SOME REMARKS FOR THE SPECTRUM OF THE 
\( p \)-LAPLACIAN ON SASAKIAN MANIFOLDS

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1. Introduction

Let \((M, g)\) be a compact manifold of dimension \(n\) with metric tensor \(g\). Let \(\Delta^p = d\delta + \delta d\) be the Laplace-Beltrami operator acting on the space of smooth \(p\)-forms. Then we have the spectrum of \(\Delta^p\) for each \(0 \leq p \leq n\)

\[ Spec^p(M, g) = \{0 \leq \lambda_{1,p} \leq \lambda_{2,p} \cdots \uparrow +\infty\}, \]

where each eigenvalue is repeated according to its multiplicity. Many authors have studied the relationship between the spectrum of \(M\) and the geometry of \(M\). And also, Z.Olszak[1], J.S.Pak, J.C.Jeong and W.T.Kim[2], S.Yamaguchi and G.Chüman[7] and others studied the spectrum of Sasakian manifolds. In this paper we shall prove;

**Theorem A.** Let \(M = (M, \phi, \xi, \eta, g)\) and \(M' = (M', \phi', \xi', \eta', g')\) be compact c-Einstein Sasakian manifolds with \(Spec^pM = Spec^pM'\) for an arbitrary fixed \(p \geq 1\) (which implies \(dimM=dimM'=n\)). If \((n,p) \notin \{(15,1),(15,2),(15,13),(15,14)\}\), then \(M\) is of constant \(\phi\)-sectional curvature \(c\) if and only if \(M'\) is of constant \(\phi'\)-sectional curvature \(c'=c\).

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Theorem B. Let $\mathcal{M} = (M, \phi, \xi, \eta, g)$ and $\mathcal{M}' = (M', \phi', \xi, \eta, g)$ be compact Sasakian manifolds with $Spec^p(\mathcal{M}) = Spec^p(\mathcal{M}')$ (which implies $\dim M = \dim M' = n$). If $n$ is given, there exists an integer $p (0 \leq p \leq n)$ such that $\mathcal{M}$ is of constant $\phi$-sectional curvature $c$ if and only if $\mathcal{M}'$ is of constant $\phi'$-sectional curvature $c = c'$.

2. Preliminaries

By $R = (R_{ijk}^l), \rho = (R_{jk}) = (R_{ijk}^l)$ and $\sigma = (g^{jk}R_{jk})$ we denote the Riemannian curvature tensor, the Ricci tensor and the scalar curvature, respectively, and $g = (g_{ij})$ is a Riemannian metric tensor on $M$, $(g^{ij}) = (g_{ij})^{-1}$. For the tensor field $T$ on $M$ we denote $|T|$ the norm of $T$ with respect to $g$. Then for each $p \leq 2m + 1(=\dim M)$ the Minakshisundaram-Pleijel-Gaffney asymptotic expansion for $Spec^p(M, g)$ is given by

$$
\sum_{\alpha=0}^{\infty} e^{\exp(-\lambda_{\alpha, p} t)} = (4\pi t)^{-\frac{2m+1}{2}} [a_{0, p} + ta_{1, p} + \cdots + t^N a_{N, p}]
$$

$$
+ o(t^{N-m+\frac{1}{2}}) \quad \text{as} \quad t \downarrow 0,
$$

where $a_{0, p}, a_{1, p}, a_{2, p}, \cdots$ are numbers which can be expressed by (see [3])

$$
a_{0, p} = \binom{2m + 1}{p} \int_M dM,
$$

(2.1)

$$
a_{1, p} = \frac{1}{6} \left[ \binom{2m + 1}{p} - 6 \binom{2m - 1}{p - 1} \right] \int_M \sigma \, dM,
$$

(2.2)

$$
a_{2, p} = \frac{1}{360} \left\{ \binom{5(2m + 1)}{p} - 60 \binom{2m - 1}{p - 1} + 180 \binom{2m - 3}{p - 2} \right\} \sigma^2
$$

$$
+ \left\{ -2 \binom{2m + 1}{p} + 180 \binom{2m - 1}{p - 1} - 720 \binom{2m - 3}{p - 2} \right\} |\rho|^2
$$

$$
+ \left\{ 2 \binom{2m + 1}{p} - 30 \binom{2m - 1}{p - 1} + 180 \binom{2m - 3}{p - 2} \right\} |R|^2 \, dM,
$$

(2.3)

where $dM$ denotes the volume element of $M$, and $\binom{k}{r} = 0$ for $k \leq 0$ or $r < 0$. 
Some remarks for the spectrum of the $p$-Laplacian

Let $\mathcal{M} = (M, \phi, \xi, \eta, g)$ be a compact Sasakian manifold (cf. [8]). This means that $M$ is a $(2m + 1)$-dimensional compact differentiable manifold with a normal contact metric structure $(\phi, \xi, \eta, g)$, where $\phi = (\phi_i^j), \xi = (\xi^i), \eta = (\eta_i)$ are tensor fields of type (1,1), (1,0), (0,1) respectively.

Now we introduce the tensor fields $H = (H_{kjh})$ and $Q = (Q_{ij})$ on $\mathcal{M}$ defined by

$$H_{kjh} = R_{kjh} - \frac{c+3}{4}(g_{kh}g_{ji} - g_{ki}g_{jh}) - \frac{c-1}{4}(\phi_{kh}\phi_{ji} - \phi_{ki}\phi_{jh})$$

$$- 2\phi_{kj}\phi_{ih} - g_{kh}\eta_j\eta_i + g_{ki}\eta_j\eta_h - g_{ji}\eta_k\eta_h + g_{jh}\eta_k\eta_i),$$

$$Q_{ij} = R_{ij} - ag_{ij} - b\eta_i\eta_j,$$

where $c = \frac{\sigma - m(3m + 1)}{m(m+1)}$, $a = \frac{\sigma}{2m} - 1$ and $b = 2m + 1 - \frac{\sigma}{2m}$.

Then we have

(2.4) $|H|^2 = |R|^2 - \frac{2}{m(m+1)}\sigma^2 + \frac{4(3m+1)}{m+1}\sigma - \frac{4m(2m+1)(3m+1)}{m+1},$

(2.5) $|Q|^2 = |\rho|^2 - \frac{1}{2m}\sigma^2 + 2\sigma - 2m(2m+1).$

A Sasakian manifold $\mathcal{M} = (M, \phi, \xi, \eta, g)$ is called a space of constant $\phi$-sectional curvature $c$ (resp. $c$-Einstein) if $H$ (resp. $Q$) vanishes identically. It is well known that a space of constant $\phi$-sectional curvature is $c$-Einstein. For any $c$-Einstein manifold of dimension $\geq 5$, the scalar curvature is necessarily constant. A 3-dimensional $c$-Einstein manifold means that the scalar curvature is constant. On any 3-dimensional Sasakian manifold the tensor field $H$ vanishes, but in this case the scalar curvature may be non-constant. Therefore, in dimension 3, it is of constant $\phi$-sectional curvature if and only if $\sigma$ is constant.

We also consider the so-called contact Bochner curvature tensor field $B = (B_{kjh})$ defined on $\mathcal{M}$ by (cf. [1,7])

$$B_{kjh} = R_{kjh} - \frac{1}{2m+4}(g_{kh}R_{ji} - g_{ki}R_{jh} - g_{jh}R_{ki} + g_{ji}R_{kh})$$

$$- \phi_{kh}R_{ji}\phi_i^l + \phi_{ki}R_{jl}\phi_h^l - \phi_{ji}R_{kl}\phi_h^l + \phi_{jh}R_{kl}\phi_i^l$$

$$+ 2\phi_{kj}R_{il}\phi_h^l + 2\phi_{ih}R_{kl}\phi_j^l - R_{kh}\eta_j\eta_i + R_{ki}\eta_j\eta_h$$
\[
-R_{ji} \eta_k \eta_h + R_{jk} \eta_k \eta_l i + \frac{r - 4}{2m + 4} (g_{kh} g_{ji} - g_{ki} g_{jh}) \\
+ \frac{r + 2m}{2m + 4} (\phi_{kh} \phi_{ji} - \phi_{ki} \phi_{jh} - 2 \phi_{kj} \phi_{ih}) \\
- \frac{r}{2m + 4} (g_{kh} \eta_j \eta_j i - g_{ki} \eta_j \eta_h + g_{ji} \eta_k \eta_h - g_{jh} \eta_k \eta_i),
\]

where \( r = \frac{\sigma + 2m}{2m + 2} \). Then we also obtain

\[
|B|^2 = |R|^2 - \frac{8}{m + 2} |\rho|^2 + \frac{2}{(m + 1)(m + 2)} \sigma^2 \\
+ \frac{4(3m^2 + 3m - 2)}{(m + 1)(m + 2)} \sigma - 24m^2 + 36m - 56 + \frac{8(13m + 14)}{m + 1}(m + 2).
\]

Moreover, it may be easily seen that \( H = 0 \) if and only if \( B = 0 \) and \( Q = 0 \). From (2.4)~(2.6), we have

\[
|R|^2 = |B|^2 + \frac{8}{m + 2} |Q|^2 + \frac{2}{m(m + 1)} \sigma^2 - \frac{4(3m + 1)}{(m + 1)} \sigma \\
+ 24m^2 - 36m + 56 + \frac{8(4m^2 - 2m - 7)}{m + 1}.
\]

For \( p \notin \{1, 2, 3, 2m, 2m + 1\} \), substituting (2.7) into (2.3) yields

\[
a_{2,p} = \alpha \int_M \left[ 4P_1 |B|^2 + \frac{8}{m + 2} P_2 |Q|^2 + \frac{4}{m(m + 1)} P_3 \sigma^2 \\
+ \frac{16}{m + 1} P_4 \sigma + \frac{16m}{m + 1} P_5 \right] dM,
\]

where

\[
P_1 := P_1(m, p) = 8m^4 - (60p + 8)m^3 + (210p^2 - 120p - 2)m^2 \\
+ (-180p^3 + 225p^2 - 75p + 2)m \\
+ 45p^4 - 90p^3 + 60p^2 - 15p,
\]

\[
P_2 := P_2(m, p) = -4m^5 + (180p + 28)m^4 - (450p^2 - 300p + 23)m^3 \\
+ (360p^3 - 465p^2 + 15p - 7)m^2 \\
- (90p^4 - 180p^3 + 45p^2 - 15p - 6)m - 30p^2 + 30p,
\]
Some remarks for the spectrum of the $p$-Laplacian

\[ P_3 := P_3(m,p) = 20m^6 - (120p + 4)m^5 + (240p^2 - 9)m^4 
- (180p^3 + 30p^2 - 120p + 11)m^3 
+ (45p^4 + 90p^3 - 180p^2 - 15p + 1)m^2 
- (45p^4 - 90p^3 + 15p^2 - 3)m - 15p^2 + 15p, \]

\[ \alpha := \frac{(2m - 3)}{(p - 2)} \]

$P_4$ and $P_5$ are also constant depending only on $m$ and $p$.

For $p \in \{1, 2, 3, 2m, 2m + 1\}$, the formula (2.3) is of the form:

\[ a_{2,p} = \beta \int_M \left[ 4Q_1|B|^2 + \frac{8}{m + 2} Q_2|Q|^2 + \frac{4}{m(m + 1)} Q_3 \sigma^2 + \frac{16}{m + 1} Q_4 \sigma + \frac{16m}{m + 1} Q_5 \right] dM, \]  

(2.9)

where for $i = 1, 2, 3, 4, 5$

(i) if $p = 1, m \geq 2$, then

\[ \beta = \frac{1}{360}, \quad Q_i = Q_i(m) = \frac{P_i(m,1)}{2m(2m-1)(2m-2)}, \]

while for $(m,p) = (1,1)$, \( Q_1 = -6, Q_2 = \frac{165}{4}, Q_3 = 9 \),

(ii) if $p = 2, m \geq 2$, then

\[ \beta = \frac{1}{2 \times 360}, \quad Q_i = Q_i(m) = \frac{P_i(m,2)}{(2m-1)(2m-2)}, \]

while for $(m,p) = (1,2)$, \( Q_1 = -12, Q_2 = \frac{165}{2}, Q_3 = 18 \),

(iii) if $p = 3, m \geq 2$, then

\[ \beta = \frac{1}{6 \times 360}, \quad Q_i = Q_i(m) = \frac{P_i(m,3)}{2m-2}, \]

while for $(m,p) = (1,3)$, \( Q_1 = 3, Q_2 = \frac{15}{2}, Q_3 = 18 \),
(iv) if \( p = 2m, m \geq 2 \), then
\[
\beta = \frac{1}{360}, \quad Q_i = Q_i(m) = \frac{P_i(m, 2m)}{2m(2m - 1)(2m - 2)},
\]

(v) if \( p = 2m + 1, m \geq 2 \), then
\[
\beta = \frac{1}{360}, \quad Q_i = Q_i(m) = \frac{P_i(m, 2m + 1)}{(2m + 1)(2m)(2m - 1)(2m - 2)}.
\]

**Remark 1.** The signs of the coefficients of \(|B|^2, |Q|^2\) and \( \sigma^2 \) in the formulae (2.8) and (2.9) are respectively determined by the polynomials \( P_1, P_2 \) and \( P_3 \) when \((m, p) \neq (1, 1), (1, 2), (1, 3)\).

**Remark 2.** In the following table we list some particular values of \( m \) for \( p \leq 100 \).

<table>
<thead>
<tr>
<th>( p )</th>
<th>the values of ( m ) such that ( P_1, P_2, P_3 &gt; 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>[8,51]</td>
</tr>
<tr>
<td>2</td>
<td>[2,4] 6 [8,93]</td>
</tr>
<tr>
<td>3</td>
<td>[2,6] [9,136]</td>
</tr>
<tr>
<td>4</td>
<td>[3,8] [12,178]</td>
</tr>
<tr>
<td>5</td>
<td>[2,10] [14,221]</td>
</tr>
<tr>
<td>6</td>
<td>[4,12] [17,263]</td>
</tr>
<tr>
<td>7</td>
<td>[3,14] [19,305]</td>
</tr>
<tr>
<td>8</td>
<td>[5,16] [22,348]</td>
</tr>
<tr>
<td>9</td>
<td>4 [6.19] [25,390]</td>
</tr>
<tr>
<td>10</td>
<td>[6,9] [11,21] [27,433]</td>
</tr>
<tr>
<td>20</td>
<td>[10,11] [13,17] [24,43] [52,857]</td>
</tr>
<tr>
<td>30</td>
<td>[15,17] [19,25] [36,66] [77,1281]</td>
</tr>
<tr>
<td>40</td>
<td>[20,23] [26,33] [49,89] [101,1705]</td>
</tr>
<tr>
<td>50</td>
<td>[25,30] [32,41] [62,112] [126,2129]</td>
</tr>
<tr>
<td>60</td>
<td>[30,36] [39,50] [75,135] [150,2553]</td>
</tr>
<tr>
<td>70</td>
<td>[35,42] [45,58] [87,158] [174,2976]</td>
</tr>
<tr>
<td>80</td>
<td>[40,48] [52,181] [198,3400]</td>
</tr>
<tr>
<td>90</td>
<td>[280,3824]</td>
</tr>
<tr>
<td>100</td>
<td>[300,4248]</td>
</tr>
</tbody>
</table>

We obtain all the values found in \([1.7]\) when \( p = 1.2 \).
From now on we shall write (2.8) and (2.9) in the following form;

\[
\begin{align*}
\alpha_{2,p} &= \gamma \int_M \left[ 4R_1|B|^2 + \frac{8}{m+2} R_2|Q|^2 + \frac{4}{m(m+1)} R_3 \sigma^2 \\
&\quad + \frac{16}{m+1} R_4 \sigma + \frac{16m}{m+1} R_5 \right] dM,
\end{align*}
\]

(2.10)

where \( \gamma \) is either \( \alpha \) or \( \beta \), and \( R_i \) is either \( P_i \) or \( Q_i \) (i=1,2,3,4,5).

**Remark 3.** The equation \( \binom{2m+1}{p} - 6 \binom{2m-1}{p-1} = 0 \) does not admit the natural roots, In fact, \( \binom{2m+1}{p} - 6 \binom{2m-1}{p-1} = 0 \) if and only if \( m(2m+1) - 3p(2m-p+1) = 0 \) if and only if \( m = \frac{u-2}{2}, p = \frac{u-1}{2} \pm v \), where \( u^2 - 12v^2 = 1 \). Therefore \( m \) can not be a natural number, because \( u \) is an odd number.

**Remark 4.** Let \( \mathcal{M} = (\mathcal{M}, \phi, \xi, \eta, g) \) and \( \mathcal{M}' = (\mathcal{M}', \phi', \xi', \eta', g') \) be compact Sasakian manifolds with \( \text{Spec}^p(\mathcal{M}) = \text{Spec}^p(\mathcal{M}') \) for an arbitrary fixed \( p \geq 1 \). Then for any \( m \in N(2m+1 \geq p) \) such that the polynomials \( R_1, R_2 \) and \( R_3 \) are strictly positive(for example, some particular values listed in Remark 2), \( \mathcal{M} \) is of constant \( \phi \)-sectional curvature \( c \) if and only if \( \mathcal{M}' \) is of constant \( \phi' \)-sectional curvature \( c' \).

**Proof.** Assume that \( \mathcal{M}' \) has constant \( \phi' \)-sectional curvature \( c' \). Then our assumption \( \text{Spec}^p(\mathcal{M}) = \text{Spec}^p(\mathcal{M}') \) and Remark 3 imply

\[
\int_M \left[ 4R_1|B|^2 + \frac{8}{m+2} R_2|Q|^2 + \frac{4}{m(m+1)} R_3 \sigma^2 \right] dM
\]

(2.11)

\[
= \int_{\mathcal{M}'} \frac{4}{m(m+1)} R_3 \sigma'^2 dM,
\]

On the other hand

\[
\int_M \sigma^2 dM \geq \int_{\mathcal{M}'} \sigma'^2 dM',
\]

because \( \int_M \sigma dM = \int_{\mathcal{M}'} \sigma' dM' \), \( \sigma' = \text{constant} \), \( \int_M dM = \int_{\mathcal{M}'} dM' \). Hence from (2.11) we obtain \( B = 0 = Q \). Q.E.D.
3. Proof of Theorems

Proof of Theorem A. If $\mathcal{M}$ and $\mathcal{M}'$ are $c$-Einstein manifolds, then $Q = 0 = Q'$, and $\sigma, \sigma'$ are constants. By Remark 3 and (2.2), we have $\sigma = \sigma'$. The assumption $Spec^p(\mathcal{M}) = Spec^p(\mathcal{M}')$ implies

$$\int_{\mathcal{M}} 4R_1|B|^2dM = \int_{\mathcal{M}'} 4R_1|B'|^2dM'$$

But for $(n, p) \notin \{(15, 1), (15, 2), (15, 13), (15, 14)\}, R_1 \neq 0$ (cf. Theorem 3.1(i) in [4]). Hence $B = 0$ if and only if $B' = 0$. Q.E.D.

Let $S^n$ be an odd dimensional unit sphere with constant curvature 1. Then $\mathcal{S} = (S^n, \phi, \xi, \eta, g)$ admits a Sasakian structure $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ which is called a natural Sasakian structure on $S^n$. Using our THEOREM A, Remark 4 and Theorem 2([5]), we can deduce the characterization.

Corollary. Let $\mathcal{M} = (M, \phi, \xi, \eta, g)$ be a compact Sasakian manifold with $Spec^p(\mathcal{M}) = Spec^p(\mathcal{S})$ for a given $p \geq 1$. If (i) the functions $R_1, R_2, R_3$ are strictly positive, or (ii) $M$ is $c$-Einstein and $(n, p) \notin \{(15, 1), (15, 2), (15, 13), (15, 14)\}$, then $\mathcal{M}$ is isomorphic to $\mathcal{S}$, that is, there is an isometry $f : (M, g) \longrightarrow (S^n, \tilde{g})$ such that $f_*\xi = \tilde{\xi}, f^*\tilde{\eta} = \eta$ and $f_* \circ \phi = \tilde{\phi} \circ f_*$.

Proof of Theorem B. By Remark 1, for $(m, p) \notin (1, 1), (1, 2), (1, 3)$, it is sufficient to show that there exists an integer $p$ such that $P_1, P_2, P_3 > 0$. This can be done as follows $(2m + 1 =: n)$;

If $n = 3, 5, 7, 9, 11$, we choose $p = 0 ([1, 7])$. If $n = 13, 17 \leq n \leq 187$, we choose $p = 2$ (Remark 2). If $n = 15$, we choose $p = 4$ (Remark 2). If $n \geq 47(n = 16k - 1$ or $16k + 1$ or $16k + 3$ or $16k + 5$ or $16k + 7$ or $16k + 9$ or $16k + 11$ or $16k + 13(k \geq 3)$), we always choose $p = k$.

To see the last statement, we calculated the following polynomials $\overline{P}_1, \overline{P}_2, \overline{P}_3$, which can be obtained from (2.3) with $2m + 1 =: n$.

$$\overline{P}_1(n, p) := 4P_1(m, p) = 2n(n - 1)(n - 2)(n - 3) - 30(n - 2)(n - 3)p(n - p) + 180p(p - 1)(n - p)(n - p - 1),$$

$$\overline{P}_2(n, p) := 4P_2(m, p) = 2n(n - 1)(n - 2)(n - 3) - 30(n - 2)(n - 3)p(n - p) + 180p(p - 1)(n - p)(n - p - 1),$$

$$\overline{P}_3(n, p) := 4P_3(m, p) = 2n(n - 1)(n - 2)(n - 3) - 30(n - 2)(n - 3)p(n - p) + 180p(p - 1)(n - p)(n - p - 1).$$
\[
\widetilde{P}_2(n, p) := 8P_2(m, p) = -n(n - 1)(n - 2)(n - 3)(n + 3) \\
+ 90(n - 2)(n - 3)(n + 3)(n - p)p \\
- 360p(p - 1)(n - p)(n - p - 1)(n + 3) \\
+ 16n(n - 1)(n - 2)(n - 3) \\
- 240p(n - p)(n - 2)(n - 3) \\
+ 1440p(p - 1)(n - p)(n - p - 1),
\]

\[
\widetilde{P}_3(n, p) := 16P_3(m, p) \\
= 5(n + 1)n(n - 1)^2(n - 2)(n - 3) \\
- 60(n + 1)(n - 1)(n - 2)(n - 3)p(n - p) \\
+ 180p(p - 1)(n - p - 1)(n - p)(n - 1)(n + 1) \\
+ 16n(n - 1)(n - 2)(n - 3) - 240(n - 2)(n - 3)p(n - p) \\
+ 1440p(p - 1)(n - p - 1)(n - p) \\
- 2(n + 1)n(n - 1)(n - 2)(n - 3) \\
+ 180(n + 1)(n - 2)(n - 3)p(n - p) \\
- 720p(p - 1)(n - p - 1)(n - p)(n + 1). \quad Q.E.D.
\]

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