A STUDY ABOUT THE CONVERSE OF R.B. HOLMES' THEOREM

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Let $M$ be a nonempty subset of a normed linear space $X$ and let $x \in X$. An element $y \in M$ satisfying

$$
\|x - y\| = d(x, M) := \inf\{\|x - m\| : m \in M\}
$$

is called a best approximation to $x$ from $M$. For a given $x \in X$, the set of all best approximations to $x$ from $M$ is denoted by

$$
P_M(x) := \{m \in M : \|x - m\| = d(x, M)\}.
$$

The set $M$ is called proximinal (resp., Chebyshev) if for every $x$ in $X$, $P_M(x)$ is nonempty (resp., a singleton). If $M$ is a proximinal subset of $X$, then the set-valued mapping $P_M : X \rightarrow 2^M$ defined by $x \mapsto P_M(x)$ is called the metric projection onto $M$.

For a given normed linear space $X$, let

$$
B(X) = \{x \in X : \|x\| \leq 1\}
$$

$$
S(X) = \{x \in X : \|x\| = 1\}.
$$

$B(X)$ is called the unit ball, and $S(X)$ is called the unit sphere in $X$.

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**Definition 1.** The modulus of convexity of a normed linear space $X$ is the function $\delta = \delta_X : [0, 2] \rightarrow [0, 1]$ defined by

$$\delta(\varepsilon) = \inf\{1 - \frac{1}{2}\|x + y\| : x, y \in S(X), \|x - y\| \geq \varepsilon\}.$$ 

$X$ is called a uniformly convex space if $\delta(\varepsilon) > 0$ for every $0 < \varepsilon \leq 2$. Geometrically, $X$ is uniformly convex if the midpoints of line segments having length at least $\varepsilon$ and its endpoints on the unit sphere must lie uniformly deep inside the unit ball.

The next result is well known.

**Lemma 2.** [1, 2] If $X$ is a finite-dimensional normed linear space, then $X$ is strictly convex if and only if it is uniformly convex.

**Definition 3.** Let $X$ be a normed linear space and let $H(X)$ denote the family of all nonempty, bounded, closed and convex subsets of $X$. Define $H : H(X) \times H(X) \rightarrow \mathbb{R}$ by

$$H(A, B) = \max\{h(A, B), h(B, A)\}$$

where $h(A, B) = \sup_{a \in A} d(a, B)$. Then $H$ is a metric on $H(X)$, called the Hausdorff metric.

In [2], we know that R.B.Holmes proved the following theorem.

**Theorem 4.** [2] Let $X$ be a uniformly convex space. For any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that for any $x, y$ in the unit ball of $X$ with $\|x - y\| < \delta$ imply that

$$\|P_M(x) - P_M(y)\| < \varepsilon$$

for every proximinal subspace $M$ of $X$. In other words, the set of metric projections

$$\{P_M : M \text{ is a proximinal subspace }\}$$

is a uniformly equicontinuous family on the unit ball.
Open Problem. Does the converse of R.B. Holmes' Theorem hold? That is, if
\[ \{ P_M : \text{M is a proximinal subspace} \} \]
is a uniformly equicontinuous family on the unit ball of a normed linear space \( X \), must \( X \) be uniformly convex?

In [2], F.R. Deutsch gave the open problem and expected that the answer is no. By the following theorem, we will give a counterexample for the above open problem.

Theorem 5. Let \( X = \mathbb{R}^2 \) be a normed linear space with \( \|(x, y)\| = |x| + |y| \). For any \( \varepsilon > 0 \) and any \( x, y \) in the unit ball of \( X \) with \( \|x - y\| < \frac{\varepsilon}{2} \),
\[ H(P_M(x), P_M(y)) < \varepsilon \]
for every proximinal subspace \( M \) of \( X \).

Proof. Let \( \varepsilon > 0 \) be given and let \( x, y \in B(X) \) be with \( \|x - y\| < \frac{\varepsilon}{2} \), that is, \( d(x, y) = \|x - y\| = |x_1 - y_1| + |x_2 - y_2| < \frac{\varepsilon}{2} \), where \( x = (x_1, x_2), y = (y_1, y_2) \). Assume that \( M = \text{span}\{(1, \alpha)\} \) if \( \alpha \in \mathbb{R} \) and \( M = \text{span}\{(0, 1)\} \) if \( \alpha = \infty \).

Now we claim that for every proximinal subspace \( M \) of \( X \),
\[ H(P_M(x), P_M(y)) \leq 2\|x - y\| \]
for every \( x, y \in B(X) \).

If \( 1 < |\alpha| \leq \infty \), then \( P_M(x) = \{(\frac{x_2}{\alpha}, x_2)\} \). So
\[ \|P_M(x) - P_M(y)\| = \frac{1}{\alpha} |x_2 - y_2| + |x_2 - y_2| < 2 |x_2 - y_2| \leq 2 \|x - y\|. \]

If \( 0 \leq |\alpha| < 1 \), then \( P_M(x) = \{(x_1, \alpha x_1)\} \). So
\[ \|P_M(x) - P_M(y)\| = |x_1 - y_1| + |\alpha| |x_1 - y_1| < 2 |x_1 - y_1| \leq 2 \|x - y\|. \]
It remains to prove the claim for $\alpha = 1$, and $\alpha = -1$.

Case 1: $\alpha = 1$. Note that for every $x = (x_1, x_2) \in X$,

$$P_M(x) = [(x_1, x_1), (x_2, x_2)]$$

where $[(x_1, x_1), (x_2, x_2)]$ is the line segment with the end points $(x_1, x_1)$ and $(x_2, x_2)$.

If $x_1 \leq x_2$, $y_1 \leq y_2$ or $(x_2 \leq x_1$, $y_2 \leq y_1$), then

$$H(P_M(x), P_M(y)) = \max\{d((x_1, x_1), (y_1, y_1)), d((x_2, x_2), (y_2, y_2))\}$$

$$= \max\{2|x_1 - y_1|, 2|x_2 - y_2|\}$$

$$\leq 2|x_1 - y_1| + 2|x_2 - y_2|$$

$$= 2\|x - y\|.$$  

If $x_1 \leq x_2$, $y_2 \leq y_1$ or $(x_2 \leq x_1$, $y_2 \leq y_1$), then

$$H(P_M(x), P_M(y)) = \max\{d((x_1, x_1), (y_2, y_2)), d((x_2, x_2), (y_1, y_1))\}$$

$$= \max\{2|x_1 - y_2|, 2|x_2 - y_1|\}.$$  

First, we will prove the claim when $(x_1 \leq x_2$, $y_2 \leq y_1$).

(1) If

$$(-\frac{1}{2} \leq x_1 \leq 0, \ \frac{1}{2} \leq x_2 \leq 1 \ \text{and} \ \frac{1}{2} \leq y_1 \leq 1, \ \ 0 \leq y_2 \leq \frac{1}{2}) \ \text{or}$$

$$(-1 \leq x_1 \leq -\frac{1}{2}, \ 0 \leq x_2 \leq \frac{1}{2} \ \text{and} \ \left\{0 \leq y_1 \leq \frac{1}{2}, \ 0 \leq y_2 \leq \frac{1}{2}, \ y_2 \leq y_1 \ \text{or} \right.$$

$$\quad \quad \left.0 \leq y_1 \leq \frac{1}{2}, \ -\frac{1}{2} \leq y_2 \leq 0\right\}) \ \text{or}$$

$$(-1 \leq x_1 \leq -\frac{1}{2}, \ -\frac{1}{2} \leq x_2 \leq 0 \ \text{and}$$

$$\quad \quad \left.-\frac{1}{2} \leq y_1 \leq 0, \ -\frac{1}{2} \leq y_2 \leq 0, \ y_2 \leq y_1 \right\}) \ \text{or}$$

$$\left(-\frac{1}{2} \leq x_1 \leq 0, \ 0 \leq x_2 \leq \frac{1}{2} \ \text{and} \right.$$

$$\quad \quad \left.0 \leq y_1 \leq \frac{1}{2}, \ 0 \leq y_2 \leq \frac{1}{2}, \ y_2 \leq y_1 \right).$$
then

\[ |x_1 - y_2| = y_2 - x_1 \leq y_1 - x_1 = |x_1 - y_1| \]
\[ |x_2 - y_1| = x_2 - y_1 \leq x_2 - y_2 = |x_2 - y_2| \quad \text{if } y_1 \leq x_2 \]
\[ |x_2 - y_1| = y_1 - x_2 \leq y_1 - x_1 = |x_1 - y_1| \quad \text{if } x_2 \leq y_1. \]

(2) If

\[
\left( 0 \leq x_1 \leq \frac{1}{2}, \quad \frac{1}{2} \leq x_2 \leq 1 \quad \text{and} \quad \frac{1}{2} \leq y_1 \leq 1, \quad 0 \leq y_2 \leq \frac{1}{2} \right) \quad \text{or} \]

\[
\left( -\frac{1}{2} \leq x_1 \leq 0, \quad \frac{1}{2} \leq x_2 \leq 1 \quad \text{and} \quad \frac{1}{2} \leq y_1 \leq 1, \quad -\frac{1}{2} \leq y_2 \leq 0 \right) \quad \text{or} \]

\[
\left( -1 \leq x_1 \leq -\frac{1}{2}, \quad 0 \leq x_2 \leq \frac{1}{2} \quad \text{and} \quad 0 \leq y_1 \leq \frac{1}{2}, \quad -1 \leq y_2 \leq -\frac{1}{2} \right) \quad \text{or} \]

\[
\left( -1 \leq x_1 \leq -\frac{1}{2}, \quad -\frac{1}{2} \leq x_2 \leq 0 \quad \text{and} \quad -\frac{1}{2} \leq y_1 \leq 0, \quad -1 \leq y_2 \leq -\frac{1}{2} \right) \quad \text{or} \]

\[
\left( 0 \leq x_1 \leq \frac{1}{2}, \quad 0 \leq x_2 \leq \frac{1}{2}, \quad x_1 \leq x_2 \quad \text{and} \quad 0 \leq y_1 \leq \frac{1}{2}, \quad 0 \leq y_2 \leq \frac{1}{2}, \quad y_2 \leq y_1 \right) \quad \text{or} \]

\[
\left( -\frac{1}{2} \leq x_1 \leq 0, \quad 0 \leq x_2 \leq \frac{1}{2} \quad \text{and} \quad 0 \leq y_1 \leq \frac{1}{2}, \quad -\frac{1}{2} \leq y_2 \leq 0 \right) \quad \text{or} \]

\[
\left( -\frac{1}{2} \leq x_1 \leq 0, \quad -\frac{1}{2} \leq x_2 \leq 0, \quad x_1 \leq x_2 \quad \text{and} \quad -\frac{1}{2} \leq y_1 \leq 0, \quad -\frac{1}{2} \leq y_2 \leq 0 \right), \quad y_2 \leq y_1, \]

then

\[ |x_1 - y_2| = x_1 - y_2 \leq x_2 - y_2 = |x_2 - y_2| \quad \text{if } y_2 \leq x_1 \]
\[ |x_1 - y_2| = y_2 - x_1 \leq y_1 - x_1 = |x_1 - y_1| \quad \text{if } x_1 \leq y_2 \]
\[ |x_2 - y_1| = x_2 - y_1 \leq x_2 - y_2 = |x_2 - y_2| \quad \text{if } y_1 \leq x_2 \]
\[ |x_2 - y_1| = y_1 - x_2 \leq y_1 - x_1 = |x_1 - y_1| \quad \text{if } x_2 \leq y_1. \]
(3) If
\[
\left( 0 \leq x_1 \leq \frac{1}{2}, \ 0 \leq x_2 \leq \frac{1}{2}, \ x_1 \leq x_2 \ \text{and} \ \frac{1}{2} \leq y_1 \leq 1, \ -\frac{1}{2} \leq y_2 \leq 0 \right) \ \text{or} \\
\left( -\frac{1}{2} \leq x_1 \leq 0, \ -\frac{1}{2} \leq x_2 \leq 0, \ x_1 \leq x_2 \ \text{and} \ 0 \leq y_1 \leq \frac{1}{2}, \ -1 \leq y_2 \leq -\frac{1}{2} \right),
\]

then
\[
|x_1 - y_2| = x_1 - y_2 \leq x_2 - y_2 = |x_2 - y_2| \\
|x_2 - y_1| = y_1 - x_2 \leq y_1 - x_1 = |x_1 - y_1|.
\]

(4) If
\[
\left( 0 \leq x_1 \leq \frac{1}{2}, \ \frac{1}{2} \leq x_2 \leq 1 \ \text{and} \ \frac{1}{2} \leq y_1 \leq 1, \ -\frac{1}{2} \leq y_2 \leq 0 \right) \ \text{or} \\
\left( 0 \leq x_1 \leq \frac{1}{2}, \ 0 \leq x_2 \leq \frac{1}{2}, \ x_1 \leq x_2 \ \text{and} \ \left\{ \begin{array}{l} 0 \leq y_1 \leq \frac{1}{2}, \ -1 \leq y_2 \leq -\frac{1}{2} \ \text{or} \\
0 \leq y_1 \leq \frac{1}{2}, \ -\frac{1}{2} \leq y_2 \leq 0 \end{array} \right\} \right) \ \text{or} \\
\left( -\frac{1}{2} \leq x_1 \leq 0, \ 0 \leq x_2 \leq \frac{1}{2} \ \text{and} \ 0 \leq y_1 \leq \frac{1}{2}, \ -1 \leq y_2 \leq -\frac{1}{2} \right) \ \text{or} \\
\left( -\frac{1}{2} \leq x_1 \leq 0, \ -\frac{1}{2} \leq x_2 \leq 0, \ x_1 \leq x_2 \ \text{and} \ -\frac{1}{2} \leq y_1 \leq 0, \ -1 \leq y_2 \leq -\frac{1}{2} \right),
\]

then
\[
|x_1 - y_2| = x_1 - y_2 \leq x_2 - y_2 = |x_2 - y_2| \\
|x_2 - y_1| = x_2 - y_1 \leq x_2 - y_2 = |x_2 - y_2| \text{ if } y_1 \leq x_2 \\
|x_2 - y_1| = y_1 - x_2 \leq y_1 - x_1 = |x_1 - y_1| \text{ if } x_2 \leq y_1.
\]

(5) If
\[
\left( -1 \leq x_1 \leq -\frac{1}{2}, \ 0 \leq x_2 \leq \frac{1}{2} \ \text{and} \ \left\{ \begin{array}{l} \frac{1}{2} \leq y_1 \leq 1, \ 0 \leq y_2 \leq \frac{1}{2} \ \text{or} \ \frac{1}{2} \leq y_1 \leq 1, \ -\frac{1}{2} \leq y_2 \leq 0 \end{array} \right\} \right) \ \text{or}
\]
\((-1 \leq x_1 \leq -\frac{1}{2}, \ -\frac{1}{2} \leq x_2 \leq 0\) and
\[
\begin{cases}
\frac{1}{2} \leq y_1 \leq 1, \ 0 \leq y_2 \leq \frac{1}{2} \quad \text{or} \\
\frac{1}{2} \leq y_1 \leq 1, \ -\frac{1}{2} \leq y_2 \leq 0 \quad \text{or} \\
0 \leq y_1 \leq \frac{1}{2}, \ 0 \leq y_2 \leq \frac{1}{2}, \ y_2 \leq y_1 \quad \text{or} \\
0 \leq y_1 \leq \frac{1}{2}, \ -\frac{1}{2} \leq y_2 \leq 0
\end{cases}
\] or

\((-\frac{1}{2} \leq x_1 \leq 0, \ 0 \leq x_2 \leq \frac{1}{2}\) and
\[
\frac{1}{2} \leq y_1 \leq 1, \ 0 \leq y_2 \leq \frac{1}{2}
\] or

\((-\frac{1}{2} \leq x_1 \leq 0, \ -\frac{1}{2} \leq x_2 \leq 0, \ x_1 \leq x_2, \ \text{and}
\[
\begin{cases}
\frac{1}{2} \leq y_1 \leq 1, \ 0 \leq y_2 \leq \frac{1}{2} \quad \text{or} \\
0 \leq y_1 \leq \frac{1}{2}, \ 0 \leq y_2 \leq \frac{1}{2}, \ y_2 \leq y_1
\end{cases}
\},
\]

then

\[
|x_1 - y_2| = y_2 - x_1 \leq y_1 - x_1 = |x_1 - y_1|
\]

\[
|x_2 - y_1| = y_1 - x_2 \leq y_1 - x_1 = |x_1 - y_1|
\]

(6) If

\((-1 \leq x_1 \leq -\frac{1}{2}, \ -\frac{1}{2} \leq x_2 \leq 0\) and
\[
0 \leq y_1 \leq \frac{1}{2}, \ -1 \leq y_2 \leq -\frac{1}{2}
\] or

\[
\begin{cases}
0 \leq x_1 \leq \frac{1}{2}, \ 0 \leq x_2 \leq \frac{1}{2}, \ x_1 \leq x_2 \quad \text{and} \\
\frac{1}{2} \leq y_1 \leq 1, \ 0 \leq y_2 \leq \frac{1}{2}
\end{cases}
\] or

\((-\frac{1}{2} \leq x_1 \leq 0, \ 0 \leq x_2 \leq \frac{1}{2}\) and
\[
\frac{1}{2} \leq y_1 \leq 1, \ -\frac{1}{2} \leq y_2 \leq 0
\] or

\((-\frac{1}{2} \leq x_1 \leq 0, \ -\frac{1}{2} \leq x_2 \leq 0, \ x_1 \leq x_2 \) and
\[
\begin{cases}
\frac{1}{2} \leq y_1 \leq 1, \ -\frac{1}{2} \leq y_2 \leq 0 \quad \text{or} \\
0 \leq y_1 \leq \frac{1}{2}, \ -\frac{1}{2} \leq y_2 \leq 0
\end{cases}
\),
then
\[ |x_1 - y_2| = x_1 - y_2 \leq x_2 - y_2 = |x_2 - y_2| \quad \text{if } y_2 \leq x_1 \]
\[ |x_1 - y_2| = y_2 - x_1 \leq y_1 - x_1 = |x_1 - y_1| \quad \text{if } x_1 \leq y_2 \]
\[ |x_2 - y_1| = y_1 - x_2 \leq y_1 - x_1 = |x_1 - y_1|. \]

(7) If
\[ \left( 0 \leq x_1 \leq \frac{1}{2}, \quad \frac{1}{2} \leq x_2 \leq 1 \right) \quad \text{and} \]
\[ \left\{ \begin{array}{l}
0 \leq y_1 \leq \frac{1}{2}, \quad -1 \leq y_2 \leq -\frac{1}{2} \quad \text{or} \\
-\frac{1}{2} \leq y_1 \leq 0, \quad -1 \leq y_2 \leq -\frac{1}{2} \quad \text{or} \\
0 \leq y_1 \leq \frac{1}{2}, \quad -\frac{1}{2} \leq y_2 \leq 0 \quad \text{or} \\
-\frac{1}{2} \leq y_1 \leq 0, \quad -\frac{1}{2} \leq y_2 \leq 0, \quad y_2 \leq y_1 \end{array} \right\} \] or
\[ \left( -\frac{1}{2} \leq x_1 \leq 0, \quad \frac{1}{2} \leq x_2 \leq 1 \right) \quad \text{and} \]
\[ \left\{ \begin{array}{l}
0 \leq y_1 \leq \frac{1}{2}, \quad -1 \leq y_2 \leq -\frac{1}{2} \quad \text{or} \\
-\frac{1}{2} \leq y_1 \leq 0, \quad -1 \leq y_2 \leq -\frac{1}{2} \quad \text{or} \\
0 \leq y_1 \leq \frac{1}{2}, \quad -\frac{1}{2} \leq y_2 \leq 0 \quad \text{or} \\
-\frac{1}{2} \leq y_1 \leq 0, \quad -\frac{1}{2} \leq y_2 \leq 0, \quad y_2 \leq y_1 \end{array} \right\} \] or
\[ \left( 0 \leq x_1 \leq \frac{1}{2}, \quad 0 \leq x_2 \leq \frac{1}{2}, \quad x_1 \leq x_2 \right) \quad \text{and} \]
\[ \left\{ \begin{array}{l}
-\frac{1}{2} \leq y_1 \leq 0, \quad -1 \leq y_2 \leq -\frac{1}{2} \quad \text{or} \\
-\frac{1}{2} \leq y_1 \leq 0, \quad -\frac{1}{2} \leq y_2 \leq 0, \quad y_2 \leq y_1 \end{array} \right\} \] or
\[ \left( -\frac{1}{2} \leq x_1 \leq 0, \quad 0 \leq x_2 \leq \frac{1}{2} \right) \quad \text{and} \]
\[ -\frac{1}{2} \leq y_1 \leq 0, \quad -1 \leq y_2 \leq -\frac{1}{2} \),

then
\[ |x_1 - y_2| = x_1 - y_2 \leq x_2 - y_2 = |x_2 - y_2| \]
\[ |x_2 - y_1| = x_2 - y_1 \leq x_2 - y_2 = |x_2 - y_2|. \]

(8) If
\[ \left( -\frac{1}{2} \leq x_1 \leq 0, \quad \frac{1}{2} \leq x_2 \leq 1 \right) \quad \text{and} \]
\[ 0 \leq y_1 \leq \frac{1}{2}, \quad 0 \leq y_2 \leq \frac{1}{2}, \quad y_2 \leq y_1 \right\} \] or
\((-1 \leq x_1 \leq -\frac{1}{2}, \ 0 \leq x_2 \leq \frac{1}{2} \text{ and} \)
\(-\frac{1}{2} \leq y_1 \leq 0, \ -\frac{1}{2} \leq y_2 \leq 0, \ y_2 \leq y_1\),
then
\[|x_1 - y_2| = y_2 - x_1 \leq y_1 - x_1 = |x_1 - y_1|\]
\[|x_2 - y_1| = x_2 - y_1 \leq x_2 - y_2 = |x_2 - y_2|.
\]

(9) If
\((0 \leq x_1 \leq \frac{1}{2}, \ \frac{1}{2} \leq x_2 \leq 1 \text{ and} \)
\(0 \leq y_1 \leq \frac{1}{2}, \ 0 \leq y_2 \leq \frac{1}{2}, \ y_2 \leq y_1\) or
\((-\frac{1}{2} \leq x_1 \leq 0, \ \frac{1}{2} \leq x_2 \leq 1 \text{ and} \)
\(\left\{0 \leq y_1 \leq \frac{1}{2}, \ -\frac{1}{2} \leq y_2 \leq 0 \text{ or} \right.\)
\(-\frac{1}{2} \leq y_1 \leq 0, \ -\frac{1}{2} \leq y_2 \leq 0, \ y_2 \leq y_1\}\)
\((-1 \leq x_1 \leq -\frac{1}{2}, \ 0 \leq x_2 \leq \frac{1}{2} \text{ and} \)
\(-\frac{1}{2} \leq y_1 \leq 0, \ -1 \leq y_2 \leq -\frac{1}{2}\) or
\((-\frac{1}{2} \leq x_1 \leq 0, \ 0 \leq x_2 \leq \frac{1}{2} \text{ and} \)
\(-\frac{1}{2} \leq y_1 \leq 0, \ -\frac{1}{2} \leq y_2 \leq 0, \ y_2 \leq y_1\),
then
\[|x_1 - y_2| = x_1 - y_2 \leq x_2 - y_2 = |x_2 - y_2| \text{ if } y_2 \leq x_1\]
\[|x_1 - y_2| = y_2 - x_1 \leq y_1 - x_1 = |x_1 - y_1| \text{ if } x_1 \leq y_2\]
\[|x_2 - y_1| = x_2 - y_1 \leq x_2 - y_2 = |x_2 - y_2|.
\]
Thus, for any case of (1), \(\cdots\), (9), we have
\[
H(P_M(x), P_M(y)) = \max\{2|x_1 - y_2|, 2|x_2 - y_1|\}
\leq \max\{2|x_1 - y_1|, 2|x_2 - y_2|\}
\leq 2|x_1 - y_1| + 2|x_2 - y_2|
= 2||x - y||.
\]
Next, we can prove the claim when \((x_2 \leq x_1, \ y_2 \leq y_1)\) by changing the role of \(x = (x_1, x_2)\) and \(y = (y_1, y_2)\).

Case 2 : \(\alpha = -1\). This is, in fact, symmetric version of case 1 with respect to \(x\)-axis or \(y\)-axis.

Hence, for a given \(\varepsilon > 0\) there exists \(\delta(\varepsilon) = \frac{\varepsilon}{2} > 0\) such that for any \(x, y\) in the unit ball of \(X\) with \(||x - y|| < \delta||\),

\[
H(P_M(x), P_M(y)) \leq 2||x - y|| < 2\delta = \varepsilon
\]

for every proximinal subspace \(M\) of \(X\). In other words, the set of metric projections

\[
\{P_M : M \text{ is a proximinal subspace }\}
\]

is a uniformly equicontinuous family on the unit ball.

**Remarks.** (1) By Theorem 5, the set of metric projections

\[
\{P_M : M \text{ is a proximinal subspace }\}
\]

is the uniformly equicontinuous family on the unit ball of \((\mathbb{R}^2, \| \cdot \|_1)\). But \((\mathbb{R}^2, \| \cdot \|_1)\) is not a strictly convex space, so \((\mathbb{R}^2, \| \cdot \|_1)\) is not uniformly convex. Thus the converse of R.B.Holmes' theorem is not true.

(2) Furthermore, by the above theorem, even though the set of metric projections

\[
\{P_M : M \text{ is a proximinal subspace }\}
\]

is a uniformly equi-Lipschitz continuous family on the unit ball of \(X\), \(X\) can not be uniformly convex.
A study about the converse Of R.B.Holmes' theorem

References

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