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SINGULAR SOLUTIONS OF SEMILINEAR PARABOLIC EQUATIONS

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1. Introduction

In this paper we discuss the existence and uniqueness of singular solutions for equations of the form

$$u_t = u_{xx} - |u|^{q-1}u_x - |u|^{p-1}u, \quad p, q > 1, \tag{F}$$

in the domain $Q = \{(x,t) : x \in \mathbb{R}, t > 0\}$. This equation represents a model of diffusion-convection with absorption.

Following Escobedo and Zuazua (1991), it is easy to see that given initial data $u_0 \in L^1(\mathbb{R})$ there exists a unique solution $u(x,t)$ of (F) such that $u \in C((0,\infty); W^{2,l}(\mathbb{R})) \cap C^1((0,\infty); L^l(\mathbb{R}))$ for every $l \in (1,\infty)$. Moreover, physical considerations lead us to assume that $u_0(x) \geq 0$, in which case $u(x,t)$ is positive unless $u \equiv 0$ in Q from the Maximum Principle and becomes C^∞ smooth in Q from standard regularity theory.

The properties of solutions are usually explained in terms of the properties of special solutions such as singular ones. A singular solution (it is also called a fundamental solution or a source-type solution) is the one corresponding to initial data a Dirac mass, $M\delta(x)$ with $M > 0$, namely a solution $u(x,t)$ such that $u(x,t)$ satisfies (F) in the classical sense and $u(x,t) \to M\delta(x)$ as $t \to 0$ in the sense of measures, namely

$$\lim_{t\to 0} \int u(x,t)\phi(x)dx = M\phi(0),$$

for all continuous, bounded functions ϕ on $\mathbb{R}$.

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Brezis and Friedman (1983) showed that the equation $u_t = u_{xx} - u^p$ admits the unique singular solution only for $1 < p < 3$. Moreover Escobedo, Vazquez and Zuazua (1991) considered equations of the type

$$u_t = u_{xx} - u^{q-1}u_x \tag{E}$$

and proved its existence and uniqueness for every $q > 1$. But it seems to us that no results are available for equations of mixed type.

As a first attempt to this direction, combining those two results we here show that the unique singular solution of (F) exists if and only if $1 < p < 3$ or $3 \le p < q + 1$. This reveals some interactions between absorption and convection for the existence of singular solution.

For the proof of existence and uniqueness we mainly use the comparison with the equation (E) and for nonexistence following [1], we take a function of the form $\eta(k(|x|^{q^*} + t))$ as a test function to lead a contradiction, see section 5 for details. This method and the main estimates (see Lemma 2) turns out to be very useful and concise and applicable to wide class of diffusion equations. This will be discussed in other space.

2. Preliminary Estimates

We first recall some basic estimates which will be used. Any solution $u(x,t)$ of (F) as mentioned in introduction satisfies

$$\frac{d}{dt}\int u(x,t)dx \le 0. \tag{1}$$

Moreover for any two solutions u, v; by the standard technique of multiplying by a sign function and integrating we get the contraction principle

$$|u(\cdot,t) - v(\cdot,t)|_{L^1(\mathbb{R})} \le |u(\cdot,\tau) - v(\cdot,\tau)|_{L^1(\mathbb{R})}, \tag{2}$$

for any $t \ge \tau \ge 0$.

For the equation (E), it is shown in [3] (see estimate (2.32) in that paper) that

$$0 \le u(x,t) \le C(M)(t^{-1/2} + t^{(1-q)/2}), \qquad \forall t > 0, \tag{3}$$

where $C(M)$ is a constant depending only on $M = |u_0(x)|_{L^1(\mathbb{R})}$. This estimate is not sharp and for $1 < q \le 2$, Escobedo, Vazquez and Zuazua obtained a better bound for u:

LEMMA 1. ([2], Lemma 1.2) *For $1 < q \leq 2$ we have*

$$0 \leq u(x,t) \leq \left(\frac{qM}{(q-1)t}\right)^{1/q}, \tag{4}$$

where M is defined above.

The main estimate in this paper is a bound for $q > 2$. Accidently we obtained a similar estimate.

LEMMA 2. *For $q > 2$ we have*

$$0 \leq u(x,t) \leq 4^{(q-1)/q^2} \left(\frac{2M}{t}\right)^{1/q}. \tag{5}$$

Proof. We define the new variable $z = u^{q-1}$. In terms of z, equation (E) reads

$$z_t + zz_x - \beta\frac{z_x^2}{z} = z_{xx} \tag{6}$$

where $\beta = (2-q)/(q-1)$.

Let $v(x,t)$ be a solution of the Burgers' equation

$$v_t + vv_x = v_{xx}$$

with initial data $v(x,0) = u^{q-1}(x,s)$, $s > 0$. For $q > 2$, $\beta < 0$ and $v(x,t)$ is a supersolution of (6) and a comparison principle implies that

$$u^{q-1}(x,t+s) \leq v(x,t), \quad t > 0.$$

Moreover from Lemma 1 we obtain

$$0 \leq v(x,t) \leq \left(\frac{2\int v(x,0)dx}{t}\right)^{1/2}$$

and

$$0 \leq u^{q-1}(x,t+s) \leq \left(\frac{2\int u^{q-1}(x,s)dx}{t}\right)^{1/2}, \quad t > 0. \tag{7}$$

We also have

$$\int u^{q-1}(x,s)dx \le |u^{q-2}(x,s)|_{L^\infty(\mathbb{R})} \int u(x,s)dx = M|u(x,s)|^{q-2}_{L^\infty(\mathbb{R})}.$$

We now let $s = t+\epsilon$, with $\epsilon > 0$ and define

$$w(t) = \sup_{0<\tau\le t} \tau^{1/q}|u(x,\tau+\epsilon)|_{L^\infty(\mathbb{R})},$$

then, for $t > 0$, $0 \le w(t) < \infty$ and

$$0 \le w(t)^{q-1} \le w(2t)^{q-1} \le 2^{(q-1)/q}(2M)^{1/2}w(t)^{(q-2)/2}.$$

Therefore we have

$$w(t) \le 2^{2(q-1)/q^2}(2M)^{1/q}, \qquad \forall t > 0. \tag{8}$$

This implies that $0 \le t^{1/q}u(x,t+\epsilon) \le 4^{(q-1)/q^2}(2M)^{1/q}$ and (5) in the limit as $\epsilon \to 0$. □

3. Existence

We denote by $E_M(x,t)$ the singular solution for (E) corresponding to initial data $M\delta(x)$. We note that $\int E_M(x,t)dx = M$ for all $t > 0$. We also denote $q^* = \max\{q,2\}$ and we prove

THEOREM 3. *For* $1 < p < q^*+1$, *there exists a fundamental solution of (F) for every* $M > 0$.

Proof. Let us define $\tilde{u}_n(x,t)$ as the solution of (F) for $t > 1/n$ with initial data $\tilde{u}_n(x,1/n) = E_M(x,1/n)$ at $t = 1/n$. By the Maximum Principle (see [5] for example) we know that

$$\tilde{u}_n(x,t) \le E_M(x,t) \qquad \forall t \ge 1/n \quad \text{and} \quad x \in \mathbb{R}.$$

Since $\tilde{u}_n(x,t)$ is monotone decreasing as $n \to \infty$, its limit

$$u(x,t) = \lim_{n\to\infty} \tilde{u}_n(x,t)$$

exists and $u(x,t)$ is a weak solution of (F) in Q. By a standard regularity result, we may conclude that $u(x,t)$ is a classical solution in Q.

It is clear that $u(x,t) \leq E_M(x,t)$ in Q and $u(x,0) = 0$ for every $x \neq 0$. Moreover, for $1/n < t \leq 1$

$$(9) \qquad \begin{aligned} I_n(t) =& \left| \int (\tilde{u}_n(x,t) - \tilde{u}_n(x,1/n))\, dx \right| \\ =& \int_{1/n}^{t} \int_{\mathbb{R}} \tilde{u}_n^p(x,s) dx ds \\ \leq& \int_{1/n}^{t} \int_{\mathbb{R}} E_M^p(x,s) dx ds \\ \leq& \int_{1/n}^{t} M \sup_{\mathbb{R}} E_M^{p-1}(\cdot, s) ds \\ \leq& C \int_{1/n}^{t} s^{-(p-1)/q^*} ds \quad \text{(from (3) and (5))} \\ \leq& \frac{Cq^*}{1-p+q^*} t^{(1-p+q^*)/q^*}. \end{aligned}$$

Note that $1 - p + q^* > 0$.

Since $\int \tilde{u}_n(x,1/n)dx = M$ by the definition of $\tilde{u}_n$, we conclude that $\tilde{u}_n(x,t)dx$ will be close to M for sufficiently small t uniformly in n and in the limit the same will be true for u. Consequently $\lim_{t\to 0} \int u(x,t)dx = M$. For every continuous bounded function ϕ,

$$0 \leq \int u(x,t)[\phi(x) - \phi(0)]_+ dx \leq \int E_M(x,t)[\phi(x) - \phi(0)]_+ dx$$

and since the second integral tends to 0 as $t \to 0$, the same will be true for the first integral. Similarly

$$\lim_{t\to 0} \int u(x,t)[\phi(x) - \phi(0)]_- dx = 0$$

and writing

$$\phi(x) = \phi(0) + [\phi(x) - \phi(0)]_+ - [\phi(x) - \phi(0)]_-,$$

we finally obtain

$$\lim_{t\to 0}\int u(x,t)\phi(x)dx = M\phi(0). \tag{10}$$

□

4. Uniqueness

For the proof of uniqueness, we prepare the following lemmas.

LEMMA 4. *Let $u(x,t)$ be a fundamental solution of (F), then*

$$\int u(x,t)dx - M = -\int_0^t \int u^p(x,s)dxds. \tag{11}$$

In particular $\lim_{t\to 0}\int u(x,t)dx = M$ *and* $\int u(x,t)dx \le M$ *for every* $t > 0$.

Proof. An integration gives

$$\int u(x,t)dx - \int u(x,\tau)dx = -\int_\tau^t \int u^p(x,s)dxds$$

for $t > \tau > 0$. By definition $\lim_{\tau\to 0}\int u(x,\tau)dx = M$ and (11) holds. The last inequality also holds obviously. □

LEMMA 5. *Let $u(x,t)$ be a fundamental solution of (F), then $u(x,t) \le E_M(x,t)$ in Q.*

Proof. Let $w_n(x,t)$ be a solution of

$$w_t = w_{xx} - (w^q/q)_x, \quad w(x,1/n) = u(x,1/n)$$

for $t > 1/n$. Similarly to the proof of Theorem 3, one can see that w_n is increasing and $u(x,t) \le w_n(x,t)$ for $t > 1/n$. Let $w = \lim_{n\to\infty} w_n$, then $u(x,t) \le w(x,t)$ for $t > 0$ and $w(x,t)$ is a weak solution of (E) and also becomes a classical solution in Q.

We now note that

$$\int w_n(x,t)dx = \int w_n(x,1/n)dx = \int u(x,1/n)dx \le M$$

for $t > 1/n$. From the Monotone Convergence Theorem, we obtain

$$\int w(x,t)dx \le M.$$

Since $\int u(x,t)dx$ goes to M as t tends to 0, the same will be true for $\int w(x,t)dx$. We now apply the same argument as the proof of Theorem 3 to conclude that

$$\lim_{t\to 0} \int w(x,t)\phi(x)dx = M\phi(0) \tag{12}$$

for every continuous bounded function ϕ. Thus $w(x,t)$ is a fundamental solution of (E) and the completion of the proof of Lemma follows from the uniqueness of the fundamental solution of (E). But here we take $\psi(x) = |\phi|_{L^\infty} - [\phi(x) - \phi(0)]_+$ as a test function. Then

$$\begin{aligned} 0 &\le \int u(x,t)\psi(x)dx \\ &\le \int w(x,t)\psi(x)dx \le M|\phi|_{L^\infty}. \end{aligned}$$

Since $\lim_{t\to 0}\int u(x,t)\psi(x)dx = M|\phi|_{L^\infty}$, we obtain

$$\lim_{t\to 0} \int w(x,t)[\phi(x) - \phi(0)]_+ dx = 0. \tag{13}$$

Similarly we have

$$\lim_{t\to 0} \int w(x,t)[\phi(x) - \phi(0)]_- dx = 0$$

and (12). □

We now prove the uniqueness.

THEOREM 6. *There exists at most one fundamental solution of (F) for each $M > 0$.*

Proof. Let u, v be fundamental solutions of (F) with initial data $M\delta(x)$. Then from the contraction principle (2) and Lemma 4 we have

$$\begin{aligned}\int |u(x,t)-v(x,t)|dx &\leq \int |u(x,1/n)-v(x,1/n)|dx \\ &\leq \int (E_M(x,1/n)-u(x,1/n))dx \\ &+ \int (E_M(x,1/n)-v(x,1/n))dx \\ &= 2M - \int u(x,1/n)dx - \int v(x,1/n)dx,\end{aligned}$$

which tends to 0 as $n \to \infty$ from lemma 4. Hence $u \equiv v$ in Q. □

5. Nonexistence

We here consider the case $p \geq q^* + 1$. Recall that $q^* = \max\{q, 2\}$. Let $u(x,t)$ be a fundamental solution of (F) with initial data $M\delta$, then from Lemma 4 we see that $u(x,t)$ is L^p-integrable over Q.

Following [1], we prove that

THEOREM 7. *There is no fundamental solution of (F).*

Proof. Let $\eta(s)$ be any smooth non-decreasing function on $\mathbb{R}$ such that

$$\eta(s) = \begin{cases} 1 & \text{for} \quad s \geq 1 \\ 0 & \text{for} \quad s \leq 0, \end{cases}$$

and set $\eta_k(s) = \eta(ks)$.

If we define $\phi_k(x,t) = \eta_k(|x|^{q^*} + t)$, then we know that

$$\int_\epsilon^T \int_{\mathbb{R}} u_t \phi_k - \int_\epsilon^T \int_{\mathbb{R}} u \phi_{k,xx} - \int_\epsilon^T \int_{\mathbb{R}} \frac{u^q}{q} \phi_{k,x} + \int_\epsilon^T \int_{\mathbb{R}} u^p \phi_k = 0$$

for $0 < \epsilon < T$. We first have

$$\int_\epsilon^T \int_{\mathbb{R}} u_t \phi_k dx dt = \int_{\mathbb{R}} u(x,T)\phi_k(x,T)dx - \int_{\mathbb{R}} u(x,\epsilon)\phi_k(x,\epsilon)dx - \int_\epsilon^T \int_{\mathbb{R}} u\phi_{k,t}$$

and in the limit as $\epsilon \to 0$ we get

$$(13)\quad \int_0^T \int u^p \phi_k \le \int_0^T \int u\phi_{k,t} + \int_0^T \int u\phi_{k,xx} + \int_0^T \int u^q/q\phi_{k,x}.$$

We claim that all the integral in the right tends to 0 as $k \to \infty$. It then follows that $\int_0^T \int u^p dxdt = 0$ and $u \equiv 0$ in Q. We have

$$\left|\int_0^T \int u\phi_{k,t}\right| \le Ck \iint_{D_k} u,$$
$$\left|\int_0^T \int u\phi_{k,xx}\right| \le Ck^{2/q^*} \iint_{D_k} u,$$
$$\left|\int_0^T \int \frac{u^q}{q}\phi_{k,x}\right| \le Ck^{1/q^*} \iint_{D_k} u^q,$$

where $D_k = \{(x,t); t > 0, \quad 0 < |x|^{q^*} + t \le 1/k\}$. By Hölder inequality we finally get:

$$\iint_{D_k} u \le \left(\iint_{D_k} u^p\right)^{1/p} |D_k|^{1-1/p}$$
$$\iint_{D_k} u^q \le \left(\iint_{D_k} u^p\right)^{q/p} |D_k|^{1-q/p}.$$

Now note that $|D_k| \le 2k^{-(1+1/q^*)}$, $2/q^* \le 1$ and

$$1 - \left(1 + \frac{1}{q^*}\right)\left(1 - \frac{1}{p}\right) = -\frac{p - q^* - 1}{pq^*} \le 0,$$
$$\frac{1}{q^*} - \left(1 + \frac{1}{q^*}\right)\left(1 - \frac{q}{p}\right) = -\frac{p(q^* - q) + p - q^* - 1}{pq^*} \le 0.$$

Our claim follows from the fact that $\iint_{D_k} u^p \to 0$ as $k \to \infty$. □

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