \\
(p) k_\lambda^\nu &= (p-1) k_\lambda^\alpha k_\alpha^\nu, \quad (p, q, r = 1, 2, \cdots)
\end{align*}
\]

We also use the scalars \( g \) and \( k \), defined by

\[
g = \frac{a}{\hbar}, \quad k = \frac{\ell}{\hbar}
\]

The following two lemmas were proved in 1.

**Lemma 4.1.** If the condition

\[
g \neq 0
\]

is satisfied in \( X_2 \), the system (2.8) of Einstein's equations admits a unique solution

\[
S_{\omega_{\mu \nu}} = \frac{1}{g} \nabla_\nu k_{\omega_{\mu}}
\]

where \( \nabla_\nu \) is the symbolic vector of the covariant derivative with respect to \( \{ \lambda_\nu^\mu \} \).

**Remark 4.2.** The condition (2.3) imposed to \( X_2 \) shows that the condition (4.4) is always satisfied. Therefore, we may remark that the system (2.8) of Einstein's equations always admit a unique solution of the form (4.5) in \( X_2 \).
LEMMA 4.3. If $T_{\omega \mu \nu}$ is a tensor skew-symmetric in the first two indices, the following recurrence relations hold in $X_2$:

\[(4.6a) \quad T^{(10)}_{\omega \mu \nu} = 0, \quad T^{11r}_{\omega \mu \nu} = k_{00r} T_{\omega \mu \nu} \]

\[(4.6b) \quad T^r_{\nu[\omega \mu]} = 0, \quad T^r_{\nu \omega} = k T^r_{\nu[\omega \mu]}, \quad (r = 0, 1, 2, \cdots) \]

LEMMA 4.4. If $T_{\omega \mu \nu}$ is a tensor skew-symmetric in the first two indices, then the following relations hold in $X_2$:

\[(4.7) \quad T^{pqr}_{\omega \mu \nu} = -T^{qpr}_{\nu \omega \mu} \]

\[(4.8) \quad T^{pqr}_{\omega \mu \nu} + T^{qrp}_{\mu \nu \omega} + T^{rpq}_{\nu \omega \mu} = 0, \quad (p, q, r = 0, 1, 2, \cdots) \]

Proof. The relation (4.7) is a direct consequence of (4.1). The relation (4.8) for the special case $p=q=r=0$, that is $T_{[\omega \mu \nu]}=0$, follows easily since all indices take the values $1, 2$ only in $X_2$ and $T_{\omega \mu \nu}$ is skew-symmetric in the first two indices. The relation (4.8) for the general case may be proved from the above special case as in the following way:

\[
0 = (T_{\alpha \beta \gamma} + T_{\beta \gamma \alpha} + T_{\gamma \alpha \beta})^{(p)} k_{\omega}^{\alpha(q)} k_{\mu}^{\beta(r)} k_{\nu}^{\gamma}
= T^{pqr}_{\omega \mu \nu} + T^{qrp}_{\mu \nu \omega} + T^{rpq}_{\nu \omega \mu}
\]

LEMMA 4.5. If the conditions (2.8) and (2.10) hold in $X_2$, they admit a unique 2-dimensional solution of the form

\[(4.9) \quad gS_{\omega \mu \nu} = 4h_{\nu[\omega k_{\mu}]}^{\alpha} X_{\alpha} - 4k_{1[\omega X_{\mu}]} \]

Proof. Employing the abbreviations (2.11) and (2.12) and using the relations (3.4), we have

\[
D_{\nu} k_{\omega \mu} = \partial_{\nu} k_{\omega \mu} - \Gamma_{\omega \mu}^{\alpha \nu} k_{\alpha \mu} - \Gamma_{\mu}^{\alpha \omega} k_{\omega \alpha}
= \partial_{\nu} k_{\omega \mu} - \left( \left\{ k_{\omega}^{\mu} \right\} + S_{\omega \nu}^{\alpha} + U_{\alpha \omega}^{\mu} \right) k_{\alpha \mu}
- \left( \left\{ k_{\mu}^{\nu} \right\} + S_{\mu \nu}^{\alpha} + U_{\alpha \nu}^{\mu} \right) k_{\omega \alpha}
= \nabla_{\nu} k_{\omega \mu} + \left( S_{\omega \nu}^{\alpha} - S_{\mu \nu}^{\alpha} \right) \quad (U_{\alpha \omega}^{\mu} k_{\mu}^{\alpha} - U_{\alpha \mu}^{\nu} k_{\nu}^{\alpha})
\]
Using (4.8) and (4.6a), the second term of the right-hand side of (4.10) may be written as

\[
\text{(second term)} = \left( 010 \quad 100 \right) S_{\nu\mu\omega} - \left( 010 \quad 100 \right) S_{\mu\nu\omega} + \left( 010 \quad 100 \right) S_{\nu\omega\mu}\nu
\]

\[
= \left( 010 \quad 100 \right) S_{\omega\mu\nu} + 2 \left( 010 \quad 010 \right) S_{\nu[\omega\mu]} = 2 S_{\nu[\omega\mu]}
\]

Using (3.5b), (4.6a) and (4.8) again, the third term of the right-hand side of (4.10) is

\[
\text{(third term)} = (-2S_{\alpha(\omega\nu)} + 4X_{(\omega h_{\nu})\alpha} - 2X_{\alpha h_{\nu\omega}})k_{\mu}^{\alpha} - (-2S_{\alpha(\mu\nu)} + 4X_{(\mu h_{\nu})\alpha} - 2X_{\alpha h_{\mu\nu}})k_{\omega}^{\alpha}
\]

\[
= -2 S_{\nu[\omega\mu]} - 4X_{\nu k_{\omega\mu} - 4h_{\nu[\omega k_{\mu}]}^{\alpha} X_{\alpha} + 4k_{\nu[\omega X_{\mu}]}^{\alpha}}
\]

We now substitute (3.1b), (4.11), and (4.12) into (4.10) to obtain

\[
\nabla_{\nu} k_{\omega\mu} = 4h_{\nu[\omega k_{\mu}]}^{\alpha} X_{\alpha} - 4k_{\nu[\omega X_{\mu}]}^{\alpha}
\]

Now, the solution (4.9) is the result of (4.13) and (4.5).

**Lemma 4.6.** If the conditions (2.8) and (2.10) hold in $X_2$, the tensor $U_{\nu\omega\mu}$ is given by

\[
U_{\nu\omega\mu} = \frac{4}{g} (k_{(\omega X_{\mu})} - \delta_{(\omega k_{\mu})}^{\alpha} X_{\alpha} - h_{\omega\nu} k_{(\omega X_{\mu})}^{\alpha})
\]

\[
+ 4X_{(\omega\delta_{\mu})}^{\nu} - 2X_{\nu} h_{\omega\mu}
\]

**Proof.** Using (4.9) we have

\[
-2g S_{\nu(\omega\mu)} = 4(k_{(\omega X_{\mu})} - \delta_{(\omega k_{\mu})}^{\alpha} X_{\alpha} - h_{\omega\mu} k_{(\omega X_{\mu})}^{\alpha})
\]

The representation (4.14) follows immediately by substituting (4.15) into (3.5)a.

Now that we have obtained the tensor $S_{\omega\mu}^{\nu}$ and $U_{\nu\omega\mu}$ in terms of $g_{\lambda\mu}$, it is possible for us to determine the 2-dimensional RE-connection $\Gamma_{\omega\nu}^{\nu}$ by only substituting for $S$ and $U$ into (3.4). We formally state our main theorem as follows:
Theorem 4.7. In $X_2$ there always exists a unique 2-dimensional RE-connection $\Gamma^\nu_\mu$ represented by

$$
\Gamma^\nu_\mu = \{^\nu_\mu\} + \frac{4}{g}(k^\nu_\omega X^\omega_\mu - \delta^\nu_\mu k^\alpha_\omega X_\omega^\alpha - h^\omega_\mu k^\nu_\alpha X^\alpha) + 4X^{(\omega\delta^\nu_\mu)} - 2X^\nu h^\omega_\mu
$$

(4.16)

References

5. V. Hlavatý, Geometry of Einstein's unified field theory, Noordhoop Ltd, 1957.

Department of Mathematics
University of Incheon
402-749, Inchon, Korea