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THE STATE SPACE OF A CANONICAL LINEAR SYSTEM

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1. Introduction

A fundamental problem is to construct linear systems with given transfer functions. This problem has a well known solution for unitary linear systems whose state spaces and coefficient spaces are Hilbert spaces. The solution is due independently to B. Sz.-Nagy and C. Foias [15] and to L. de Branges and J. Ball and N. Cohen [4]. Such a linear system is essentially uniquely determined by its transfer function. The de Branges-Rovnyak construction makes use of the theory of square summable power series with coefficients in a Hilbert space. The construction also applies when the coefficient space is a Krein space [7].

A general construction of unitary linear systems with given transfer functions was announced without proof by Azizov [3]. A proof was later supplied by Dijksma, Langer, and de Snoo [10]. The state spaces and coefficient spaces of these linear systems are Krein spaces.

D. Alpay [1] has shown that a canonical linear system is not uniquely determined by its transfer function when the state space is a Krein space. A uniqueness theorem is derived from the work of Sorjonen [14] when the state space of the linear system has finite Pontryagin index. In the general case conditions for the uniqueness of a canonical linear system with given transfer function have been given by de Branges [6].

The present construction of unitary linear systems assumes that multiplication by the transfer function is an everywhere defined transformation in the space of square summable power series with vector coefficients. Transfer functions with this property were first considered by Ch. Davis and C. Foias [9] when the state space is a Hilbert space. A generalization of their results has been obtained by M. Möller [13]

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when the state space is a Krein space. Our construction is similar to that of Dijksma, Langer and de Snoo [10] in that it depends on the same extension theorem of Mark Krein. A canonical linear system which is conjugate isometric is obtained under a stronger hypothesis, in which case a uniqueness theorem of de Branges [6] applies.

A vector space $\mathcal{K}$ over the complex numbers with a scalar product $\langle\ ,\ \rangle_{\mathcal{K}}$ is called a Krein space if there are two spaces $\mathcal{K}_+$ and $\mathcal{K}_-$ such that $\mathcal{K}_+$ is a Hilbert space with respect to the scalar product $\langle\ ,\ \rangle_{\mathcal{K}}$, $\mathcal{K}_-$ is a Hilbert space with respect to the scalar product $-\langle\ ,\ \rangle_{\mathcal{K}}$ and $\mathcal{K}$ is an orthogonal sum of a Hilbert space $\mathcal{K}_+$ and the anti-space of a Hilbert space $\mathcal{K}_-$. In genenal, such decompositions are not unique. The choice of orthogonal decomposition induces a Hilbert space strong topology on $\mathcal{K}$. The strong topology of this Hilbert space is called the Mackey topology of $\mathcal{K}$. The norm of the Hilbert space depends on the choice of orthogonal decomposition, but two such norms are equivalent.

Let $\mathcal{H}$ and $\mathcal{C}$ be Krein spaces. A continuous linear transformation

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} : \begin{matrix} \mathcal{H} \\ \oplus \\ \mathcal{C} \end{matrix} \longrightarrow \begin{matrix} \mathcal{H} \\ \oplus \\ \mathcal{C} \end{matrix}$$

is called a linear system. The underlying Krein space $\mathcal{H}$ is called the state space and the auxiliary Krein space $\mathcal{C}$ is called the coefficient space or the external space. The transformation A is called the main transformation. The transformation B is called the input transformation. The transformation C is called the output transformation. The operator D is called the external operator.

A linear system is said to be contractive if the matrix is contractive, unitary if the matrix is unitary, and conjugate isometric if the matrix has an isometric adjoint. The transfer function $W(z)$ of the linear system is defined by

$$W(z) = D + zC(I - zA)^{-1}B \quad \text{where } z \in \{z : z^{-1} \in \rho(A)\}.$$

A linear system is said to be observable if there is no nonzero element f of the state space such that $CA^n f = 0$ for every nonnegative integer n. Let r be a positive real number. An observable linear system is said to be in a canonical form if the elements of the state space are power series with vector coefficients in such a way that the identity $a_n = r^{-n}CA^nf$ holds whenever $f(z) = \sum_{n=0}^{\infty} a_n z^n$.

If an observable linear system is in a canonical form, then the elements of the state space are power series which converge in some neighborhood of the origin. For this linear system the main transformation $A(f(z)) = r[f(z) - f(0)]/z$, $B(c) = r[W(r^{-1}z) - W(0)]c/z$, $C(f(z)) = f(0)$, and $D(c) = W(0)c$, where $W(z)$ is the transfer function of the linear system.

The theory of canonical linear systems which are conjugate isometric is a generalization of the theory of square summable power series with vector coefficients. Assume that the coefficient space $\mathcal{C}$ is a Krein space. Write $\mathcal{C}$ as the orthogonal sum of a Hilbert space $\mathcal{C}_+$ and the anti-space $\mathcal{C}_-$ of a Hilbert space. Let J be the operator which is the identity on $\mathcal{C}_+$ and which is minus the identity on $\mathcal{C}_-$. Let r be a positive real number. Let

$$\mathcal{C}_r(z) = \{f : f(z) = \sum_{n=0}^{\infty} a_n z^n, a_n \in \mathcal{C}, \sum_{n=0}^{\infty} r^{2n} \langle Ja_n, a_n \rangle_{\mathcal{C}} < \infty\}.$$

The condition does not depend on the choice of decompositions of $\mathcal{C}$. The space $\mathcal{C}_r(z)$ is considered as a Krein space with the unique scalar product such that

$$\langle f(z), f(z) \rangle_{\mathcal{C}_r(z)} = \sum_{n=0}^{\infty} r^{2n} \langle a_n, a_n \rangle_{\mathcal{C}}.$$

The identity for difference-quotients

$$\begin{aligned} & r^2 \langle [f(z) - f(0)]/z, [g(z) - g(0)]/z \rangle_{\mathcal{C}_r(z)} \\ = & \ \langle f(z), g(z) \rangle_{\mathcal{C}_r(z)} - \langle f(0), g(0) \rangle_{\mathcal{C}} \end{aligned}$$

holds for all $f(z)$ and $g(z)$ in $\mathcal{C}_r(z)$. These properties imply that the space $\mathcal{C}_r(z)$ is the state space of a canonical linear system which is conjugate isometric and has transfer function identically zero.

The construction of linear systems in Krein spaces made by Ando [2] makes use of a Krein space generalization of complementation theory [5].

THEOREM 1.1. *If a Krein space $\mathcal{P}$ is contained continuously and contractively in a Krein space $\mathcal{H}$, then a unique Krein space $\mathcal{Q}$ exists, which is contained continuously and contractively in $\mathcal{H}$, such that the inequality*

$$\langle c, c\rangle_{\mathcal{H}} \leq \langle a, a\rangle_{\mathcal{P}} + \langle b, b\rangle_{\mathcal{Q}}$$

holds whenever $c = a + b$ with a in $\mathcal{P}$ and b in $\mathcal{Q}$ and such that every element c of $\mathcal{H}$ admits some such decomposition for which equality holds.

The space $\mathcal{Q}$ is called the complementary space to $\mathcal{P}$ in $\mathcal{H}$. A unique minimal decomposition is obtained when equality holds. If

$$\langle c, c\rangle_{\mathcal{H}} = \langle a, a\rangle_{\mathcal{P}} + \langle b, b\rangle_{\mathcal{Q}}$$

where $c = a + b$, then a is obtained from c under the adjoint of the inclusion of $\mathcal{P}$ in $\mathcal{H}$ and b is obtained from c under the adjoint of the inclusion of $\mathcal{Q}$ in $\mathcal{H}$.

Complementation theory can be used to give new proofs of theorems of Dritschel [11] and of Dritschel and Rovnyak [12] which generalize the commutant lifting theorem to Krein spaces [8].

THEOREM 1.2. *Let $B(z)$ be a power series with operator coefficients such that multiplication by $B(z)$ is a contractive transformation in $\mathcal{C}_1(z)$. There exists a Krein space $\mathcal{H}(B)$ which is the state space of a conjugate isometric canonical linear system with transfer function $B(z)$.*

2. Conjugate isometric linear systems

THEOREM 2.1. *Assume that*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} : \begin{matrix} \mathcal{H} \\ \oplus \\ \mathcal{C} \end{matrix} \longrightarrow \begin{matrix} \mathcal{H} \\ \oplus \\ \mathcal{C} \end{matrix}$$

is a conjugate isometric linear system with a transfer function $W(z)$. There is a unique continuous linear transformation of $\mathcal{C}_r(z)$ into $\mathcal{H}$

which takes polynomial $h(z) = \sum_{n=0}^{k} a_n z^n$ *into* $\sum_{n=0}^{k} r^n A^{*n} C^* a_n$. *The adjoint transformation takes* f *of* $\mathcal{H}$ *into* $\sum_{n=0}^{\infty} r^{-n} C A^n f z^n$.

Proof. See Möller [13].

A property of the formal adjoint of multiplication by $B(z)$ is implicit in the work of de Branges [6]. A proof is now given as an introduction to the theory of canonical linear systems.

THEOREM 2.2. *Assume that* $\mathcal{H}$ *is the state space of a canonical linear system which is conjugate isometric and has transfer function* $B(z)$. *If* $h(z)$ *and* $g(z)$ *are polynomials with vector coefficients such that the formal adjoint of multiplication by* $B(r^{-1}z)$ *in* $\mathcal{C}_r(z)$ *takes* $h(z)$ *into* $g(z)$, *then* $h(z) - B(r^{-1}z)g(z)$ *belongs to* $\mathcal{H}$. *If* $h(z) = \sum_{n=0}^{k} a_n z^n$, *then the identity*

$$\sum_{n=0}^{k} r^{2n} \langle c_n, a_n \rangle_{\mathcal{C}} = \langle f(z), h(z) - B(r^{-1}z) g(z) \rangle_{\mathcal{H}}$$

holds for every element $f(z) = \sum_{n=0}^{\infty} c_n z^n$ *of* $\mathcal{H}$.

Proof. Consider $\mathcal{H}_1 = \{ f : [f(z) - f(0)]/z \in \mathcal{H} \}$ as a Krein space with the scalar product

$$\langle f(z), f(z) \rangle_{\mathcal{H}_1} = \langle r^2 [f(z) - f(0)]/z, [f(z) - f(0)]/z \rangle_{\mathcal{H}} + \langle f(0), f(0) \rangle_{\mathcal{C}}.$$

The space $\mathcal{H}_1$ is the state space of a canonical linear system which is conjugate isometric and has transfer function $B_1(z) = zB(z)$. There is a partially isometric transformation $T_1 : \mathcal{H} \times \mathcal{C} \longrightarrow \mathcal{H}_1$ defined by $T_1((f(z), c)) = f(z) + B(r^{-1}z)c$. Every element of $\mathcal{H}_1$ is of the form $f(z) + B(r^{-1}z)c$ with $f(z)$ an element of $\mathcal{H}$ and c an element of $\mathcal{C}$.

Consider $\mathcal{H}_2 = \{ f(z) : [f(z) - f(0)]/z \in \mathcal{H}_1 \}$ as a Krein space with the scalar product

$$\langle f(z), f(z) \rangle_{\mathcal{H}_2} = r^2 \langle [f(z) - f(0)]/z, [f(z) - f(0)]/z \rangle_{\mathcal{H}_1} + \langle f(0), f(0) \rangle_{\mathcal{C}}.$$

A partially isometric transformation of the Cartesian product $\mathcal{H}_1 \times \mathcal{C}$ onto $\mathcal{H}_2$ exists which is defined by taking a pair consisting of an element $f(z)$ of $\mathcal{H}_1$ and an element c of $\mathcal{C}$ into the element $f(z) + r^{-1}zB(r^{-1}z)c$

of $\mathcal{H}_2$. Every element of $\mathcal{H}_1$ is of the form $f(z)+B(r^{-1}z)c$ with $f(z) \in \mathcal{H}$ and $c \in \mathcal{C}$.

Define the Krein space

$$\mathcal{P}_2 = \{p\colon p(z) = \sum_{k=0}^{1} a_k z^k, a_k \in \mathcal{C}\}$$

with scalar product inherited from $\mathcal{C}_r(z)$. There is a partial isometric trasnsformation $T_2\colon \mathcal{H} \times \mathcal{P}_2 \longrightarrow \mathcal{H}_2$ defind by $T_2(f(z), g(z)) = f(z) + B(r^{-1}z)g(z)$. The space $\mathcal{H}_2$ is the state space of a canonical linear system which is conjugate isometric and has transfer function $B_1(z) = zB(z)$. The space $\mathcal{H}_1$ is contained continuously in the space $\mathcal{H}_2$ because the space $\mathcal{H}$ is contained continuously in the space $\mathcal{H}_1$.

The construction can be iterated. For any nonnegative integer n an n-times augmented space $\mathcal{H}_n$ is defined inductively starting with $\mathcal{H}_0 = \mathcal{H}$. Once $\mathcal{H}_n$ has been defined, define $\mathcal{H}_{n+1}$ to be the augmented space of $\mathcal{H}_n$. The space $\mathcal{H}_n$ is the state space of a canonical linear system with transfer function $B_n(z) = z^n B(z)$.

Define the Krein space

$$\mathcal{P}_n = \{p\colon p(z) = \sum_{k=0}^{n-1} a_k z^k, a_k \in \mathcal{C}\}$$

with scalar product inherited from $\mathcal{C}_r(z)$. A partially isometric transformation $T_n\colon \mathcal{H} \times \mathcal{P}_n \longrightarrow \mathcal{H}_n$ exists defined $T_n(f(z), g(z)) = f(z) + B(r^{-1}z)g(z)$.

Let $h(z)$ be a polynomial element of $\mathcal{C}_r(z)$ of degree less than n. Since $h(z) \in \mathcal{H}_n$, there exists unique polynomial $g(z) \in \mathcal{P}_n$ such that $(h(z)-B(r^{-1}z)g(z), g(z))$ is in the orthogonal complement of the kernel of T_n. It follows that the identity

$$\langle h(z), B(r^{-1}z)p(z)\rangle_{\mathcal{H}_n} = \langle g(z), p(z)\rangle_{\mathcal{C}_r(z)}$$

holds for every polynomial element $p(z) \in \mathcal{C}_r(z)$. The identity implies that $g(z)$ is obtained from $h(z)$ under the formal adjoint of multiplication by $B(r^{-1}z)$ in $\mathcal{C}_r(z)$.

By this construction the series $h(z) - B(r^{-1}z)g(z)$ belongs to $\mathcal{H}$ and the identity

$$\langle h(z) - B(r^{-1}z)g(z), f(z)\rangle_{\mathcal{H}} = \langle h(z), f(z)\rangle_{\mathcal{H}_n}$$

holds for every element $f(z)$ of $\mathcal{H}$. The identity stated in the theorem now follows because $h(z)$ is a polynomial of degree less than n.

This completes the proof of the theorem.

3. Krein completion

A construction of Krein spaces results from the spectral theory for self-adjoint transformations on Hilbert spaces. Let $\mathcal{H}$ be a Hilbert space and J be a self-adjoint transformation on $\mathcal{H}$. Let $\mathcal{H}_0$ be the kernel of J. Let $\mathcal{H}_+$ be the largest invariant subspace of J, which is orthogonal to $\mathcal{H}_0$, such that the restriction of J to the subspace has nonnegative spectrum. Let $\mathcal{H}_-$ be the largest invariant subspace of J, which is orthogonal to $\mathcal{H}_0$, such that the restriction of J to the subspace has nonpositive spectrum. Then $\mathcal{H}_+$, $\mathcal{H}_-$ and $\mathcal{H}_0$ are orthogonal subspaces of $\mathcal{H}$ which span $\mathcal{H}$. Define π to be the orthogonal projection of $\mathcal{H}$ onto the vector span of $\mathcal{H}_+$ and $\mathcal{H}_-$.

The Krein completion $\hat{\mathcal{H}}_+$ of $\mathcal{H}_+$ with respect to J is the essentially unique Hilbert space, which contains $\mathcal{H}_+$ as a dense vector subspace, such that the identity

$$\langle a, b\rangle_{\hat{\mathcal{H}}_+} = \langle J\ a, b\rangle_{\mathcal{H}_+}$$

holds for all elements a and b of $\mathcal{H}_+$. The Krein completion $\hat{\mathcal{H}}_-$ of $\mathcal{H}_-$ with respect to J is the essentially unique anti-space of a Hilbert space, which contains $\mathcal{H}_-$ as a dense vector subspace, such that the identity

$$\langle a, b\rangle_{\hat{\mathcal{H}}_-} = \langle J\ a, b\rangle_{\mathcal{H}_-}$$

holds for all elements a and b of $\mathcal{H}_-$. The Krein completion $\hat{\mathcal{H}}$ of $\mathcal{H}$ is the essentially unique Krein space which is obtained as the orthogonal sum of such spaces $\hat{\mathcal{H}}_+$ and $\hat{\mathcal{H}}_-$. The range of π is a dense vector subspace of $\hat{\mathcal{H}}$, and the identity

$$\langle \pi\ a, \pi\ b\rangle_{\hat{\mathcal{H}}} = \langle J\ a, b\rangle_{\mathcal{H}}$$

holds for all elements a and b of $\mathcal{H}$.

A construction of continuous transformations in Krein spaces is due to Mark Krein [10].

THEOREM 3.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be Hilbert spaces. Assume that $J_{\mathcal{A}}$ is a self-adjoint transformation of $\mathcal{A}$ into itself, that $J_{\mathcal{B}}$ is a self-adjoint transformation of $\mathcal{B}$ into itself, that U is a continuous transformation of $\mathcal{A}$ into $\mathcal{B}$ and that V is a continuous transformation of $\mathcal{B}$ into $\mathcal{A}$. Let $\hat{\mathcal{A}}$ be the Krein completion of $\mathcal{A}$ with respect to $J_{\mathcal{A}}$ and let $\hat{\mathcal{B}}$ be the Krein completion of $\mathcal{B}$ with respect to $J_{\mathcal{B}}$. Let $\pi_{\mathcal{A}}$ be the projection of $\mathcal{A}$ into $\hat{\mathcal{A}}$ and let $\pi_{\mathcal{B}}$ be the projection of $\mathcal{B}$ into $\hat{\mathcal{B}}$. If the identity*

$$\langle J_{\mathcal{B}}\, U\, a, b\rangle_{\mathcal{B}} \;=\; \langle J_{\mathcal{A}}\, a, V\, b\rangle_{\mathcal{A}}$$

is satisfied for all elements a of $\mathcal{A}$ and b of $\mathcal{B}$, then unique adjoint transformations $\hat{U}$ of $\hat{\mathcal{A}}$ into $\hat{\mathcal{B}}$ and $\hat{V}$ of $\hat{\mathcal{B}}$ into $\hat{\mathcal{A}}$ exist such that the identities $\hat{U}\ \pi_{\mathcal{A}} = \pi_{\mathcal{B}}\ U$ and $\hat{V}\ \pi_{\mathcal{B}} = \pi_{\mathcal{A}}\ V$ are satisfied.

4. Existence of a unitary linear system

If $B(z)$ is a given power series with operator coefficients, define $\mathcal{G}(B)$ to be the graph of the adjoint of multiplication by $B(r^{-1}z)$ in $\mathcal{C}_r(z)$. Consider $\mathcal{G}(B)$ as a Hilbert space with the unique scalar product such that the identity

$$\begin{aligned}&\langle (h(z), g(z)), (h(z), g(z))\rangle_{\mathcal{G}(B)}\\ =\;& \langle J\ h(z), h(z)\rangle_{\mathcal{C}_r(z)} + \langle J\ g(z), g(z)\rangle_{\mathcal{C}_r(z)}\end{aligned}$$

is satisfied.

Define $core(B) = \{h(z) - B(r^{-1}z)g(z) \colon\ (h(z), g(z)) \in \mathcal{G}(B)\}$. Then $core(B)$ admits a unique scalar product such that the identity

$$\begin{aligned}&\langle h(z) - B(r^{-1}z)g(z), h(z) - B(r^{-1}z)g(z)\rangle_{core(B)}\\ =\;& \langle h(z), h(z)\rangle_{\mathcal{C}_r(z)} - \langle g(z), g(z)\rangle_{\mathcal{C}_r(z)}\end{aligned}$$

is satisfied.

Let

$$ext\ \mathcal{C}_r(z) = \{f \colon f(z) = \sum_{-\infty}^{\infty} a_n z^n, a_n \in \mathcal{C}, \sum_{-\infty}^{\infty} r^{2n}\langle J a_n, a_n\rangle_{\mathcal{C}} < \infty\}.$$

The condition does not depend of the choice of decompositions of $\mathcal{C}$. The space $\mathcal{C}_r(z)$ is considered as a Krein space with the unique scalar product such that

$$\langle f(z), f(z)\rangle_{ext\ \mathcal{C}_r(z)} = \sum_{-\infty}^{\infty} r^{2n}\langle a_n, a_n\rangle_{\mathcal{C}}.$$

Assume that multiplication by $B(r^{-1}z)$ is an everywhere defined transformation in $\mathcal{C}_r(z)$. The space *ext* $\mathcal{G}(B)$ is the set of pairs $(u(z), v(z))$ of elements of *ext* $\mathcal{C}_r(z)$ such that $u(z) - rB(r^{-1}z)z^{-1}v(r^2z^{-1})$ and $v(z) - B^*(r^{-1}z)z^{-1}u(r^2z^{-1})$ belong to $\mathcal{C}_r(z)$. The space *ext* $\mathcal{G}(B)$ becomes a Hilbert space when considered with the unique scalar product such that the identity

$$\begin{aligned}&\langle (u(z), v(z)), (u(z), v(z))\rangle_{ext\ \mathcal{G}(B)} \\ &= \langle J\ u(z), u(z)\rangle_{ext\ \mathcal{C}_r(z)} + \langle J\ v(z), v(z)\rangle_{ext\ \mathcal{C}_r(z)}\end{aligned}$$

holds for any $(u(z), v(z))$ in ext $\mathcal{G}(B)$. Then $(u(z), z^{-1}v(z^{-1}))$ belongs to *ext* $\mathcal{G}(B)$ for $(u(z), r^{-1}v(r^{-2}z)) \in \mathcal{G}(B)$ and $(z^{-1}u(z^{-1}), v(z)))$ belongs to *ext* $\mathcal{G}(B)$ for $(v(z), r^{-1}u(r^{-2}z)) \in \mathcal{G}(B^*)$.

Define the space *ext core*(B) to be the set of pairs

$$(u(z) - rB(r^{-1}z)z^{-1}v(r^2z^{-1}), -v(z) + rB^*(r^{-1}z)z^{-1}u(r^2z^{-1}))$$

with$(u(z), v(z)) \in ext\mathcal{G}(B)$. A unique scalar product is defined in the space so that the identity

$$\begin{aligned}&\langle (u(z) - rB(r^{-1}z)z^{-1}v(r^2z^{-1}), -v(z) + rB^*(r^{-1}z)z^{-1}u(r^2z^{-1})), \\ &(u(z) - rB(r^{-1}z)z^{-1}v(r^2z^{-1}), \\ &-v(z) + rB^*(r^{-1}z)z^{-1}u(r^2z^{-1}))\rangle_{ext\ core(B)} \\ =&\langle u(z) - rB(r^{-1}z)z^{-1}v(r^2z^{-1}), u(z)\rangle_{ext\ \mathcal{C}_r(z)} \\ &+ \langle v(z) - rB^*(r^{-1}z)z^{-1}u(r^2z^{-1}), v(z))\rangle_{ext\ \mathcal{C}_r(z)}\end{aligned}$$

is satisfied. The symmetry and the nondegeneracy of a scalar product can be easily verified.

Let π be a transformation of *ext* $\mathcal{G}(B)$ onto *ext core*(B) which takes $(u(z), v(z))$ into

$$(u(z) - rB(r^{-1}z)z^{-1}v(r^2z^{-1}), -v(z) + rB^*(r^{-1}z)z^{-1}u(r^2z^{-1})).$$

A construction will now be made of a unitary linear system with transfer function $B(z)$.

THEOREM 4.1. *Assume that $B(z)$ is a power series with operator coefficients such that multiplication by $B(r^{-1}z)$ is an everywhere defined transformation in $\mathcal{C}_r(z)$. Then a Krein space $\mathcal{D}(B)$ exists which is the state space of a unitary linear system with transfer function $B(z)$ and which contains ext core(B) isometrically.*

Proof. The Hilbert space *ext* $\mathcal{G}(B)$ will be denoted $\mathcal{G}$ in the proof. Let $J_{\mathcal{G}}$ be the unique self-adjoint transformation of $\mathcal{G}$ into itself such that the identity

$$\begin{aligned}&\langle J_{\mathcal{G}}\ (u(z), v(z)), (u(z), v(z))\rangle_{\mathcal{G}}\\ =&\langle u(z) - rB(r^{-1}z)z^{-1}v(r^2z^{-1}), u(z)\rangle_{ext\ \mathcal{C}_r(z)}\\ &+\langle v(z) - rB^*(r^{-1}z)z^{-1}u(r^2z^{-1}), v(z)\rangle_{ext\ \mathcal{C}_r(z)}\end{aligned}$$

holds for every element $(u(z), v(z))$ of $\mathcal{G}$. Let $\hat{\mathcal{G}}$ be the Krein completion of $\mathcal{G}$ with respect to $J_{\mathcal{G}}$. Let π be the projection of $\mathcal{G}$ into $\hat{\mathcal{G}}$.

Define a continuous transformation

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} : \mathcal{G}\times\mathcal{C} \longrightarrow \mathcal{G}\times\mathcal{C}$$

by:

$$\begin{aligned}&A(u(z), v(z)) = (r[u(z) - a_0]/z, r^{-1}zv(z)),\\ &B(c) = (-rB(0)cz^{-1}, -c),\\ &C(u(z), v(z)) = a_0,\\ &D(c) = B(0)c,\end{aligned}$$

where a_0 is the constant coefficient in $u(z) - B(z)z^{-1}v(z^{-1})$. Define a continuous transformation

$$\begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix} : \mathcal{G}\times\mathcal{C} \longrightarrow \mathcal{G}\times\mathcal{C}$$

by:

$$\begin{aligned}&A^*(u(z), v(z)) = (r^{-1}zu(z), r[v(z) - b_0]/z),\\ &B^*((u(z), v(z))) = -b_0,\\ &C^*(c) = (c, z^{-1}B^*(0)c),\\ &D^*(c) = B^*(0)c\end{aligned}$$

where b_0 is the constant coefficient in $v(z) - rB^*(r^{-1}z)z^{-1}u(r^2z^{-1})$.

The identities

$$\begin{aligned}
\langle J_{\mathcal{G}}\ A\ (u(z), v(z)), (u(z), v(z))\rangle_{\mathcal{G}} &= \langle J_{\mathcal{G}}\ (u(z), v(z)), A^*(u(z), v(z))\rangle_{\mathcal{G}},\\
\langle J_{\mathcal{G}}\ B\ c, (u(z), v(z))\rangle_{\mathcal{G}} &= \langle c, B^*\ (u(z), v(z))\rangle_{\mathcal{C}}, \text{ and}\\
\langle C\ (u(z), v(z)), c\rangle_{\mathcal{C}} &= \langle J_{\mathcal{G}}\ (u(z), v(z)), C^*\ c\rangle_{\mathcal{G}}
\end{aligned}$$

hold for all $(u(z), v(z)) \in \mathcal{G}$ and $c \in \mathcal{C}$. By Theorem 3.1 unique transformations $\hat{A}, \hat{A}^* : \hat{\mathcal{G}} \longrightarrow \hat{\mathcal{G}}$ and $\hat{B}^*, \hat{C} : \hat{\mathcal{G}} \longrightarrow \mathcal{C}$ exist such that the identities $\hat{A}\ \pi = \pi\ A$, $\hat{A}^*\pi = \pi\ A^*$, $\hat{B}^*\ \pi = B^*$ and $\hat{C}\ \pi = C$ are satisfied.

The identities

$$\begin{aligned}
&\hat{A}\ \pi(u(z), v(z)) = (r[f(z) - f(0)]/z, r^{-1}zg(z) - B^*(r^{-1}z)f(0)),\\
&\hat{A}^*\ \pi(u(z), v(z)) = (r^{-1}zf(z) - B(r^{-1}z)g(0), r[g(z) - g(0)]/z),\\
&\pi\ B\ c = (r[B(r^{-1}z) - B(0)]c/z, [1 - B^*(r^{-1}z)B(0)]c),\\
&\hat{B}^*\ \pi_{\mathcal{G}}\ (u(z), v(z)) = g(0), \text{and}\\
&\hat{C}\ \pi_{\mathcal{G}}\ (u(z), v(z)) = f(0)
\end{aligned}$$

hold for all $(u(z), v(z)) \in \mathcal{G}$, where $\pi\ (u(z), v(z)) = (f(z), g(z))$.

It will be shown that there is no nonzero element k of $\hat{\mathcal{G}}$ such that $\hat{C}\hat{A}^n\ k$ and $\hat{B}^*\hat{A^*}^n\ k$ vanish for every nonnegative integer n. These conditions imply that k is orthogonal in $\hat{\mathcal{G}}$ to every element of $\mathcal{G}$. Since $\mathcal{G}$ is dense in $\hat{\mathcal{G}}$ by construction, $k = 0$.

A pair $(f(z), g(z))$ of power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ is associated with every element k of $\hat{\mathcal{G}}$ by $a_n = r^{-n}\hat{C}\hat{A}^n\ k$ and $b_n = r^{-n}\hat{B}^*\hat{A^*}^n\ k$ for every nonnegative integer n. The element k is uniquely determined by a knowledge of these power series. These power series are

$$\begin{aligned}
f(z) = u(z) - rB(r^{-1}z)z^{-1}v(r^2z^{-1}) \text{ and } g(z)\\
= -v(z) + rB^*(r^{-1}z)z^{-1}u(r^2z^{-1})
\end{aligned}$$

whenever $k = \pi_{\mathcal{G}}\ (u(z), v(z))$ for any $(u(z), v(z))$ in ext $\mathcal{G}(B)$.

The Krein space $\hat{\mathcal{G}}$ can therefore be realized as the state space of an extended canonical linear system $\mathcal{D}(B)$ in such a way that ext $core(B)$ is contained densely and isometrically in the space.

This completes the proof of the theorem.

A construction of canonical linear systems which are conjugate isometric and which have given transfer functions is due to Yang [16]. A construction of such linear systems is obtained by the present methods.

THEOREM 4.2. *Assume that $B(z)$ is a power series with operator coefficients such that multiplication by $B(r^{-1}z)$ is an everywhere defined transformation in $\mathcal{C}_r(z)$. A sufficient condition for the existence of a space $\mathcal{H}(B)$ is that the set of elements $(f(z), g(z))$ of $\mathcal{D}(B)$ such that $f(z) = 0$ is a Krein space which is contained continuously and isometrically in $\mathcal{D}(B)$. The space is contained continuously in $\mathcal{C}(z)$. A partially isometric transformation of $\mathcal{D}(B)$ onto $\mathcal{H}(B)$ is then defined by taking $(f(z), g(z))$ into $f(z)$.*

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