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SEPARATING SETS AND SYSTEMS OF SIMULTANEOUS EQUATIONS IN THE PREDUAL OF AN OPERATOR ALGEBRA

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1. Introduction

Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded linear operaors on $\mathcal{H}$. A *dual algebra* is a subalgebra of $\mathcal{L}(\mathcal{H})$ that contains the identity operator $I_{\mathcal{H}}$ and is closed in the weak* topology on $\mathcal{L}(\mathcal{H})$. Note that the ultraweak operator topology coincides with the weak* topology on $\mathcal{L}(\mathcal{H})$ (see [5]). Bercovici-Foiaş-Pearcy [3] studied the problem of solving systems of simultaneous equations in the predual of a dual algebra. The theory of dual algebras has been applied to the topics of invariant subspace, dilation theory and reflexivity (see [4]), and is closely related with properties $(\mathbf{A}_{m,n})$ which are defined below. Hadwin-Nordgren [8] studied relationships between separating vectors and property $(\mathbf{A}_{1,1})$, and M. Marsalli [12] proved that a von Neumann algebra $\mathcal{A}$ has property $(\mathbf{A}_{1,1})$ if and only if $\mathcal{A}$ has property $(\mathbf{A}_{1,\aleph_0})$. In this paper, we obtain a relationship between separating sets and properties $(\mathbf{A}_{m,n})$. Finally, using some results of section 3, we obtain some characterizations for von Neumann algebras with property $(\mathbf{A}_{m,n})$ and prove that those algebras with properties $(\mathbf{A}_{n,n})$ are distinct one from another.

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2. Notation and preliminaries

The notation and terminology employed herein agree with those in [4]. The class $\mathcal{C}_1(\mathcal{H})$ is the Banach space of trace-class operators on $\mathcal{H}$ equipped with the trace norm. The dual algebra $\mathcal{A}$ can be identified with the dual space of $\mathcal{Q}_{\mathcal{A}} = \mathcal{C}_1(\mathcal{H})/{}^{\perp}\mathcal{A}$, where ${}^{\perp}\mathcal{A}$ is the preannihilator in $\mathcal{C}_1(\mathcal{H})$ of $\mathcal{A}$, under the pairing

$$< T, [L]_{\mathcal{A}} >= tr(TL), \quad T \in \mathcal{A}, \quad [L]_{\mathcal{A}} \in \mathcal{Q}_{\mathcal{A}}.$$

We write $[L]$ for $[L]_{\mathcal{A}}$ when there is no possibility of confusion. If x and y are vectors in $\mathcal{H}$, we define a rank one operator $x \otimes y$ by $(x \otimes y)u = (u, y)x$ for all u in $\mathcal{H}$. Throughout this paper, let $\mathbf{N}$ is the set of natural numbers.

DEFINITION 2.1. Suppose that m and n are cardinal numbers such that $1 \leq m, n \leq \aleph_0$. A dual algebra $\mathcal{A}$ will be said to have property $(\mathbf{A}_{m,n})$ if every $m \times n$ system of simultaneous equations of the form

$$[x_i \otimes y_j] = [L_{ij}], \quad 0 \leq i < m, \quad 0 \leq j < n,$$

where $\{[L_{ij}]\}_{\substack{0\leq i<m \\ 0\leq j<n}}$ is an arbitrary $m \times n$ array from $\mathcal{Q}_{\mathcal{A}}$, has a solution $\{x_i\}_{0\leq i<m}$, $\{y_j\}_{0\leq j<n}$ consisting of a pair of sequences of vectors from $\mathcal{H}$. For a brief notation, we shall denote $(\mathbf{A}_{n,n})$ by $(\mathbf{A}_n)$.

Suppose $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ is a dual algebra and $n \in \mathbf{N}$. We write $\mathcal{M}_n(\mathcal{A})$ for the subalgebra of $\mathcal{L}(\mathcal{H}^{(n)})$ consisting of all $n \times n$ matrices with entries from $\mathcal{A}$, where $\mathcal{H}^{(n)} = \underbrace{\mathcal{H} \oplus \cdots \oplus \mathcal{H}}_{(n)}$. Then it follows from Proposition 2.2 of [4] that $\mathcal{M}_n(\mathcal{A})$ is a dual algebra. In particular, the predual $\mathcal{Q}_{\mathcal{M}_n(\mathcal{A})}$ is identified with the Banach space $\mathcal{M}_n(\mathcal{Q}_{\mathcal{A}})$ consisting of all $n \times n$ matrices with entries from $\mathcal{Q}_{\mathcal{A}}$. The duality is given by the pairing

$$< (T_{ij}), ([L_{ij}]) >= \sum_{i,j=1}^{n} < T_{ij}, [L_{ij}] >.$$

$(T_{ij}) \in \mathcal{M}_n(\mathcal{A})$, $([L_{ij}]) \in \mathcal{M}_n(\mathcal{Q}_{\mathcal{A}})$. If $\tilde{x} = (x_1, \cdots, x_n)$ and $\tilde{y} = (y_1, \cdots, y_n)$ belong to $\mathcal{H}^{(n)}$, then $[\tilde{x} \otimes \tilde{y}]_{\mathcal{M}_n(\mathcal{A})}$ is identified with the $n \times n$ matrix $([x_j \otimes y_i]_{\mathcal{A}})$. The following theorem comes from Proposition 1.3 of [1].

THEOREM 2.2. *Let $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ be a dual algebra and let $k \in \mathbf{N}$. Then the dual algebra $\mathcal{M}_k(\mathcal{A})$ has property $(\mathbf{A}_1)$ if and only if $\mathcal{A}$ has property $(\mathbf{A}_k)$.*

For $T \in \mathcal{L}(\mathcal{H})$, we write $\mathcal{A}_T$ for the dual algebra generated by T and $T^{(n)} = \underbrace{T \oplus \cdots \oplus T}_{(n)}$.

3. n-Separating sets of operator algebras

Let $\mathcal{A}$ be a subalgebra of $\mathcal{L}(\mathcal{H})$ and let $\{x_i\}_{i=1}^n$ be a linearly independent subset of $\mathcal{H}$, where $n \in \mathbf{N}$. Then $\{x_i\}_{i=1}^n$ is said to be an *n-separating set* for $\mathcal{A}$ if $\sum_{i=1}^n T_i x_i = 0$ for $T_i \in \mathcal{A}$ implies $T_i = 0$, $1 \leq i \leq n$. And we call that $\mathcal{A}$ has an n-separating set $\{x_i\}_{i=1}^n$ (see [7] and [13]). It is obvious that if $m \leq n$, then an algebra with an n–separating set has an m–separating set. Let $\mathcal{A}$ be a subalgebra of $\mathcal{L}(\mathcal{H})$ and let $x \in \mathcal{H}$. Then we write $\mathcal{A}x = \{Tx | T \in \mathcal{A}\}$. Note that $\mathcal{A}x$ is not always closed in $\mathcal{H}$.

LEMMA 3.1. *Let $\mathcal{A}$ be a dual algebra of $\mathcal{L}(\mathcal{H})$ and let $x \in \mathcal{H}$ be a separating vector for $\mathcal{A}$ such that $\mathcal{A}x$ is closed subspace of $\mathcal{H}$. Then every element $[L] \in \mathcal{Q}_{\mathcal{A}}$ can be written as $[L] = [x \otimes y]$ for some $y \in \mathcal{H}$. Furthermore, $\mathcal{A}$ has property $(\mathbf{A}_{1,\aleph_0})$.*

Proof. Since x is a separating vector for $\mathcal{A}$ and $\mathcal{A}x$ is closed, it follows from Theorem 5.1 of [8] that every element $[L] \in \mathcal{Q}_{\mathcal{A}}$ can be written as $[L] = [x \otimes y]$ for some $y \in \mathcal{H}$. Furthermore, it is obvios that $\mathcal{A}$ has property $(\mathbf{A}_{1,\aleph_0})$, and the proof is complete.

We state a lemma whose proof is elementary and will be omitted.

LEMMA 3.2. *Suppose $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ is a dual algebra and $n \in \mathbf{N}$. Then $\mathcal{A}$ has an n-separating set $\{x_i\}_{i=1}^n$ if and only if $\mathcal{M}_n(\mathcal{A})$ has a separating vector $\tilde{x} = x_1 \oplus \cdots \oplus x_n \in \mathcal{H}^{(n)}$.*

The following lemma is an improvement of Theorem 2.2.

LEMMA 3.3. *Let $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ be a dual algebra. Suppose $k \in \mathbf{N}$ and $1 \leq n \leq \aleph_0$. Then the dual algebra $\mathcal{M}_k(\mathcal{A})$ has property $(\mathbf{A}_{n,1})$ if and only if $\mathcal{A}$ has property $(\mathbf{A}_{kn,k})$.*

Proof. First we shall prove this lemma when $1 \le n < \aleph_0$. Assume that the dual algebra $\mathcal{M}_k(\mathcal{A})$ has property $(\mathbf{A}_{n,1})$. If we give a sequence $\{[L_{ij}]\}_{\substack{1\le i\le m\\ 1\le j\le n}}$ in $Q_{\mathcal{A}}$, then there exist $\tilde{x}_i, \tilde{y} \in \mathcal{H}^{(k)}, 1 \le i \le n$, such that

$$(3.1) \qquad [\tilde{x}_i \otimes \tilde{y}] = \begin{pmatrix} [L_{(i-1)k+1,1}] & [L_{(i-1)k+2,1}] & \cdots & [L_{ik,1}] \\ [L_{(i-1)k+1,2}] & [L_{(i-1)k+2,2}] & \cdots & [L_{ik,2}] \\ \vdots & \vdots & \ddots & \vdots \\ [L_{(i-1)k+1,k}] & [L_{(i-1)k+2,k}] & \cdots & [L_{ik,k}] \end{pmatrix}.$$

Putting $\tilde{x}_i = x_{1i} \oplus \cdots \oplus x_{ki}, 1 \le i \le n$, and $\tilde{y} = y_1 \oplus \cdots \oplus y_k$, we get

$$[\tilde{x}_i \otimes \tilde{y}] = [(x_{1i} \oplus \cdots \oplus x_{ki}) \otimes (y_1 \oplus \cdots \oplus y_k)]$$

$$(3.2) \qquad = \begin{pmatrix} [x_{1i} \otimes y_1] & \cdots & [x_{ki} \otimes y_1] \\ \vdots & \ddots & \vdots \\ [x_{1i} \otimes y_k] & \cdots & [x_{ki} \otimes y_k] \end{pmatrix}.$$

Therefore we obtain

$$(3.3) \qquad \begin{pmatrix} [L_{(i-1)k+1,1}] & [L_{(i-1)k+1,2}] & \cdots & [L_{(i-1)k+1,k}] \\ [L_{(i-1)k+2,1}] & [L_{(i-1)k+2,2}] & \cdots & [L_{(i-2)k+2,k}] \\ \vdots & \vdots & \ddots & \vdots \\ [L_{ik,1}] & [L_{ik,2}] & \cdots & [L_{ik,k}] \end{pmatrix}$$

$$= \begin{pmatrix} [x_{1i} \otimes y_1] & [x_{1i} \otimes y_2] & \cdots & [x_{1i} \otimes y_k] \\ [x_{2i} \otimes y_1] & [x_{2i} \otimes y_2] & \cdots & [x_{2i} \otimes y_k] \\ \vdots & \vdots & \ddots & \vdots \\ [x_{ki} \otimes y_1] & [x_{ki} \otimes y_2] & \cdots & [x_{ki} \otimes y_k] \end{pmatrix},$$

for $1 \le i \le n$. Now suppose that we change the notation for convenience (for example, we put $u_1 = x_{11}, \cdots, u_k = x_{k1}, u_{k+1} = x_{12}, \cdots, u_{2k} = x_{k2}, u_{(n-1)k} = x_{1n}, \ldots, u_{nk} = x_{kn}$). Then it is easy to show that $[L_{ij}] = [u_i \otimes y_j]$ for $1 \le i \le kn, 1 \le j \le k$, which implies that $\mathcal{A}$ has property $(\mathbf{A}_{kn,k})$.

Conversely, repeating the above method, we can solve the required simultaneous systems for the property $(\mathbf{A}_{n,1})$. Because the calculation is similar with that of the above, we omit it here.

Finally, if we consider infinite matrices instead of the finite matrices in (3.1) and (3.2), then we can solve the required systems of simultaneous equations for the statement in the case of $n = \aleph_0$ by the similar method with the above. Hence the proof is complete.

The following theorem shows the relationship between n-separating sets and properties $(\mathbf{A}_{n,\aleph_0})$. And the following should be compaired with Lemma 3.2 and Proposition 2.13 of [11].

THEOREM 3.4. *Let $\mathcal{A}$ be a dual algebra of $\mathcal{L}(\mathcal{H})$ and let $n \in \mathbf{N}$. If $\mathcal{A}$ has an n-separating set $\{x_i\}_{i=1}^n$ such that $\sum_{i=1}^n \mathcal{A}x_i$ is closed, then $\mathcal{A}$ has property $(\mathbf{A}_{n,\aleph_0})$. Conversely, if $\mathcal{A}$ has property $(\mathbf{A}_{n,\aleph_0})$, then $\mathcal{A}$ has an n-separating set.*

Proof. For the proof of the first statement, we put $\tilde{x} = x_1 \oplus x_2 \oplus \cdots \oplus x_n \in \mathcal{H}^{(n)}$. Then by Lemma 3.2 $\tilde{x}$ is a separating vector for $\mathcal{M}_n(\mathcal{A})$ and

$$(3.4) \quad \mathcal{M}_n(\mathcal{A})\tilde{x} = \{y_1 \oplus y_2 \oplus \cdots \oplus y_n | y_i = \sum_{j=1}^n T_{ij}x_j, T_{ij} \in \mathcal{A}, 1 \leq i \leq n\}$$

is closed in $\mathcal{H}^{(n)}$. According to Lemma 3.1, every element $\widetilde{[L]} \in \mathcal{M}_n(\mathcal{Q}_\mathcal{A})$ can be written as

$$(3.5) \qquad \widetilde{[L]} = [\tilde{x} \otimes \tilde{y}]$$

for some $\tilde{y}$ in $\mathcal{H}^{(n)}$. Furthermore, for any sequence $\{\widetilde{[L]}_k\}_{k=1}^\infty$ in $\mathcal{M}_n(\mathcal{Q}_\mathcal{A})$ there exists a sequence $\{\tilde{y}_k\}_{k=1}^\infty$ of vectors from $\mathcal{H}^{(n)}$ such that $\widetilde{[L]}_k = [\tilde{x} \otimes \tilde{y}_k], 1 \leq k < \infty$. Now we claim that $\mathcal{A}$ has property $(\mathbf{A}_{n,\aleph_0})$. This proof is similar to that of Lemma 3.3. For convenient of the readers, we sketch the proof. First we give a system $\{[L_{ij}]\}_{\substack{1 \leq i \leq m \\ 1 \leq j \leq \infty}}$ in $\mathcal{Q}_\mathcal{A}$. Then by (3.5) there exists $\{\tilde{y}_k\}_{k=1}^\infty$ in $\mathcal{H}^{(n)}$ such that

$$(3.6) \qquad [\tilde{x} \otimes \tilde{y}_k] = \begin{pmatrix} [L_{1,(k-1)n+1}] & [L_{2,(k-1)n+1}] & \cdots & [L_{n,(k-1)n+1}] \\ [L_{1,(k-1)n+2}] & [L_{2,(k-1)n+2}] & \cdots & [L_{n,(k-1)n+2}] \\ \vdots & \vdots & \ddots & \vdots \\ [L_{1,kn}] & [L_{2,nk}] & \cdots & [L_{n,kn}] \end{pmatrix}.$$

If we put $\tilde{y}_k = y_{1k} \oplus \cdots \oplus y_{nk} \in \mathcal{H}^{(n)}$, then for $1 \le k < \aleph_0$, the matrix in the right side of (3.6) equals the following matrix

$$(3.7) \qquad \begin{pmatrix} [x_1 \otimes y_{1k}] & [x_2 \otimes y_{1k}] & \cdots & [x_n \otimes y_{1k}] \\ [x_1 \otimes y_{2k}] & [x_2 \otimes y_{2k}] & \cdots & [x_n \otimes y_{2k}] \\ \vdots & \vdots & \ddots & \vdots \\ [x_1 \otimes y_{nk}] & [x_2 \otimes y_{2k}] & \cdots & [x_n \otimes y_{2k}] \end{pmatrix}.$$

Now, if we change the notation as the proof of Lemma 3.3, we can solve the required system.

Conversely, if $\mathcal{A}$ has property $(\mathbf{A}_{n,\aleph_0})$, Lemma 3.3 and Proposition 2.3 in [11] imply that $\mathcal{M}_n(\mathcal{A})$ has property $(\mathbf{A}_{1,\aleph_0})$ and a separating vector $\tilde{x}$. Set $\tilde{x} = x_1 \oplus x_2 \oplus \cdots \oplus x_n \in \mathcal{H}^{(n)}$. It follows from Lemma 3.3, $\{x_i\}_{i=1}^n$ is an n-separating set for $\mathcal{A}$. Therefore the proof is complete.

Note that the condition of closure appearing in the hypothesis of Theorem 3.4 is essential (for example, consider a backward unilateral shift of multiplicity n).

The following is an immediate corollary of Theorem 3.4.

COROLLARY3.5. *Assume that $T \in \mathcal{L}(\mathcal{H})$ and a subset $\{x_i\}_{i=1}^n$ of $\mathcal{H}$ is an n-separating set for $\mathcal{A}_T$ such that $\sum_{i=1}^n \mathcal{A}_T x_i$ is closed, where $n \in \mathbf{N}$. Then $\mathcal{A}_{T^{(m)}}$ has property $(\mathbf{A}_{mn,\aleph_0})$ for any $m \in \mathbf{N}$.*

4. Von Neumann algebras with properties $(\mathbf{A}_{m,n})$

The idea of this section comes from [10] originally.

LEMMA 4.1. [12] *Let $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ be a von Neumann algebra. Then the following are equivalent:*

(1) *$\mathcal{A}$ has a separating vector.*
(2) *$\mathcal{A}$ has property $(\mathbf{A}_1)$.*
(3) *$\mathcal{A}$ has property $(\mathbf{A}_{1,\aleph_0})$.*

By Lemma 3.2, Lemma 3.3, and Lemma 4.1, we can prove the following Theorem easily.

THEOREM 4.2. *Suppose that n is a cardinal number such that $1 \leq n \leq \aleph_0$. Let $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ be a von Neumann algebra. Then the following are equivalent:*

(1) *$\mathcal{A}$ has property $(\mathbf{A}_n)$.*
(2) *$\mathcal{A}$ has property $(\mathbf{A}_{n,\aleph_0})$.*
(3) *$\mathcal{A}$ has an n-separating set.*
(4) *$\mathcal{M}_n(\mathcal{A})$ has a separating vector.*
(5) *$\mathcal{A}$ has property $(\mathbf{A}_{\aleph_0,n})$.*

Jung [9] proved that dual algebras with properties $(\mathbf{A}_{m,n})$ are distinct one from another. As a sequel study, we prove the following theorem.

THEOREM 4.3. *Suppose n is any cardinal number such that $1 \leq n \leq \aleph_0$. Then von Neumann algebras with properties $(\mathbf{A}_n)$ are distinct one from another.*

Before we prove this theorem, we need some lemmas.

LEMMA 4.4. *Suppose $n \in \mathbf{N}$. If a von Neumann algebra $\mathcal{A}$ has property $(\mathbf{A}_1)$, then the ampliation $\mathcal{A}^{(n)} = \{\underbrace{A \oplus A \oplus \cdots \oplus A}_{(n)} | A \in \mathcal{A}\}$ has property $(\mathbf{A}_n)$.*

Proof. By Theorem 4.2, $\mathcal{A}$ has property $(\mathbf{A}_{1,n})$. Then by a simple calculation, it is easy to show that $\mathcal{A}^{(n)}$ has property $(\mathbf{A}_n)$.

LEMMA 4.5. [11, Proposition 2.9] *Let $\mathcal{A}$ be a maximal abelian von Neumann algebra. Then $\mathcal{A}$ has property $(\mathbf{A}_1)$, but it does not have property $(\mathbf{A}_2)$.*

LEMMA 4.6. *If $\mathcal{A}$ is a commutative von Neumamm algebra in $\mathcal{L}(\mathcal{H})$, then it has property $(\mathbf{A}_1)$.*

Proof. Since every commutative von Neumann algebra in $\mathcal{L}(\mathcal{H})$ is contained in a maximal abelian von Neumann algebra $\mathcal{B}$ in $\mathcal{L}(\mathcal{H})$ (see [13, p.85]), $\mathcal{A}$ is contained in a maximal von Neumann algebra $\mathcal{B}$ in $\mathcal{L}(\mathcal{H})$. By Lemma 4.5 and Proposition 2.04 of [4], imply that $\mathcal{A}$ has property $(\mathbf{A}_1)$. Hence the proof is complete.

We write $\mathcal{W}_T$ for the von Neumann algebra generated by an operator $T \in \mathcal{L}(\mathcal{H})$. The following lemma is an immediate corollary of Lemma 4.6.

LEMMA 4.7. *If U is a unitary operator on $\mathcal{H}$, then $\mathcal{W}_U$ has property* $(\mathbf{A}_1)$.

Note that if U is a bilateral shift of multiplicity one, then it followes from [6] that $\mathcal{A}_{U^{(n)}}$ has property $(\mathbf{A}_n)$ but not property $(\mathbf{A}_{n+1})$, where $n \in \mathbf{N}$.

The following proposition proves Theorem 4.3.

PROPOSITION 4.8. *Let U be a bilateral shift of multiplicity one and let $n \in \mathbf{N}$. Then the von Neumann algebra $\mathcal{W}_U^{(n)}$ has property $(\mathbf{A}_n)$ but not property $(\mathbf{A}_{n+1})$.*

Proof. It follows from Lemma 4.7 and Lemma 4.4 that the n-th ampliation $\mathcal{W}_U^{(n)}$ has property $(\mathbf{A}_n)$. Moreover, since $\mathcal{W}_{U^{(n)}} \subset \mathcal{W}_U^{(n)}$, $\mathcal{W}_{U^{(n)}}$ has property $(\mathbf{A}_n)$ (see [4. Proposition 2.04]). However $\mathcal{W}_{U^{(n)}}$ does not have property $(\mathbf{A}_{n+1})$. (Otherwise, since $\mathcal{A}_{U^{(n)}} \subset \mathcal{W}_{U^{(n)}}, \mathcal{A}_{U^{(n)}}$ has property $(\mathbf{A}_{n+1})$. Hence $\mathcal{A}_{U^{(n)}}$ has property $(\mathbf{A}_{n+1})$, which induces a contradiction). Hence the proof is complete.

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