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SOME REMARKS FOR THE SPECTRUM OF THE p–LAPLACIAN ON SASAKIAN MANIFOLDS

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1. Introduction

Let (M, g) be a compact manifold of dimension n with metric tensor g. Let $\Delta^p = d\delta + \delta d$ be the Laplace-Beltrami operator acting on the space of smooth p-forms. Then we have the spectrum of Δ^p for each $0 \leq p \leq n$

$$Spec^p(M, g) = \{0 \leq \lambda_{1,p} \leq \lambda_{2,p} \cdots \uparrow +\infty\},$$

where each eigenvalue is repeated according to its multiplicity. Many authors have studied the relationship between the spectrum of M and the geometry of M. And also, Z.Olszak[1], J.S.Pak, J.C.Jeong and W.T.Kim[2], S.Yamaguchi and G.Chūman[7] and others studied the spectrum of Sasakian manifolds. In this paper we shall prove ;

THEOREM A. *Let $\mathcal{M} = (M, \phi, \xi, \eta, g)$ and $\mathcal{M}' = (M', \phi', \xi', \eta', g')$ be compact c-Einstein Sasakian manifolds with $Spec^p\mathcal{M} = Spec^p\mathcal{M}'$ for an arbitrary fixed $p \geq 1$ (which implies $dimM$=$dimM'$=n). If $(n,p) \notin \{(15,1),(15,2),(15,13),(15,14)\}$, then $\mathcal{M}$ is of constant ϕ-sectional curvature c if and only if $\mathcal{M}'$ is of constant ϕ'-sectional curvature c'=c.*

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THEOREM B. *Let* $\mathcal{M} = (M, \phi, \xi, \eta, g)$ *and* $\mathcal{M}' = (M', \phi', \xi, \eta, g)$ *be compact Sasakian manifolds with* $Spec^p(\mathcal{M}) = Spec^p(\mathcal{M}')$*(which implies* $dim M = dim M' = n$*). If* n *is given, there exists an integer* $p(0 \leq p \leq n)$ *such that* $\mathcal{M}$ *is of constant* ϕ*- sectional curvature* c *if and only if* $\mathcal{M}'$ *is of constant* ϕ'*- sectional curvature* $c = c'$.

2. Preliminaries

By $R = (R_{ijk}^{\ \ l}), \rho = (R_{jk}) = (R_{ljk}^{\ \ l})$ and $\sigma = (g^{jk}R_{jk})$ we denote the Riemannian curvature tensor, the Ricci tensor and the scalar curvature, respectively, and $g = (g_{ij})$ is a Riemannian metric tensor on M, $(g^{ij}) = (g_{ij})^{-1}$. For the tensor field T on M we denote $|T|$ the norm of T with respect to g. Then for each $p \leq 2m + 1(=\dim M)$ the Minakshisundaram-Pleijel-Gaffney asymptotic expansion for $Spec^p(M, g)$ is given by

$$\sum_{\alpha=0}^{\infty} exp(-\lambda_{\alpha,p}t) = (4\pi t)^{-\frac{2m+1}{2}}[a_{0,p} + ta_{1,p} + \cdots + t^N a_{N,p}] + o(t^{N-m+\frac{1}{2}}) \quad \text{as} \quad t \downarrow 0,$$

where $a_{0,p}, a_{1,p}, a_{2,p}, \cdots$ are numbers which can be expressed by (see [3])

$$a_{0,p} = \binom{2m+1}{p} \int_M dM, \tag{2.1}$$

$$a_{1,p} = \frac{1}{6}\left[\binom{2m+1}{p} - 6\binom{2m-1}{p-1}\right] \int_M \sigma dM, \tag{2.2}$$

$$\begin{aligned} a_{2,p} = \frac{1}{360} \int_M [&\left\{5\binom{2m+1}{p} - 60\binom{2m-1}{p-1} + 180\binom{2m-3}{p-2}\right\}\sigma^2 \\ &+ \left\{-2\binom{2m+1}{p} + 180\binom{2m-1}{p-1} - 720\binom{2m-3}{p-2}\right\}|\rho|^2 \\ &+ \left\{2\binom{2m+1}{p} - 30\binom{2m-1}{p-1} + 180\binom{2m-3}{p-2}\right\}|R|^2]\, dM, \end{aligned} \tag{2.3}$$

where dM denotes the volume element of M, and $\binom{k}{r} = 0$ for $k \leq 0$ or $r < 0$.

Let $\mathcal{M} = (M, \phi, \xi, \eta, g)$ be a compact Sasakian manifold (cf. [8]). This means that M is a $(2m+1)$-dimensional compact differentiable manifold with a normal contact metric structure (ϕ, ξ, η, g), where $\phi = (\phi_i^j), \xi = (\xi^i), \eta = (\eta_i)$ are tensor fields of type (1,1), (1,0), (0,1) respectively.

Now we introduce the tensor fields $H = (H_{kjih})$ and $Q = (Q_{ij})$ on $\mathcal{M}$ defined by

$$H_{kjih} = R_{kjih} - \frac{c+3}{4}(g_{kh}g_{ji} - g_{ki}g_{jh}) - \frac{c-1}{4}(\phi_{kh}\phi_{ji} - \phi_{ki}\phi_{jh} - 2\phi_{kj}\phi_{ih} - g_{kh}\eta_j\eta_i + g_{ki}\eta_j\eta_h - g_{ji}\eta_k\eta_h + g_{jh}\eta_k\eta_i),$$

$$Q_{ij} = R_{ij} - ag_{ij} - b\eta_i\eta_j,$$

where $c = \dfrac{\sigma - m(3m+1)}{m(m+1)}, a = \dfrac{\sigma}{2m} - 1$ and $b = 2m + 1 - \dfrac{\sigma}{2m}$. Then we have

(2.4)

$$|H|^2 = |R|^2 - \frac{2}{m(m+1)}\sigma^2 + \frac{4(3m+1)}{m+1}\sigma - \frac{4m(2m+1)(3m+1)}{m+1},$$

(2.5) $$|Q|^2 = |\rho|^2 - \frac{1}{2m}\sigma^2 + 2\sigma - 2m(2m+1).$$

A Sasakian manifold $\mathcal{M} = (M, \phi, \xi, \eta, g)$ is called a *space of constant ϕ-sectional curvature* c(resp.*c-Einstein*) if H(resp.Q) vanishes identically. It is well known that a space of constant ϕ-sectional curvature is c-Einstein. For any c-Einstein manifold of dimension ≥ 5, the scalar curvature is necessarily constant. A 3-dimensional c-Einstein manifold means that the scalar curvature is constant. On any 3-dimensional Sasakian manifold the tensor field H vanishes, but in this case the scalar curvature may be non-constant. Therefore, in dimension 3, it is of constant ϕ-sectional curvature if and only if σ is constant.

We also consider the so-called *contact Bochner curvature tensor field* $B = (B_{kjih})$ defined on $\mathcal{M}$ by (cf.[1,7])

$$B_{kjih} = R_{kjih} - \frac{1}{2m+4}(g_{kh}R_{ji} - g_{ki}R_{jh} - g_{jh}R_{ki} + g_{ji}R_{kh} - \phi_{kh}R_{jl}\phi_i^l + \phi_{ki}R_{jl}\phi_h^l - \phi_{ji}R_{kl}\phi_h^l + \phi_{jh}R_{kl}\phi_i^l + 2\phi_{kj}R_{il}\phi_h^l + 2\phi_{ih}R_{kl}\phi_j^l - R_{kh}\eta_j\eta_i + R_{ki}\eta_j\eta_h$$

$$- R_{ji}\eta_k\eta_h + R_{jh}\eta_k\eta_i) + \frac{r-4}{2m+4}(g_{kh}g_{ji} - g_{ki}g_{jh})$$
$$+ \frac{r+2m}{2m+4}(\phi_{kh}\phi_{ji} - \phi_{ki}\phi_{jh} - 2\phi_{kj}\phi_{ih})$$
$$- \frac{r}{2m+4}(g_{kh}\eta_j\eta_i - g_{ki}\eta_j\eta_h + g_{ji}\eta_k\eta_h - g_{jh}\eta_k\eta_i),$$

where $r = \dfrac{\sigma+2m}{2m+2}$. Then we also obtain

(2.6)
$$|B|^2 = |R|^2 - \frac{8}{m+2}|\rho|^2 + \frac{2}{(m+1)(m+2)}\sigma^2 + \frac{4(3m^2+3m-2)}{(m+1)(m+2)}\sigma - 24m^2 + 36m - 56 + \frac{8(13m+14)}{(m+1)(m+2)}.$$

Moreover, it may be easily seen that $H=0$ if and only if $B=0$ and $Q=0$. From (2.4)~(2.6), we have

$$|R|^2 = |B|^2 + \frac{8}{m+2}|Q|^2 + \frac{2}{m(m+1)}\sigma^2 - \frac{4(3m+1)}{(m+1)}\sigma + 24m^2 - 36m + 56 + \frac{8(4m^2-2m-7)}{m+1}. \tag{2.7}$$

For $p \notin \{1,2,3,2m,2m+1\}$, substituting (2.7) into (2.3) yields

$$a_{2,p} = \alpha \int_M \Big[4P_1|B|^2 + \frac{8}{m+2}P_2|Q|^2 + \frac{4}{m(m+1)}P_3\sigma^2 + \frac{16}{m+1}P_4\sigma + \frac{16m}{m+1}P_5\Big]\, dM, \tag{2.8}$$

where

$$\begin{aligned} P_1 := P_1(m,p) = {} & 8m^4 - (60p+8)m^3 + (210p^2 - 120p - 2)m^2 \\ & + (-180p^3 + 225p^2 - 75p + 2)m \\ & + 45p^4 - 90p^3 + 60p^2 - 15p, \end{aligned}$$

$$\begin{aligned} P_2 := P_2(m,p) = {} & -4m^5 + (180p+28)m^4 - (450p^2 - 300p + 23)m^3 \\ & + (360p^3 - 465p^2 + 15p - 7)m^2 \\ & - (90p^4 - 180p^3 + 45p^2 - 15p - 6)m - 30p^2 + 30p, \end{aligned}$$

$$
\begin{aligned}
P_3 := P_3(m,p) = 20m^6 &- (120p+4)m^5 + (240p^2-9)m^4 \\
&- (180p^3+30p^2-120p+11)m^3 \\
&+ (45p^4+90p^3-180p^2-15p+1)m^2 \\
&- (45p^4-90p^3+15p^2-3)m - 15p^2+15p,
\end{aligned}
$$

$$
\alpha := \frac{\binom{2m-3}{p-2}}{360p(p-1)(2m-p+1)(2m-p)},
$$

P_4 and P_5 are also constant depending only on m and p.

For $p \in \{1,2,3,2m,2m+1\}$, the formula (2.3) is of the form ;

$$
\begin{aligned}
a_{2,p} = \beta \int_M [4Q_1|B|^2 &+ \frac{8}{m+2}Q_2|Q|^2 + \frac{4}{m(m+1)}Q_3\sigma^2 \\
&+ \frac{16}{m+1}Q_4\sigma + \frac{16m}{m+1}Q_5]\, dM,
\end{aligned}
\tag{2.9}
$$

where for $i = 1,2,3,4,5$

(i) if $p = 1, m \geq 2$, then

$$
\beta = \frac{1}{360}, \quad Q_i = Q_i(m) = \frac{P_i(m,1)}{2m(2m-1)(2m-2)},
$$

while for $(m,p) = (1,1)$, $\quad Q_1 = -6, Q_2 = \frac{165}{4}, Q_3 = 9,$

(ii) if $p = 2, m \geq 2$, then

$$
\beta = \frac{1}{2 \times 360}, \quad Q_i = Q_i(m) = \frac{P_i(m,2)}{(2m-1)(2m-2)},
$$

while for $(m,p) = (1,2)$, $\quad Q_1 = -12, Q_2 = \frac{165}{2}, Q_3 = 18,$

(iii) if $p = 3, m \geq 2$, then

$$
\beta = \frac{1}{6 \times 360}, \quad Q_i = Q_i(m) = \frac{P_i(m,3)}{2m-2},
$$

while for $(m,p) = (1,3)$, $\quad Q_1 = 3, Q_2 = \frac{15}{2}, Q_3 = 18,$

(iv) if $p = 2m, m \geq 2$, then

$$\beta = \frac{1}{360}, \quad Q_i = Q_i(m) = \frac{P_i(m, 2m)}{2m(2m-1)(2m-2)},$$

(v) if $p = 2m+1, m \geq 2$, then

$$\beta = \frac{1}{360}, \quad Q_i = Q_i(m) = \frac{P_i(m, 2m+1)}{(2m+1)(2m)(2m-1)(2m-2)}.$$

REMARK 1. The signs of the coefficients of $|B|^2, |Q|^2$ and σ^2 in the formulae (2.8) and (2.9) are respectively determined by the polynomials P_1, P_2 and P_3 when $(m,p) \neq (1,1), (1,2), (1,3)$.

REMARK 2. In the following table we list some particular values of m for $p \leq 100$.

p	the values of m such that $P_1, P_2, P_3 > 0$			
1	[8,51]			
2	[2,4]	6	[8,93]	
3	[2,6]	[9,136]		
4	[3,8]	[12,178]		
5	[2,10]	[14,221]		
6	[4,12]	[17,263]		
7	[3,14]	[19,305]		
8	[5,16]	[22,348]		
9	4	[6,19]	[25,390]	
10	[6,9]	[11,21]	[27,433]	
20	[10,11]	[13,17]	[24,43]	[52,857]
30	[15,17]	[19,25]	[36,66]	[77,1281]
40	[20,23]	[26,33]	[49,89]	[101,1705]
50	[25,30]	[32,41]	[62,112]	[126,2129]
60	[30,36]	[39,50]	[75,135]	[150,2553]
70	[35,42]	[45,58]	[87,158]	[174,2976]
80	[40,48]	[52,181]	[198,3400]	
90	[280,3824]			
100	[300,4248]			

We obtain all the values found in [1,7] when $p = 1, 2$.

From now on we shall write (2.8) and (2.9) in the following form ;

$$\begin{aligned} a_{2,p} = \gamma \int_M [4R_1|B|^2 + \frac{8}{m+2}R_2|Q|^2 + \frac{4}{m(m+1)}R_3\sigma^2 \\ + \frac{16}{m+1}R_4\sigma + \frac{16m}{m+1}R_5]\, dM, \end{aligned} \tag{2.10}$$

where γ is either α or β, and R_i is either P_i or Q_i (i=1,2,3,4,5).

REMARK 3. The equation $\binom{2m+1}{p} - 6\binom{2m-1}{p-1} = 0$ does not admit the natural roots, In fact, $\binom{2m+1}{p} - 6\binom{2m-1}{p-1} = 0$ if and only if $m(2m+1) - 3p(2m-p+1) = 0$ if and only if $m = \frac{u-2}{2}, p = \frac{u-1}{2} \pm v$, where $u^2 - 12v^2 = 1$. Therefore m can not be a natural number, because u is an odd number.

REMARK 4. Let $\mathcal{M} = (M, \phi, \xi, \eta, g)$ and $\mathcal{M}' = (M', \phi', \xi', \eta', g')$ be compact Sasakian manifolds with $Spec^p(\mathcal{M}) = Spec^p(\mathcal{M}')$ for an arbitrary fixed $p \geq 1$. Then for any $m \in N(2m+1 \geq p)$ such that the polynomials R_1, R_2 and R_3 are strictly positive(for example, some particular values listed in Remark 2), $\mathcal{M}$ is of constant ϕ-sectional curvature c if and only if $\mathcal{M}'$ is of constant ϕ'-sectional curvature $c = c'$.

Proof. Assume that $\mathcal{M}'$ has constant ϕ'-sectional curvature c'. Then our assumption $Spec^p(\mathcal{M}) = Spec^p(\mathcal{M}')$ and Remark 3 imply

$$\begin{aligned} \int_M \left[4R_1|B|^2 + \frac{8}{m+2}R_2|Q|^2 + \frac{4}{m(m+1)}R_3\sigma^2 \right] dM \\ = \int_{M'} \frac{4}{m(m+1)} R_3'\sigma'^2 dM, \end{aligned} \tag{2.11}$$

On the other hand

$$\int_M \sigma^2 dM \geq \int_{M'} \sigma'^2 dM',$$

because $\int_M \sigma\, dM = \int_{M'} \sigma'\, dM', \sigma' = $ constant, $\int_M dM = \int_{M'} dM'$. Hence from (2.11) we obtain $B = 0 = Q$. Q.E.D.

3. Proof of Theorems

Proof of Theorem A. If $\mathcal{M}$ and $\mathcal{M}'$ are c-Einstein manifolds, then $Q = 0 = Q'$, and σ, σ' are constants. By Remark 3 and (2.2), we have $\sigma = \sigma'$. The assumption $Spec^p(\mathcal{M}) = Spec^p(\mathcal{M}')$ implies

$$\int_M 4R_1|B|^2 dM = \int_{M'} 4R_1|B'|^2 dM'$$

But for $(n,p) \notin \{(15,1),(15,2),(15,13),(15,14)\}, R_1 \neq 0$ (cf. Theorem 3.1(i) in [4]). Hence $B = 0$ if and only if $B' = 0$. Q.E.D.

Let S^n be an odd dimensional unit sphere with constant curvature 1. Then $\mathcal{S} = (S^n, \bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$ admits a Sasakian structure $(\bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$ which is called a ***natural Sasakian structure*** on S^n. Using our THEOREM A, Remark 4 and Theorem 2([5]), we can deduce the characterization.

COROLLARY. *Let $\mathcal{M} = (M, \phi, \xi, \eta, g)$ be a compact Sasakian manifold with $Spec^p(\mathcal{M}) = Spec^p(\mathcal{S})$ for a given $p \geq 1$. If (i) the functions R_1, R_2, R_3 are strictly positive, or (ii) M is c-Einstein and $(n,p) \notin \{(15,1),(15,2),(15,13),(15,14)\}$, then $\mathcal{M}$ is isomorphic to $\mathcal{S}$, that is, there is an isometry $f : (M,g) \longrightarrow (S^n, \bar{g})$ such that $f_*\xi = \bar{\xi}, f^*\bar{\eta} = \eta$ and $f_* \circ \phi = \bar{\phi} \circ f_*$.*

Proof of Theorem B. By Remark 1, for $(m,p) \notin (1,1),(1,2),(1,3)$, it is sufficient to show that there exists an integer p such that $P_1, P_2, P_3 > 0$. This can be done as follows $(2m+1 =: n)$;

If $n = 3,5,7,9,11$, we choose $p = 0$ ([1,7]). If $n = 13, 17 \leq n \leq 187$, we choose $p = 2$ (Remark 2). If $n = 15$, we choose $p = 4$ (Remark 2). If $n \geq 47 (n = 16k-1$ or $16k+1$ or $16k+3$ or $16k+5$ or $16k+7$ or $16k+9$ or $16k+11$ or $16k+13 (k \geq 3))$, we always choose $p = k$.

To see the last statement, we calculated the following polynomials $\widetilde{P_1}, \widetilde{P_2}, \widetilde{P_3}$, which can be obtained from (2.3) with $2m+1 =: n$.

$$\begin{aligned} \widetilde{P_1}(n,p) := 4P_1(m,p) = \; & 2n(n-1)(n-2)(n-3) \\ & - 30(n-2)(n-3)p(n-p) \\ & + 180p(p-1)(n-p)(n-p-1), \end{aligned}$$

$$\begin{aligned}\widetilde{P_2}(n,p) := 8P_2(m,p) = & -n(n-1)(n-2)(n-3)(n+3) \\ & + 90(n-2)(n-3)(n+3)(n-p)p \\ & - 360p(p-1)(n-p)(n-p-1)(n+3) \\ & + 16n(n-1)(n-2)(n-3) \\ & - 240p(n-p)(n-2)(n-3) \\ & + 1440p(p-1)(n-p)(n-p-1),\end{aligned}$$

$$\begin{aligned}\widetilde{P_3}(n,p) & := 16P_3(m,p) \\ = & \; 5(n+1)n(n-1)^2(n-2)(n-3) \\ & - 60(n+1)(n-1)(n-2)(n-3)p(n-p) \\ & + 180p(p-1)(n-p-1)(n-p)(n-1)(n+1) \\ & + 16n(n-1)(n-2)(n-3) - 240(n-2)(n-3)p(n-p) \\ & + 1440p(p-1)(n-p-1)(n-p) \\ & - 2(n+1)n(n-1)(n-2)(n-3) \\ & + 180(n+1)(n-2)(n-3)p(n-p) \\ & - 720p(p-1)(n-p-1)(n-p)(n+1). \quad Q.E.D.\end{aligned}$$

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