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ON THE GAUSS MAP OF HYPERSURFACES IN THE SPACE FORM

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1. Introduction

Let M^n be a hypersurface in the Euclidean space R^{n+1} with position vector field x and unit normal vector field G. Then it can be shown (Proposition 2.2) that M^n satisfies the condition

$$G = Ax + b \tag{1.1}$$

for some fixed endomorphism A of R^{n+1} and for some fixed vector b in R^{n+1} if and only if M^n is an isoparametric hypersurface or equivalently, M^n is an open piece of one of the following hypersurfaces: hyperplane R^n, hypersphere $S^n(r)$, generalized cylinder $S^p(r) \times R^{n-p}$.

Generalizing above result, in Section 2, we classify the hypersurfaces M^n in the space form $\bar{M}^{n+1}(c)$ which satisfy the condition (1.1) for some suitable A and b.

Note that the condition (1.1) is equivalent to the condition

$$S(X) = A(X),\ X \in TM, \tag{1.2}$$

where S is the shape operator of M^n with respect to G. Generalizing the condition (1.2), in Section 3, we classify the hypersurfaces in R^{n+1} which satisfies the condition

$$S(X) = (AX)^T,\ X \in TM, \tag{1.3}$$

where $(\ )^T$ denotes the tangential component.

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2. Hypersurfaces satisfying $G = Ax + b$

Note that $\bar{M}^{n+1}(0)$ is the Euclidean space R^{n+1} and for $c > 0$, the sphere $S^{n+1}(1/\sqrt{c})$ is the natural embedding of $\bar{M}^{n+1}(c)$ into the Euclidean space R^{n+2}. Let R_1^{n+2} be the Lorentz space with the metric $< x, y > = x_1y_1 + \cdots + x_{n+1}y_{n+1} - x_{n+2}\ y_{n+2}$. Then for $c < 0$, $H^{n+1}(1/\sqrt{-c})$ is the natural embedding of $\bar{M}^{n+1}(c)$ into the Lorentz space R_1^{n+2}, where $H^{n+1}(1/\sqrt{-c}) = \{x = (x_1, \cdots, x_{n+2}) \in R^{n+2} | x_{n+2} > 0, < x, x >= \frac{1}{c}\}$.

Hereafter, we assume that $c = 0$ or ± 1.

LEMMA 2.1. *Let M^n be a hypersurface in the space form $\bar{M}^{n+1}(c)$, $c = 0$ or ± 1, with unit normal vector field G. Then we have*

$$\Delta G = n\nabla\alpha + \|S\|^2 G - nc\alpha x, \tag{2.1}$$

where α is the mean curvature of M^n with respect to G, Δ is the Laplacian of M^n acting on vector valued functions and $\|S\|^2 = tr(S^2)$.

Proof. We will give the proof for the case $c = 1$ and the other cases can be proved in a similar manner. Choose a local orthonormal frame $e_1, \cdots, e_{n+2}$ in such a way that $e_1, \cdots, e_n$ are tangent to M^n and $e_{n+1} = G$ and $e_{n+2} = x$. Moreover we may assume that $e_1, \cdots, e_n$ are eigenvectors of the shape operator S with the eigenvalues $\mu_1, \cdots, \mu_n$. Then we have

$$\begin{aligned}
\Delta G &= \sum_{j=1}^{n} \{\bar{\nabla}_{\nabla_{e_j} e_j} G - \bar{\nabla}_{e_j} \bar{\nabla}_{e_j} G\} \\
&= \sum_{j=1}^{n} \{e_j(\mu_j)e_j + \mu_j \bar{\nabla}_{e_j} e_j - S(\nabla_{e_j}^{e_j})\} \\
&= \sum_{k=1}^{n} \{e_k(\mu_k) + \sum_{j=1}^{n} (\mu_j - \mu_k) w_j^k(e_j)\} e_k + \sum_{j=1}^{n} \mu_j^2 G - \sum_{j=1}^{n} \mu_j x \\
&= \sum_{k=1}^{n} e_k(\sum_j \mu_j) e_k + \sum_{j=1}^{n} \mu_j^2 G - \sum_{j=1}^{n} \mu_j x,
\end{aligned}$$

where the third equality follows from the fact $\bar{\nabla}_{e_j} e_j = \nabla_{e_j} e_j + \mu_j G - x$

and the fourth equality follows from the Codazzi equation. Since $n\alpha = \sum_{j=1}^{n} \mu_j$, $\|S\|^2 = \sum_{j=1}^{n} \mu_j^2$, the lemma follows.

PROPOSITION 2.2. *A hypersurface M^n in R^{n+1} satisfies the condition (1.1) if and only if M^n is an open piece of one of the following hypersurfaces: hyperplane R^n, hypersphere $S^n(r)$, generalized cylinder $S^p(r) \times R^{n-p}$.*

Proof. Choose an orthonormal frame $e_1, \cdots, e_n$, $e_{n+1} = G$ such that $e_1, \cdots, e_n$ are eigenvectors of S with eigenvalues $\mu_1, \cdots \mu_n$. By differentiating (1.1) we have

$$Ae_i = -\mu_i e_i, \quad i = 1, \cdots, n. \tag{2.2}$$

and from (1.1) and (2.1), using the fact $\Delta x = -n\alpha G$, we obtain

$$-n\alpha AG = n\nabla\alpha + \|S\|^2 G. \tag{2.3}$$

Let $U = \{p \in M | \alpha(p) \neq 0\}$. Then on U we have

$$AG = -\frac{1}{\alpha}\nabla\alpha - \frac{1}{n\alpha}\|S\|^2 G. \tag{2.4}$$

Since A is a fixed endomorphism, the eigenvalues of A are constant. Hence $-\mu_1, \cdots, -\mu_n$, $-\frac{1}{n\alpha}\|S\|^2$ are constant on U. Therefore U is an isoparametric hypersurface and α is constant on U. Hence we have either $U = \phi$ or $U = M$. If $U = \phi$, then (2.3) shows that M is an open piece of a hyperplane R^n. If $U = M$, then by the classification theorem of isoparametric hypersurface of R^{n+1}, M is an open piece of one of the following hypersurfaces: $S^n(r), S^p(r) \times R^{n-p}$.

The converse follows immediately.

Now we give the classification theorem for the hypersurfaces in the spherical space form $S^{n+1}(1) \subset R^{n+2}$.

THEOREM 2.3. *A hypersurface M^n in $S^{n+1}(1) \subset R^{n+2}$ satisfies (1.1) for some $A \in L(R^{n+2})$ and $b \in R^{n+2}$ if and only if M^n is an open piece of one of the following hypersurfaces:*

1) $S^n(r) \subset S^{n+1}(1)$,

2) $S^p(r_1) \times S^{n-p}(r_2) \subset S^{n+1}(1), r_1^2 + r_2^2 = 1$.

Proof. As in the proof of Lemma 2.1, we choose an orthonormal frame $e_1, \cdots, e_n$. Then from (1.1), (2.1) we have

$$Ae_i = -\mu_i e_i, \quad i = 1, \cdots, n, \tag{2.5}$$

$$-n\alpha AG = n\nabla\alpha + (\|S\|^2 - n)G - n\alpha x + nb, \tag{2.6}$$

$$Ax = G - b. \tag{2.7}$$

First, we consider the case $b = 0$. Suppose that $U = \{p \in M | \alpha(p) \neq 0\} \neq \phi$. Then on U, with respect to the frame $e_1, \cdots, e_n$, G and x, A is represented by the following matrix:

$$\begin{pmatrix} -S & * & * \\ 0 & \beta & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

where $\beta = \frac{-1}{n\alpha}(\|S\|^2 - n)$.

Hence the characteristic polynomial $p(t)$ of A satisfies

$$p(t) = (t + \mu_1) \cdots (t + \mu_n)(t^2 - \beta t - 1).$$

Since the eigenvalues of A are constant, $\mu_1, \cdots, \mu_n$ must be constant on U. Hence we have either $U = \phi$ or $U = M$. Suppose that $U = \phi$, then M is minimal and $\|S\|^2 = n$. Hence a theorem of Yano-Kon ([6]) implies that M is locally a Clifford minimal hypersurface $S^p(\sqrt{\frac{p}{n}}) \times S^{n-p}(\sqrt{\frac{n-p}{n}})$. Suppose that $U = M$, then M^n is an isoparametric hypersurface of $S^{n+1}(1)$. Let $\mu_1, \cdots, \mu_g$ be the distinct principal curvatures of M^n and let $TM = E_1 \oplus \cdots \oplus E_g$ be the eigen-decomposition corresponding to $\mu_1, \cdots, \mu_g$. And let $E_1', \cdots E_g'$ be the eigenspace of A corresponding to the eigenvalues $-\mu_1, \cdots, -\mu_g$. Then we have $E_i(p) \subset E_i'$ for all $p \in M$. Since the leaf F_i of E_i is a sphere and since $F_i \subset E_i' + a$ for some $a \in R^{n+2}$, we have $\dim E_i < \dim E_i'$. Hence from the fact that $n = \sum_{i=1}^{g} \dim E_i$ and $n = \sum_{i=1}^{g} \dim E_i' \leq n + 2$, we obtain $g = 1$ or 2. Thus M is locally a hypersphere $S^n(r)$ or a product of two spheres $S^p(r_1) \times S^{n-p}(r_2), r_1^2 + r_2^2 = 1$.

Now, we consider the case $b \neq 0$. Let $U = \{p \in M | \alpha(p) \neq 0\}$. Then on U (2.5) $\sim$ (2.7) imply that A is represented by the following matrix:

$$\begin{pmatrix} -S & * & * \\ 0 & \beta & \delta \\ 0 & \gamma & \epsilon \end{pmatrix},$$

where

$$\begin{aligned} \beta &= -\frac{1}{n\alpha}(\|S\|^2 - n), \quad & \gamma &= 1 - \frac{1}{\alpha} < x, b >, \\ \delta &= 1- < G, b >, & \epsilon &= - < x, b > . \end{aligned} \tag{2.8}$$

Since $P_A(t) = (t + \mu_1) \cdots (t + \mu_n)\ \{t^2 - (\beta + \epsilon)t + \beta\epsilon - \gamma\delta\}$, we have $\mu_1, \cdots, \mu_n$, $\beta + \epsilon$ and $\beta\epsilon - \gamma\delta$ are constant on U. Hence we have either $U = \phi$ or $U = M$. Suppose that $U = \phi$, then (2.6) implies that $M = S^n(1) \subset S^{n+1}(1)$. Suppose that $U = M$, then for some constant c_1 and c_2, $< G, b >$ and $< x, b >$ satisfy the following equations:

$$< G, b > +\alpha < x, b >= c_1, \tag{2.9}$$

$$n\alpha < G, b > +\|S\|^2 < x, b >= c_2.$$

Note that the determinant D of the coefficient matrix of (2.8) satisfies

$$D = \|S\|^2 - n\alpha^2 \geq 0, \quad D = 0 \text{ iff } \mu_1 = \cdots = \mu_n. \tag{2.10}$$

Suppose that $V = \{p \in M | D(p) > 0\} \neq \phi$. Then on V, $< G, b >$ and $< x, b >$ are constant, hence V is totally umbilic. This contradiction shows that $V = \phi$, that is, M is totally umbilic. Thus we have $M = S^n(r) \subset S^{n+1}(1)$.

The converse follows immediately.

Suppose that a hypersurface M^n in the hyperbolic space form $H^{n+1}(1) \subset R_1^{n+2}$ satisfies $G = Ax + b$, $A \in L(R_1^{n+2})$, $b \in R_1^{n+2}$. Then as in the proof of Theorem 2.2, we may prove that M is isoparametric. Hence by the classification theorem of Cartan ([1]), we have the following theorem.

THEOREM 2.4. *A hypersurface M^n in $H^{n+1}(1) \subset R_1^{n+2}$ satisfies $G = Ax + b$ if and only if M is isoparametric, or equivalently M is an open piece of one of the following hypersurfaces:*

(1) $S^n(\sinh\theta) \subset H^{n+1}(1)$,
(2) $H^n(\cosh\theta) \subset H^{n+1}(1)$,
(3) $S^p(\sinh\theta) \times H^{n-p}(\cosh\theta) \subset H^{n+1}(1)$,
(4) $\{x = (y, \frac{1}{2}|y|^2, \frac{1}{2}|y|^2 + 1) | y \in R^n\} \subset H^{n+1}(1)$.

3. Hypersurfaces satisfying $SX = (AX)^T$

Suppose that a hypersurface M^n in R^{n+1} satisfies the condition (1.3) for some $A \in L(R^{n+1})$ and $b \in R^{n+1}$. This condition is equivalent to the following:

$$< SX, Y > = < AX, Y > \tag{3.1}$$

for any tangent vector X, Y. Since S is selfadjoint, taking $\frac{1}{2}(A + A^t)$ instead of A, we may assume that A is selfadjoint. Choose a local orthonormal frame $e_1, \cdots, e_n$ such that $Se_i = \mu_i e_i$, $i = 1, \cdots, n$. Then we can write

$$Ae_i = \mu_i e_i + b_i G, \quad i = 1, \cdots, n, \tag{3.2}$$

$$AG = \sum_{k=1}^{n} b_k e_k + fG. \tag{3.3}$$

Note that

$$\bar\nabla_{e_i}(Ae_j) = A(\bar\nabla_{e_i} e_j), \; i, j = 1, \cdots, n. \tag{3.4}$$

From (3.2) $\sim$ (3.4), we have

$$b_j \mu_i e_i = e_i(\mu_j) e_j + \sum_k (\mu_j - \mu_k) w_j^k(e_i) e_k - \mu_i \delta_{ij} \sum_k b_k e_k, \tag{3.5}$$

$$f \mu_i \delta_{ij} = \mu_i \mu_j \delta_{ij} + e_i(b_j) - \sum_k w_j^k(e_i) b_k. \tag{3.6}$$

Note that the equation of Codazzi can be written down as
(3.7)
$$e_i(\mu_j)e_j + \sum_k (\mu_j - \mu_k)w_j^k(e_i)e_k = e_j(\mu_i)e_i + \sum_k (\mu_i - \mu_k)w_i^k(e_j)e_k,$$

which in particular for $i \neq j$ gives

$$e_j(\mu_i) = (\mu_j - \mu_i)w_j^i(e_i). \tag{3.8}$$

From (3.5) we obtain

$$e_i(\mu_i) = 2b_i\mu_i, \tag{3.9}$$

$$e_j(\mu_i) = 0 \text{ for } i \neq j. \tag{3.10}$$

Furthermore (3.5) and the Codazzi equation (3.7) imply that

$$b_j\mu_i e_i = b_i\mu_j e_j, \tag{3.11}$$

$$b_j\mu_i = 0 \text{ for } i \neq j. \tag{3.12}$$

Now, let V be the set of points where the multiplicities of the principal curvatures do not change. Then V is an open dense subset of M, where the principal curvatures $\mu_1, \cdots, \mu_n$ are differentiable ([5]).

Let p be any point of V. Then there are two possibilities.

1) rank $S_p \geq 2$. In this case we may assume that $\mu_1 \neq 0$, $\mu_2 \neq 0$. Then from (3.12) we have

$$b_i = 0, i = 1, \cdots, n, \tag{3.13}$$

hence $A = \text{diag}\,(\mu_1, \cdots, \mu_n, f)$. Thus $\mu_1, \cdots, \mu_n$ are constant so that M^n is an open piece of a generalized cylinder $S^p \times R^{n-p} (p \geq 2)$.

2)rank $S_p \leq 1$. In this case we may assume that

$$\mu_2 = \mu_3 = \cdots = \mu_n = 0. \tag{3.14}$$

Suppose that $\mu_1 = 0$, then M^n is locally a hyperplane. Now suppose that $\mu_1 \neq 0$. Then (3.12) implies

$$b_2 = \cdots = b_n = 0. \tag{3.15}$$

From (3.8), (3.10) and (3.14) we obtain

$$w_1^j(e_1) = 0. \tag{3.16}$$

And (3.7), (3.10), (3.14) and (3.16) imply

$$w_1^k(e_j) = 0. \tag{3.17}$$

Hence e_1 is parallel and $e_1 \wedge G$ is constant so that $\{e_1, G\}$ generates a fixed plane (say R^2), the integral curve $\gamma(s)$ of e_1 lies in R^2 and M is locally a cylindrical hypersurface on the plane curve γ. Note that $\gamma(s)$ also satisfies (1.3) for $\bar{A} = A|_{R^2}$ as follows:

$$SX = (\bar{A}X)^T. \tag{3.18}$$

Now first we give the following classification theorem for plane curves. Note that for a curve $\gamma(s)$ in R^2 the condition (1.3) is equivalent to

$$< AT, T >= \kappa, \tag{3.19}$$

where T is the unit tangent vector, $\{T, G\}$ is a positively oriented frame along γ and κ is the curvature of $\gamma(s)$.

THEOREM 3.1. *A unit speed curve $\gamma(s)$ in R^2 satisfies (1.3) if and only if up to Euclidean motions of R^2, $\gamma(s)$ is an open part of one of the following plane curves:*

(1) *a straight line,*
(2) *a circle,*
(3) *a catenary : $y = \frac{1}{a}\cosh(ax)$, where $a > 0$,*
(4) *a closed curve with equation: $\sqrt{a}\cos(\sqrt{bc}y) = \sqrt{b}\cosh(\sqrt{ac}x)$, where $a > b > 0$, $c = a - b$,*
(5) *a curve with equation : $\sqrt{a}\cosh(\sqrt{bc}y) = \sqrt{b}\sinh(\sqrt{ac}x)$, where $a, b > 0$ and $c = a + b$.*

Proof. Since A is selfadjoint, we can choose Euclidean coordinates on R^2 such that A is of the following form:

$$A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix},$$

where a and b are eigenvalues of A such that $a \geq b$. Let $T(s) = (\cos\theta(s), \sin\theta(s))$. Then $\kappa(s) = \theta'(s)$, so that (3.19) beomes $\theta'(s) = a\cos^2\theta + b\sin^2\theta$.

Case 1 : $a = b = 0$. In this case, γ is a part of a straight line.

Case 2 : $a = b \neq 0$. In this case, γ is a part of a circle.

Case 3: $a > b$. On the interval of s such that $\theta'(s) \neq 0$, s is a function of θ. Let $\beta(\theta) = \gamma(s(\theta))$. Then we have

$$\dot{\beta}(\theta) = \frac{1}{a\cos^2\theta + b\sin^2\theta}(\cos\theta, \sin\theta). \tag{3.20}$$

We consider three subcases.

Case 3.1 : $a > b = 0$ (or equivalently $a = 0 > b$). From (3.20) we obtain

$$\beta(\theta) = \frac{1}{a}\,(\ln\ |\sec\theta + \tan\theta|.\ \sec\theta),$$

which is a catenary.

Case 3.2 : $a > b > 0$ (or equivalently, $0 > a > b$). From (3.20) we obtain for $c = a - b$

$$\beta(\theta) = -\left(\frac{1}{2\sqrt{ac}}\ln\ \left(\frac{\sqrt{a} - \sqrt{c}\sin\theta}{\sqrt{a} + \sqrt{c}\sin\theta}\right), \frac{1}{\sqrt{bc}}\tan^{-1}\left(\frac{\sqrt{c}\cos\theta}{\sqrt{b}}\right)\right),$$

which is of the form (4).

Case 3.3 : $a > 0 > b$. In this case, from (3.20) we have for $c = a - b$

$$\beta(\theta) = -\left(\frac{1}{2\sqrt{ac}}\ln\ \left|\frac{\sqrt{a} - \sqrt{c}\sin\theta}{\sqrt{a} + \sqrt{c}\sin\theta}\right|, \frac{1}{2\sqrt{-bc}}\ln\ \left|\frac{\sqrt{-b} - \sqrt{c}\cos\theta}{\sqrt{-b} + \sqrt{c}\cos\theta}\right|\right),$$

which is of the form (5) if we interchange b by $-b$.

In conclusion, we prove the following theorem.

THEOREM 3.2. *A hypersurface M^n in R^{n+1} satisfies the condition* (1.3) *if and only if M^n is an open part of one of the following hypersurfaces:*

(1) *a hypersphere $S^n(r)$,*
(2) *a generalized cylinder $S^p(r) \times R^{n-p}$ $(p \geq 2)$,*
(3) *a cylinder on a plane curve in Theorem 3.1.*

Proof. We consider the following sets, whose union is dense in M^n. $V_1 =$ int $\{p \in M | \text{rank } S_p \geq 2\}$, $V_2 =$ int $\{p \in M | \text{rank } S_p \leq 1\}$. If $p \in V_1$, then from (3.2), (3.3), (3.6) and (3.13) we have rank $A \geq 3$. If $p \in V_2$, then from (3.2), (3.3), (3.15) and (3.16) we obtain rank $A \leq 2$. Since A is constant, it follows that either $V_1 = M$ or $V_2 = M$. Suppose that $V_1 = M$, then M is an open part of a product $S^p \times R^{n-p}$, $p = 2, \cdots, n$. If $V_2 = M$, then M is locally a cylindrical hypersurface on the plane curve in Theorem 3.1. Since the curves in Theorem 3.1 cannot be pasted to be a smooth curve, we obtain that M^n itself is an open part of a cylindrical hypersurface on the plane curve in Theorem 3.1.

References

1. É. Cartan, *Sur les variétés de courbure constante dans l'éspace euclidien ou non euclidien I jour Bull. Soc. Math. France.*
2. ______, *Sur les variétés de courbure constante dans l'éspace euclidien ou non euclidien II*, Bull. Soc. Math. France **48** (1920), 132-208.
3. B. Y. Chen, F. Dillen, L. Verstraelen and L. Vracken, *Submanifolds of ristricted type*, J. of Geometry **46** (1993), 20-32.
4. H. F. Münzner, *Isoparametrische Hyperflächen in spachären, I*, Math. Ann. **251** (1980), 57-71.
5. J. Park, *Geometric and analytic characterizations of isoparametric submanifolds*, Ph.D Thesis, Brandeis University, 1992.
6. P. J. Ryan, *Homogeneity and some curvature conditions for hypersurfaces*, Tôhoku Math. J. **21** (1969), 363-388.
7. K. Yano and M. Kon, *Generic submanifolds*, Annali Mat. **CXXIII** (1980), 59-92.

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