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A STUDY ABOUT THE CONVERSE OF R.B.HOLMES' THEOREM

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Let M be a nonempty subset of a normed linear space X and let $x \in X$. An element $y \in M$ satisfying

$$\|x - y\| = d(x, M) := \inf\{\|x - m\| : m \in M\}$$

is called a *best approximation* to x from M. For a given $x \in X$, the set of all best approximations to x from M is denoted by

$$P_M(x) := \{m \in M : \|x - m\| = d(x, M)\}.$$

The set M is called *proximinal* (resp, *Chebyshev*) if for every x in X, $P_M(x)$ is nonempty (resp, a singleton). If M is a proximinal subset of X, then the set-valued mapping $P_M : X \longrightarrow 2^M$ defined by $x \longmapsto P_M(x)$ is called the *metric projection* onto M.

For a given normed linear space X, let

$$B(X) = \{x \in X : \|x\| \leq 1\}$$

$$S(X) = \{x \in X : \|x\| = 1\}.$$

$B(X)$ is called the unit ball, and $S(X)$ is called the unit sphere in X.

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DEFINITION 1. The modulus of convexity of a normed linear space X is the function $\delta = \delta_X : [0,2] \longrightarrow [0,1]$ defined by

$$\delta(\varepsilon) = \inf\{1 - \frac{1}{2}\|x+y\| : x, y \in S(X), \|x-y\| \geq \varepsilon\}.$$

X is called a *uniformly convex* space if $\delta(\varepsilon) > 0$ for every $0 < \varepsilon \leq 2$. Geometrically, X is uniformly convex if the midpoints of line segments having length at least ε and its endpoints on the unit sphere must lie uniformly deep inside the unit ball.

The next result is well known.

LEMMA 2. [1, 2] *If X is a finite−dimensional normed linear space, then X is strictly convex if and only if it is uniformly convex.*

DEFINITION 3. Let X be a normed linear space and let $H(X)$ denote the family of all nonempty, bounded, closed and convex subsets of X. Define $H : H(X) \times H(X) \longrightarrow \mathbb{R}$ by

$$H(A, B) = \max\{h(A, B), h(B, A)\}$$

where $h(A, B) = \sup_{a \in A} d(a, B)$. Then H is a metric on $H(X)$, called the Hausdorff metric.

In [2], we know that R.B.Holmes proved the following theorem.

THEOREM 4. [2] *Let X be a uniformly convex space. For any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that for any x, y in the unit ball of X with $\|x - y\| < \delta$ imply that*

$$\|P_M(x) - P_M(y)\| < \varepsilon$$

for every proximinal subspace M of X. In other words, the set of metric projections

$$\{P_M : \ M \text{ is a proximinal subspace } \}$$

is a uniformly equicontinuous family on the unit ball.

OPEN PROBLEM. *Does the converse of R.B.Holmes' Theorem hold ? That is, if*

$$\{P_M : M \text{ is a proximinal subspace }\}$$

is a uniformly equicontinuous family on the unit ball of a normed linear space X, must X be uniformly convex ?

In [2], F.R.Deutsch gave the open problem and expected that the answer is no. By the following theorem, we will give a counterexample for the above open problem.

THEOREM 5. *Let $X = \mathbb{R}^2$ be a normed linear space with $\|(x,y)\| = |x|+|y|$. For any $\varepsilon > 0$ and any x, y in the unit ball of X with $\|x-y\| < \frac{\varepsilon}{2}$,*

$$H(P_M(x), P_M(y)) < \varepsilon$$

for every proximinal subspace M of X.

Proof. Let $\varepsilon > 0$ be given and let $x, y \in B(X)$ be with $\|x-y\| < \frac{\varepsilon}{2}$, that is, $d(x,y) = \|x-y\| = |x_1 - y_1| + |x_2 - y_2| < \frac{\varepsilon}{2}$, where $x = (x_1, x_2), y = (y_1, y_2)$. Assume that $M = \text{span}\{(1,\alpha)\}$ if $\alpha \in \mathbb{R}$ and $M = \text{span}\{(0,1)\}$ if $\alpha = \infty$.

Now we claim that for every proximinal subspace M of X,

$$H(P_M(x), P_M(y)) \leq 2\|x-y\|$$

for every $x, y \in B(X)$.

If $1 < |\alpha| \leq \infty$, then $P_M(x) = \{(\frac{x_2}{\alpha},\ x_2)\}$. So

$$\begin{aligned}\|P_M(x) - P_M(y)\| &= |\frac{1}{\alpha}|\ |x_2 - y_2| + |x_2 - y_2| \\ &< 2\ |x_2 - y_2| \\ &\leq 2\ \|x - y\|.\end{aligned}$$

If $0 \leq |\alpha| < 1$, then $P_M(x) = \{(x_1,\ \alpha x_1)\}$. So

$$\begin{aligned}\|P_M(x) - P_M(y)\| &= |x_1 - y_1| + |\alpha|\ |x_1 - y_1| \\ &< 2\ |x_1 - y_1| \\ &\leq 2\|x - y\|.\end{aligned}$$

It remains to prove the claim for $\alpha = 1$, and $\alpha = -1$.

Case 1 : $\alpha = 1$. Note that for every $x = (x_1, x_2) \in X$,

$$P_M(x) = [(x_1, x_1), (x_2, x_2)]$$

where $[(x_1, x_1), (x_2, x_2)]$ is the line segment with the end points (x_1, x_1) and (x_2, x_2).
If $(x_1 \leq x_2,\ y_1 \leq y_2)$ or $(x_2 \leq x_1,\ y_2 \leq y_1)$, then

$$\begin{aligned} H(P_M(x), P_M(y)) &= \max\{d((x_1, x_1), (y_1, y_1)),\ d((x_2, x_2), (y_2, y_2))\} \\ &= \max\{2\ |x_1 - y_1|,\ 2\ |x_2 - y_2|\} \\ &\leq 2\ |x_1 - y_1| + 2\ |x_2 - y_2| \\ &= 2\ \|x - y\|. \end{aligned}$$

If $(x_1 \leq x_2,\ y_2 \leq y_1)$ or $(x_2 \leq x_1,\ y_1 \leq y_2)$, then

$$\begin{aligned} H(P_M(x), P_M(y)) &= \max\{d((x_1, x_1), (y_2, y_2)),\ d((x_2, x_2), (y_1, y_1))\} \\ &= \max\{2\ |x_1 - y_2|,\ 2\ |x_2 - y_1|\}. \end{aligned}$$

First, we will prove the claim when $(x_1 \leq x_2,\ y_2 \leq y_1)$.

(1) If

$$\begin{aligned} &\Big(-\frac{1}{2} \leq x_1 \leq 0,\ \frac{1}{2} \leq x_2 \leq 1 \quad \text{and} \\ &\qquad \frac{1}{2} \leq y_1 \leq 1,\ 0 \leq y_2 \leq \frac{1}{2}\Big) \quad \text{or} \\ &\Big(-1 \leq x_1 \leq -\frac{1}{2},\ 0 \leq x_2 \leq \frac{1}{2} \quad \text{and} \\ &\qquad \Big\{0 \leq y_1 \leq \frac{1}{2},\ 0 \leq y_2 \leq \frac{1}{2},\ y_2 \leq y_1 \quad \text{or} \\ &\qquad 0 \leq y_1 \leq \frac{1}{2},\ -\frac{1}{2} \leq y_2 \leq 0\Big\}\Big) \quad \text{or} \\ &\Big(-1 \leq x_1 \leq -\frac{1}{2},\ -\frac{1}{2} \leq x_2 \leq 0 \quad \text{and} \\ &\qquad -\frac{1}{2} \leq y_1 \leq 0,\ -\frac{1}{2} \leq y_2 \leq 0,\ y_2 \leq y_1\Big) \quad \text{or} \\ &\Big(-\frac{1}{2} \leq x_1 \leq 0,\ 0 \leq x_2 \leq \frac{1}{2} \quad \text{and} \\ &\qquad 0 \leq y_1 \leq \frac{1}{2},\ 0 \leq y_2 \leq \frac{1}{2},\ y_2 \leq y_1\Big), \end{aligned}$$

then

$$
\begin{aligned}
&|x_1 - y_2| = y_2 - x_1 \leq y_1 - x_1 = |x_1 - y_1| && \\
&|x_2 - y_1| = x_2 - y_1 \leq x_2 - y_2 = |x_2 - y_2| && \text{if } y_1 \leq x_2 \\
&|x_2 - y_1| = y_1 - x_2 \leq y_1 - x_1 = |x_1 - y_1| && \text{if } x_2 \leq y_1.
\end{aligned}
$$

(2) If

$$
\begin{aligned}
&\Big(0 \leq x_1 \leq \frac{1}{2},\ \frac{1}{2} \leq x_2 \leq 1 \quad \text{and} \\
&\qquad \frac{1}{2} \leq y_1 \leq 1,\ 0 \leq y_2 \leq \frac{1}{2}\Big) \quad \text{or} \\
&\Big(-\frac{1}{2} \leq x_1 \leq 0,\ \frac{1}{2} \leq x_2 \leq 1 \quad \text{and} \\
&\qquad \frac{1}{2} \leq y_1 \leq 1,\ -\frac{1}{2} \leq y_2 \leq 0\Big) \quad \text{or} \\
&\Big(-1 \leq x_1 \leq -\frac{1}{2},\ 0 \leq x_2 \leq \frac{1}{2} \quad \text{and} \\
&\qquad 0 \leq y_1 \leq \frac{1}{2},\ -1 \leq y_2 \leq -\frac{1}{2}\Big) \quad \text{or} \\
&\Big(-1 \leq x_1 \leq -\frac{1}{2},\ -\frac{1}{2} \leq x_2 \leq 0 \quad \text{and} \\
&\qquad -\frac{1}{2} \leq y_1 \leq 0,\ -1 \leq y_2 \leq -\frac{1}{2}\Big) \quad \text{or} \\
&\Big(0 \leq x_1 \leq \frac{1}{2},\ 0 \leq x_2 \leq \frac{1}{2},\ x_1 \leq x_2 \quad \text{and} \\
&\qquad 0 \leq y_1 \leq \frac{1}{2},\ 0 \leq y_2 \leq \frac{1}{2},\ y_2 \leq y_1\Big) \quad \text{or} \\
&\Big(-\frac{1}{2} \leq x_1 \leq 0,\ 0 \leq x_2 \leq \frac{1}{2} \quad \text{and} \\
&\qquad 0 \leq y_1 \leq \frac{1}{2},\ -\frac{1}{2} \leq y_2 \leq 0\Big) \quad \text{or} \\
&\Big(-\frac{1}{2} \leq x_1 \leq 0,\ -\frac{1}{2} \leq x_2 \leq 0,\ x_1 \leq x_2 \quad \text{and} \\
&\qquad -\frac{1}{2} \leq y_1 \leq 0,\ -\frac{1}{2} \leq y_2 \leq 0,\ y_2 \leq y_1\Big),
\end{aligned}
$$

then

$$
\begin{aligned}
&|x_1 - y_2| = x_1 - y_2 \leq x_2 - y_2 = |x_2 - y_2| && \text{if } y_2 \leq x_1 \\
&|x_1 - y_2| = y_2 - x_1 \leq y_1 - x_1 = |x_1 - y_1| && \text{if } x_1 \leq y_2 \\
&|x_2 - y_1| = x_2 - y_1 \leq x_2 - y_2 = |x_2 - y_2| && \text{if } y_1 \leq x_2 \\
&|x_2 - y_1| = y_1 - x_2 \leq y_1 - x_1 = |x_1 - y_1| && \text{if } x_2 \leq y_1.
\end{aligned}
$$

(3) If

$$\Big(0 \le x_1 \le \frac{1}{2},\ 0 \le x_2 \le \frac{1}{2},\ \ x_1 \le x_2 \ \text{ and } \ \frac{1}{2} \le y_1 \le 1,\ \ -\frac{1}{2} \le y_2 \le 0\Big) \ \text{ or}$$

$$\Big(-\frac{1}{2} \le x_1 \le 0,\ \ -\frac{1}{2} \le x_2 \le 0,\ \ x_1 \le x_2 \ \text{ and } \ 0 \le y_1 \le \frac{1}{2},\ \ -1 \le y_2 \le -\frac{1}{2}\Big),$$

then

$$|x_1 - y_2| = x_1 - y_2 \le x_2 - y_2 = |x_2 - y_2|$$
$$|x_2 - y_1| = y_1 - x_2 \le y_1 - x_1 = |x_1 - y_1|.$$

(4) If

$$\Big(0 \le x_1 \le \frac{1}{2},\ \frac{1}{2} \le x_2 \le 1 \quad \text{and} \quad \frac{1}{2} \le y_1 \le 1,\ \ -\frac{1}{2} \le y_2 \le 0\Big) \ \text{ or}$$

$$\Big(0 \le x_1 \le \frac{1}{2},\ 0 \le x_2 \le \frac{1}{2},\ \ x_1 \le x_2 \quad \text{and} \quad \Big\{0 \le y_1 \le \frac{1}{2},\ \ -1 \le y_2 \le -\frac{1}{2} \ \text{ or } \ 0 \le y_1 \le \frac{1}{2},\ \ -\frac{1}{2} \le y_2 \le 0\Big\}\Big) \ \text{ or}$$

$$\Big(-\frac{1}{2} \le x_1 \le 0,\ 0 \le x_2 \le \frac{1}{2} \quad \text{and} \quad 0 \le y_1 \le \frac{1}{2},\ \ -1 \le y_2 \le -\frac{1}{2}\Big) \ \text{ or}$$

$$\Big(-\frac{1}{2} \le x_1 \le 0,\ \ -\frac{1}{2} \le x_2 \le 0,\ \ x_1 \le x_2 \quad \text{and} \quad -\frac{1}{2} \le y_1 \le 0,\ \ -1 \le y_2 \le -\frac{1}{2}\Big),$$

then

$$|x_1 - y_2| = x_1 - y_2 \le x_2 - y_2 = |x_2 - y_2|$$
$$|x_2 - y_1| = x_2 - y_1 \le x_2 - y_2 = |x_2 - y_2| \quad \text{if} \quad y_1 \le x_2$$
$$|x_2 - y_1| = y_1 - x_2 \le y_1 - x_1 = |x_1 - y_1| \quad \text{if} \quad x_2 \le y_1.$$

(5) If

$$\Big(-1 \le x_1 \le -\frac{1}{2},\ 0 \le x_2 \le \frac{1}{2} \quad \text{and}$$
$$\Big\{\frac{1}{2} \le y_1 \le 1,\ 0 \le y_2 \le \frac{1}{2} \ \text{ or } \frac{1}{2} \le y_1 \le 1,\ \ -\frac{1}{2} \le y_2 \le 0\Big\}\Big) \ \text{ or}$$

$$\Big(-1 \le x_1 \le -\frac{1}{2}, \quad -\frac{1}{2} \le x_2 \le 0 \qquad \text{and}$$

$$\Big\{\frac{1}{2} \le y_1 \le 1, \quad 0 \le y_2 \le \frac{1}{2} \quad \text{or}$$

$$\frac{1}{2} \le y_1 \le 1, \quad -\frac{1}{2} \le y_2 \le 0 \quad \text{or}$$

$$0 \le y_1 \le \frac{1}{2}, \quad 0 \le y_2 \le \frac{1}{2}, \quad y_2 \le y_1 \quad \text{or}$$

$$0 \le y_1 \le \frac{1}{2}, \quad -\frac{1}{2} \le y_2 \le 0\Big\}\Big) \quad \text{or}$$

$$\Big(-\frac{1}{2} \le x_1 \le 0, \quad 0 \le x_2 \le \frac{1}{2} \qquad \text{and}$$

$$\frac{1}{2} \le y_1 \le 1, \quad 0 \le y_2 \le \frac{1}{2}\Big) \quad \text{or}$$

$$\Big(-\frac{1}{2} \le x_1 \le 0, \quad -\frac{1}{2} \le x_2 \le 0, \quad x_1 \le x_2, \qquad \text{and}$$

$$\Big\{\frac{1}{2} \le y_1 \le 1, \quad 0 \le y_2 \le \frac{1}{2} \quad \text{or}$$

$$0 \le y_1 \le \frac{1}{2}, \quad 0 \le y_2 \le \frac{1}{2}, \quad y_2 \le y_1\Big\}\Big),$$

then

$$|x_1 - y_2| = y_2 - x_1 \le y_1 - x_1 = |x_1 - y_1|$$

$$|x_2 - y_1| = y_1 - x_2 \le y_1 - x_1 = |x_1 - y_1|.$$

(6) If

$$\Big(-1 \le x_1 \le -\frac{1}{2}, \quad -\frac{1}{2} \le x_2 \le 0 \qquad \text{and}$$

$$0 \le y_1 \le \frac{1}{2}, \quad -1 \le y_2 \le -\frac{1}{2}\Big) \quad \text{or}$$

$$\Big(0 \le x_1 \le \frac{1}{2}, \quad 0 \le x_2 \le \frac{1}{2}, \quad x_1 \le x_2 \qquad \text{and}$$

$$\frac{1}{2} \le y_1 \le 1, \quad 0 \le y_2 \le \frac{1}{2}\Big) \quad \text{or}$$

$$\Big(-\frac{1}{2} \le x_1 \le 0, \quad 0 \le x_2 \le \frac{1}{2} \qquad \text{and}$$

$$\frac{1}{2} \le y_1 \le 1, \quad -\frac{1}{2} \le y_2 \le 0\Big) \quad \text{or}$$

$$\Big(-\frac{1}{2} \le x_1 \le 0, \quad -\frac{1}{2} \le x_2 \le 0, \quad x_1 \le x_2 \qquad \text{and}$$

$$\Big\{\frac{1}{2} \le y_1 \le 1, \quad -\frac{1}{2} \le y_2 \le 0 \quad \text{or}$$

$$0 \le y_1 \le \frac{1}{2}, \quad -\frac{1}{2} \le y_2 \le 0\Big\}\Big),$$

then

$$
\begin{aligned}
|x_1 - y_2| &= x_1 - y_2 \le x_2 - y_2 = |x_2 - y_2| \quad \text{if } y_2 \le x_1 \\
|x_1 - y_2| &= y_2 - x_1 \le y_1 - x_1 = |x_1 - y_1| \quad \text{if } x_1 \le y_2 \\
|x_2 - y_1| &= y_1 - x_2 \le y_1 - x_1 = |x_1 - y_1|.
\end{aligned}
$$

(7) If

$$
\begin{aligned}
&\Big(0 \le x_1 \le \tfrac{1}{2},\ \tfrac{1}{2} \le x_2 \le 1 \quad \text{and} \\
&\qquad \Big\{0 \le y_1 \le \tfrac{1}{2},\ -1 \le y_2 \le -\tfrac{1}{2} \ \text{or} \\
&\qquad -\tfrac{1}{2} \le y_1 \le 0,\ -1 \le y_2 \le -\tfrac{1}{2} \ \text{or} \\
&\qquad 0 \le y_1 \le \tfrac{1}{2},\ -\tfrac{1}{2} \le y_2 \le 0 \ \text{or} \\
&\qquad -\tfrac{1}{2} \le y_1 \le 0,\ -\tfrac{1}{2} \le y_2 \le 0,\ y_2 \le y_1\Big\}\Big) \ \text{or} \\
&\Big(-\tfrac{1}{2} \le x_1 \le 0,\ \tfrac{1}{2} \le x_2 \le 1 \quad \text{and} \\
&\qquad \Big\{0 \le y_1 \le \tfrac{1}{2},\ -1 \le y_2 \le -\tfrac{1}{2} \ \text{or} \\
&\qquad -\tfrac{1}{2} \le y_1 \le 0,\ -1 \le y_2 \le -\tfrac{1}{2}\Big\}\Big) \ \text{or} \\
&\Big(0 \le x_1 \le \tfrac{1}{2},\ 0 \le x_2 \le \tfrac{1}{2},\ x_1 \le x_2 \ \text{and} \\
&\qquad \Big\{-\tfrac{1}{2} \le y_1 \le 0,\ -1 \le y_2 \le -\tfrac{1}{2} \ \text{or} \\
&\qquad -\tfrac{1}{2} \le y_1 \le 0,\ -\tfrac{1}{2} \le y_2 \le 0,\ y_2 \le y_1\Big\}\Big) \ \text{or} \\
&\Big(-\tfrac{1}{2} \le x_1 \le 0,\ 0 \le x_2 \le \tfrac{1}{2} \quad \text{and} \\
&\qquad -\tfrac{1}{2} \le y_1 \le 0,\ -1 \le y_2 \le -\tfrac{1}{2}\Big),
\end{aligned}
$$

then

$$
\begin{aligned}
|x_1 - y_2| &= x_1 - y_2 \le x_2 - y_2 = |x_2 - y_2| \\
|x_2 - y_1| &= x_2 - y_1 \le x_2 - y_2 = |x_2 - y_2|.
\end{aligned}
$$

(8) If

$$
\begin{aligned}
&\Big(-\tfrac{1}{2} \le x_1 \le 0,\ \tfrac{1}{2} \le x_2 \le 1 \quad \text{and} \\
&\qquad 0 \le y_1 \le \tfrac{1}{2},\ 0 \le y_2 \le \tfrac{1}{2},\ y_2 \le y_1\Big) \ \text{or}
\end{aligned}
$$

$$\Big(-1 \le x_1 \le -\frac{1}{2},\ 0 \le x_2 \le \frac{1}{2} \ \text{ and } \ -\frac{1}{2} \le y_1 \le 0,\ -\frac{1}{2} \le y_2 \le 0,\ y_2 \le y_1\Big),$$

then

$$|x_1 - y_2| = y_2 - x_1 \le y_1 - x_1 = |x_1 - y_1|$$
$$|x_2 - y_1| = x_2 - y_1 \le x_2 - y_2 = |x_2 - y_2|.$$

(9) If

$$\Big(0 \le x_1 \le \frac{1}{2},\ \frac{1}{2} \le x_2 \le 1 \quad \text{and} \quad 0 \le y_1 \le \frac{1}{2},\ 0 \le y_2 \le \frac{1}{2},\ y_2 \le y_1\Big) \ \text{ or}$$
$$\Big(-\frac{1}{2} \le x_1 \le 0,\ \frac{1}{2} \le x_2 \le 1 \quad \text{and} \quad \Big\{0 \le y_1 \le \frac{1}{2},\ -\frac{1}{2} \le y_2 \le 0 \ \text{ or } \ -\frac{1}{2} \le y_1 \le 0,\ -\frac{1}{2} \le y_2 \le 0,\ y_2 \le y_1\Big\}\Big)$$
$$\Big(-1 \le x_1 \le -\frac{1}{2},\ 0 \le x_2 \le \frac{1}{2} \quad \text{and} \quad -\frac{1}{2} \le y_1 \le 0,\ -1 \le y_2 \le -\frac{1}{2}\Big) \ \text{ or}$$
$$\Big(-\frac{1}{2} \le x_1 \le 0,\ 0 \le x_2 \le \frac{1}{2} \ \text{ and } \ -\frac{1}{2} \le y_1 \le 0,\ -\frac{1}{2} \le y_2 \le 0,\ y_2 \le y_1\Big),$$

then

$$|x_1 - y_2| = x_1 - y_2 \le x_2 - y_2 = |x_2 - y_2| \quad \text{if} \quad y_2 \le x_1$$
$$|x_1 - y_2| = y_2 - x_1 \le y_1 - x_1 = |x_1 - y_1| \quad \text{if} \quad x_1 \le y_2$$
$$|x_2 - y_1| = x_2 - y_1 \le x_2 - y_2 = |x_2 - y_2|.$$

Thus, for any case of $(1), \cdots, (9)$, we have

$$\begin{aligned} H(P_M(x), P_M(y)) &= \max\{2|x_1 - y_2|, 2|x_2 - y_1|\} \\ &\le \max\{2|x_1 - y_1|, 2|x_2 - y_2|\} \\ &\le 2|x_1 - y_1| + 2|x_2 - y_2| \\ &= 2\|x - y\|. \end{aligned}$$

Next, we can prove the claim when ($x_2 \leq x_1,\ y_2 \leq y_1$) by changing the role of $x = (x_1, x_2)$ and $y = (y_1, y_2)$.

Case 2 : $\alpha = -1$. This is, in fact, symmetric version of case 1 with respect to $x-$axis or $y-$axis.

Hence, for a given $\varepsilon > 0$ there exists $\delta(\varepsilon) = \frac{\varepsilon}{2} > 0$ such that for any x, y in the unit ball of X with $\|x - y\| < \delta$,

$$\begin{aligned} H(P_M(x), P_M(y)) &\leq 2\|x - y\| \\ &< 2\delta \\ &= \varepsilon \end{aligned}$$

for every proximinal subspace M of X. In other words, the set of metric projections

$$\{P_M : \text{ M is a proximinal subspace } \}$$

is a uniformly equicontinuous family on the unit ball.

REMARKS. (1) By Theorem 5, the set of metric projections

$$\{P_M : \text{ M is a proximinal subspace } \}$$

is the uniformly equicontinuous family on the unit ball of $(\mathbb{R}^2, \|\cdot\|_1)$. But $(\mathbb{R}^2, \|\cdot\|_1)$ is not a strictly convex space, so $(\mathbb{R}^2, \|\cdot\|_1)$ is not uniformly convex. Thus the converse of R.B.Holmes' theorem is not true.

(2) Furthermore, by the above theorem, even though the set of metric projections

$$\{P_M : \text{ M is a proximinal subspace } \}$$

is a uniformly equi-Lipschitz continuous family on the unit ball of X, X can not be uniformly convex.

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