NUMERICAL SOLUTION FOR NONLINEAR KLEIN–GORDON EQUATION BY COLLOCATION METHOD WITH RESPECT TO SPECTRAL METHOD

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1. Introduction

The nonlinear Klein Gordon equation

\[ \frac{\partial^2 u}{\partial t^2} - \Delta u + V_u(u) = f \]

where $\Delta$ is the Laplacian operator in $\mathbb{R}^d \ (d = 1, 2, 3)$, $V_u(u)$ is the derivative of the "potential function" $V$, and $f$ is a source term independent of the solution $u$, in various areas of mathematical physics. Among the particular cases which are the practical relevance, we take $V_u(u) = |u|^\alpha u$ with $\alpha > 0$ (quantum mechanics), refer to [5].

The convergence of the Galerkin finite element method for second order hyperbolic equations has been studied by many authors: cf. among others Dupont[3], who obtained error estimates for time-discrete and time continuous approximations of linear problems, and Dendy [2], who examined nonlinear problems as well as various modified Galerkin methods. To compute the nonlinear term $V_u(u)$, the product approximation is used by Yves Tourigny [6]. This approximation is a technique which consists of replacing the nonlinear term by its interpolant in the finite-dimensional subspaces. This provides an interesting alternative to numerical quadrature and greatly eases the implementation of the Galerkin method.

In this paper, by the collocation method, when $n = 2$, we get the numerical solutions of (1). We shall show the stability and convergence for using spectral method technic.
To define a collocation method with spectral method techics, we give as many distinct points
\[ x_k \quad k \in J \]  
(a set of indices)
in the domain \( \Omega \) or in its boundary \( \partial \Omega \) as the dimension of the space \( Pol_N(\Omega) \) in which the spectral solution is sought. At the number of these points, located on \( \partial \Omega \) the boundary conditions are imposed. The remaining points are used to enforce the differential equation.

We assume that for any \( k \in J \), there exists a polynomial \( \phi_k \in Pol_N(\Omega) \), necessarily unique, such that
\[ \phi_k(x_m) = \begin{cases} 1 & \text{if } k = m, \\ 0 & \text{if } k \neq m. \end{cases} \]
The \( \phi_k \)'s form a basis for the polynomials of degree \( N \), since \( v(x) = \sum_{k \in J} v(x_k)\phi_k(x) \) for all \( v \in Pol_N(\Omega) \). Let \( J \) be divided into two disjoint subsets \( J_e \) and \( J_b \), such that if \( k \in J_b \), the \( x_k \)'s are on the part \( \partial \Omega \) of the boundary. Moreover, let \( L_N \) be an approximation to the operator \( L \) in which derivatives are taken via collocation at the points \( x_k \)'s. The collocation solution is a polynomial \( u^N \in Pol_N(\Omega) \) which satisfies the equations
\[
\begin{align*}
L_Nu^N(x_k) &= f(x_k) \quad \text{for all } k \in J_e, \\
Bu^N(x_k) &= 0 \quad \text{for all } k \in J_b.
\end{align*}
\]
The unknowns in a collocation method are the values of \( u^N \) at the points \( x_k \), i.e., the coefficients of \( u^N \) with respect to the Lagrange basis. We introduce a bilinear form \( (u,v)_N \) on the space \( Z = C^0(\Omega) \) of the functions continuous up to the boundary of \( \Omega \) by fixing a family of weights \( w_k > 0 \) and setting
\[
(u,v) = \sum_{k \in J} u(x_k)v(x_k)w_k
\]
The existence of the Lagrange basis ensures that \( (u,v)_N \) is an inner product on \( Pol_N(\Omega) \). Consequently, we define a **discrete norm** on \( Pol_N(\Omega) \) as
\[
\|u\|_N = \{(u,u)_N\}^{1/2} \quad \text{for } u \in Pol_N(\Omega)
\]
The basis of the $\phi_k$'s is orthogonal under the discrete inner product. We make the assumption that the nodes $\{x_k\}$ and the weights $\{w_k\}$ are such that

$$(u, v)_N = (u, v) \quad \text{for all } u, v \text{ such that } uv \in Pol_{2N-1}(\Omega).$$

In all the applications, this assumption is fulfilled since the $x_k$'s are the knots of quadrature formulas of Gaussian type.

Let $X_N$ be the space of the polynomials of degree less than or equal to $N$ which satisfy the boundary conditions, i.e.,

$$X_N = \{ v \in Pol_N(\Omega) | Bv(x_k) = 0 \quad \text{for all } k \in J_b \}$$

Then the collocation method is equivalently written as

$$\begin{cases}
    u^N \in X_N \\
    (L_N u^N, \phi_k) = (f, \phi_k)_N
\end{cases} \quad \text{for all } k \in J_e.$$

If $Y_N$ is the space spanned by the $\phi_k$'s with $k \in J_e$, i.e.,

$$Y_N = \{ v \in Pol_N(\Omega) | v(x_k) = 0 \quad \text{for all } k \in J_b \}$$

then can be written as

$$\begin{cases}
    u^N \in X_N \\
    (L_N u^N, v)_N = (f, v)_N \quad \text{for all } v \in Y_N.
\end{cases}$$

2. Stability

Let $\Omega$ be an interval $[-1, 1]$. We would like to approximate the solution of the following problem

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + |u|^{\alpha} u = f \quad \text{in } Q = \Omega \times [0, T]$$

$$\begin{aligned}
    u(., 0) &= u_0 & \text{and} & \left( \frac{\partial u}{\partial t} \right)(., 0) &= u_1 & \text{in } \Omega \\
    u(-1, t) &= u(1, t) = 0 & \text{for } t & \in [0, T]
\end{aligned}$$
where $u_0 \in H_0(\Omega)$, $u_1 \in L(Q)$ and $f \in L(Q)$ are given functions, $T > 0$.

The solution $u^N(x, t)$ of the Legendre Tau approximation of this problem is for all $t > 0$ a polynomial of degree $N$ in $x$, which is zero at $x = \pm 1$ and satisfies the equations,

$$
\int_{-1}^{1} \left[ u_{tt}^{N}(x, t) - u_{xx}^{N}(x, t) + |u^N(x, t)|^{\alpha} u^N(x, t) \right] v(x)\,dx \\
= \int_{-1}^{1} f(x, t)v(x)\,dx \quad t > 0, \quad \text{for all } t \in P_{N-2}
$$

(3)

$$
\int_{-1}^{1} \left[ u^{N}(x, 0) - u_0(x) \right] v(x)\,dx = 0 \\
\int_{-1}^{1} \left[ u_{t}^{N}(x, 0) - u_1(x) \right] v(x)\,dx = 0
$$

Let we set $X_N = \{ u \in P_N | u(-1) = u(1) = 0 \}$, $Y_N = P_{N-2}$ and $(u, v) = \int_{-1}^{1} u(x)v(x)\,dx$. For all $u \in X_N$ we have

$$
- \int_{-1}^{1} u_{xx} P_{N-2} u\,dx = - \int_{-1}^{1} u_{xx}^{N} u\,dx = \int_{-1}^{1} (u_x)^2 \,dx
$$

But, we know that the degree of $|u^N|^{\alpha} u^N$ is greater than $2N - 1$. Here, we shall use the approximation of $|u^N|^{\alpha} u^N$ in (3). We substitute $I_N|u^N|^{\alpha} u^N$ instead of $|u^N|^{\alpha} u^N$ where $I_N : C(\Omega) \to X_N$ is the interpolation operator.

We shall find the approximate solution $u^N \in X_N$ such that

$$
\int_{-1}^{1} \left[ u_{tt}^{N}(x, t) - u_{xx}^{N}(x, t) + I_N|u^N(x, t)|^{\alpha} u^N(x, t) \right] v(x)\,dx \\
= \int_{-1}^{1} f(x, t)v(x)\,dx \quad t > 0, \quad \text{for all } v \in P_{N-2}
$$

(4)

$$
\int_{-1}^{1} \left[ u^{N}(x, 0) - u_0(x) \right] v(x)\,dx = 0 \\
\int_{-1}^{1} \left[ u_{t}^{N}(x, 0) - u_1(x) \right] v(x)\,dx = 0.
$$
Theorem 1. For some $T > 0$,

\[
\left\| P_{N-2}u_t^N(t) \right\|_{L^2(-1,1)}^2 + \left\| u_t^N(t) \right\|_{L^2(-1,1)}^2 + (2/P \beta) \left\| u^N(t) \right\|_{L^p(-1,1)}^p
\leq \left\{ \left\| P_{N-2}u_t^N(0) \right\|_{L^2(-1,1)}^2 + \left\| u_t^N(0) \right\|_{L^2(-1,1)}^2 + (2/P \beta) \left\| u^N(0) \right\|_{L^p(-1,1)}^p \right\} e^{T}
\]

Proof. Take $v = P_{N-2}u_t^N$, from the left hand side first term in (4)

\[
\int_{-1}^{1} u_{tt}(x,t) P_{N-2}u_t^N(x,t) dx
= \int_{-1}^{1} P_{N-2}u_{tt}(x,t) P_{N-2}u_t^N(x,t)dx(1-x^2)dx
= (1/2)\frac{d}{dt} \left\| P_{N-2}u_t^N(t) \right\|_{L^2(-1,1)}^2,
\]

and the second term,

\[- \int_{-1}^{1} u_{xx}^N(x,t) P_{N-2}u_t^N(x,t) dx
= \int_{-1}^{1} u_{xx}^N(x,t) P_{N-2} \frac{d}{dt} u^N(x,t) dx
= \int_{-1}^{1} u_x^N(x,t) \frac{d}{dt} u_x^N(x,t) dx
= (1/2) \frac{d}{dt} \left\| u_x^N(t) \right\|^2.
\]

Now, for $p = \alpha + 2$, refer to [4],

\[
\frac{1}{p} \frac{d}{dt} \left\| u^N(x,t) \right\|_{L^p(-1,1)}^p
= \int_{-1}^{1} |u^N(x,t)|^\alpha u^N(x,t) u_t^N(x,t) dx
= \int_{-1}^{1} |u^N(x,t)|^\alpha u^N(x,t)(1-x^2) P_{N-2}u_t^N(x,t) dx
\]
We can choose the $\beta$ which satisfies $\|u^N - I_N u^N\|_w^2 = \|u^N - P_N u^N\|_w^2 + \|R_N u^N\|_w^2$

$$(1/p) \frac{d}{dt} \|u^N(x, t)\|_{L^p(-1,1)}^p \leq \beta \int_{-1}^{1} I_N \{ |u^N(x, t)|^\alpha u^N(x, t) \} P_{N-2} u^N_t(x, t) dx$$

Therefore, from the equation

$$\int_{-1}^{1} [u^N_{tt}(x, t) - u^N_{xx}(x, t) + I_N |u^N(x, t)|^\alpha u^N(x, t)] P_{N-2} u^N_t(x, t) dx$$

$$= \int_{-1}^{1} f(x, t) P_{N-2} u^N_t(x, t) dx$$

we obtain

$$(1/2) \frac{d}{dt} \|P_{N-2} u^N_t(t)\|_{L^2(-1,1)}^2 + (1/2) \frac{d}{dt} \|u^N_x(t)\|_2^2$$

$$+ (1/P\beta) \frac{d}{dt} \|u^N(t)\|_{L^p(-1,1)}^p \leq \int_{-1}^{1} [u^N_{tt}(x, t) - u^N_{xx}(x, t) + I_N |u^N(x, t)|^\alpha u^N(x, t)] P_{N-2} u^N_t(x, t) dx$$

$$\leq \frac{1}{2} \|f(t)\|_{L^2(-1,1)}^2 + \frac{1}{2} \|P_{N-2} u^N_t(x)\|_{L^2(-1,1)}^2$$

$$\|P_{N-2} u^N_t(t)\|_{L^2(-1,1)}^2 + \|u^N_x(t)\|_2^2 + (2/P\beta) \|u^N(t)\|_{L^p(-1,1)}^p$$

$$\leq \|P_{N-2} u^N_t(0)\|_{L^2(-1,1)}^2 + \|u^N_x(0)\|_2^2 + (2/P\beta) \|u^N(0)\|_{L^p(-1,1)}^p$$

$$+ \int_0^t \|f(s)\|_{L^2(-1,1)}^2 ds + \int_0^t \|P_{N-2} u^N_t(s)\|_2^2 ds$$

Applying Gronwall’s inequality we complete the proof.

This theorem shows the stability of the approximate solution of $u^N$ for

$$0 = \int_{-1}^{1} (u^N(x, 0) - u_0(x)) u^N_{0xx} dx = - \int_{-1}^{1} (u^N_x(x, 0) - u_0_x(x)) u^N_{0x} dx$$
\[
\int_{-1}^{1} u_x^N(x,0)u_{0x}^N dx = \int_{-1}^{1} u_{0x}(x)u_{0x}^N dx \\
\leq c \int_{-1}^{1} u_{0x}(x)u_{0x}(x) dx < c\|u_0\|_{H^1_0(\Omega)}^2
\]

3. Convergence

Let \( R_N \) be a projection operator from a dense subspace \( W \) of \( D_B \) upon \( X_N \), where \( D_B \) is a set which satisfies the boundary condition of (2). For each \( u \in W \), we further require \( R_N u \) to satisfy the exact boundry conditions, i.e.,

\[ R_N : W \to X_N \cap D_B. \]

We define the norm \( \|g\|_{E^*} = \sup_{u \in E, u \neq 0} \frac{(g, u)}{\|u\|_E} \) for all \( g \in E^* \) that is dual of \( E \).

Let \( e(x,t) = u^N(x,t) - R_N u \). We obtain the following theorem.

**Theorem 2.** Assume that \( \|u\|_0 \in H^1(-1,1) \)

\[
\| P_{N-2} e_t(t) \|_{L^2(-1,1)}^2 + \| e_x(t) \|^2 \\
\leq \left\{ \| P_{N-2} e_t(0) \|_{L^2(-1,1)}^2 + \| e_x(0) \|^2 + M^2 T \right\} e^T \\
\leq \left\{ \| P_{N-2} e_t(0) \|_{L^2(-1,1)}^2 + c\|e_0\|_{H^1_0(\Omega)}^2 + M^2 T \right\} e^T
\]

**Proof.** From (3), we have

\[
\int_{-1}^{1} \left[ u_{tt}^N - u_{xx}^N + I_N|u|^N(x,t)|^\alpha u^N(x,t) - f(x,t) \right] v(x) dx = 0 \\
\text{ for all } v \in P_{N-2}
\]

Take \( v = e_t(x,t) \)

\[
0 = \int_{-1}^{1} u_{tt}^N - u_{xx}^N + I_N|u|^N|u|^\alpha u^N - (u_{tt} - u_{xx} + |u|^\alpha u)e_t dx
\]
\begin{align*}
&= \int_{-1}^{1} (u_{tt}^N - R_N u_{tt}^N + R_N u_{tt}^N - u_{tt}) e_t dx \\
&\quad - \int_{-1}^{1} (u_{xx}^N - R_N u_{xx}^N + R_N u_{xx}^N - u_{xx}) e_t dx \\
&\quad + \int_{-1}^{1} (I_N |u^N|^\alpha u^N - |u|^\alpha u) e_t dx
\end{align*}

We get
\begin{equation}
\frac{1}{2} \frac{d}{dt} \|P_{N-2} e_t(t)\|_{L^2(-1,1)}^2 + \frac{1}{2} \frac{d}{dt} \|e_x(t)\|_{L^2(-1,1)}^2
= \int_{-1}^{1} (u_{tt} - R_N u_{tt}^N) e_t + (R_N u_{tt}^N - u_{xx}) e_t + (|u|^\alpha u - I_N |u^N|^\alpha u^N) e_t dx
= \|P_{N-2} (u_{tt} - R_N u_{tt}^N)\|_{L^2(-1,1)}^2 + \|P_{N-2} (R_N u_{tt}^N - u_{xx})\|_{L^2(-1,1)}^2 + \||u|^\alpha u - I_N |u^N|^\alpha u^N\|_{L^2(-1,1)}^2
\end{equation}

We refer to [1]: For each \( v \in H_0^1(-1,1) \)
\begin{align*}
(P_{N-2}(u_{tt} - R_N u_{tt}), v) \\
= (u_{tt} - R_N u_{tt}, v) - (u_{tt} - R_N u_{tt}, v - P_{N-2} v) \\
= ((u_{tt} - R_N u_{tt})_x, (0 - R_N \theta)_x) - (u_{tt} - R_N u_{tt}, v - P_{N-2} v)
\end{align*}

where \( \theta \) is the only function in \( H_0^1(-1,1) \) satisfying \(-\theta_{xx} = v\) then we obtain
\[
\|P_{N-2}(u_{tt} - R_N u_{tt})\|_{E^*} \leq C_{N}^{1-m} \|u_{tt}\|_{H^{m-2}(-1,1)}.
\]

For each \( v \in H_0^1(-1,1) \)
\begin{align*}
(P_{N-2}(u - R_n u)_{xx}, v) \\
= - ((u - R_n u)_x, v_x) - ((u - R_n u)_{xx}, v - P_{N-2} v) \\
= ((u - R_n u)_x, v_x) - (u_{xx} - P_{n-2} u_{xx}, v - P_{n-2} v)
\end{align*}
here we have used the fact that both $P_{N-2}u_{xx}$ and $(R_{n}u)_{xx}$ are orthogonal to $v - P_{N-2}v$. Using the same approximation results as before, we deduce

$$\|P_{N-2}(u - R_{n}u)_{xx}\|_{E^*} \leq CN^{1-m}\|u\|_{H^m(-1,1)}.$$ 

In Legendre approximations, for all $u \in H^m(-1,1)$

$$\|u - I_{N}u\|_{H^{1}(-1,1)} \leq CN^{2l+\frac{1}{2}-m}\|u\|_{H^m(-1,1)} \text{ for } 0 \leq l \leq m \text{ with } m > \frac{1}{2}.$$ 

Assume that $|u|^\alpha u \in H^1(-1,1)$ let $l = 0$. We get

$$\|\|u|^\alpha u - I_{N}|u|^N|u|^N\|_{L^2(-1,1)} = \|u|^\alpha u - I_{n}|u|^\alpha\|_{L^2(-1,1)} \leq CN^{\frac{1}{2}-m}\|u|^\alpha u\|_{H^m(-1,1)}.$$ 

We may assume that $m \geq 2$.

Let $M = CN^{1-m}\|u_{tt}\|_{H^{-2}(-1,1)} + CN^{1-m}\|u\|_{H^m(-1,1)} + CN^{-\frac{1}{2}}\|u|^\alpha\|_{H^{-2}(-1,1)}$ clearly $M \to 0$ as $N \to \infty$. From (5)

$$(1/2)\frac{d}{dt}\|P_{N-2}e_t(t)\|_{L^2_w(-1,1)}^2 + (1/2)\frac{d}{dt}\|e_x(t)\|^2 \leq (1/2)M^2 + (1/2)\|P_{N-2}e_t(t)\|_{L^2_w(-1,1)}^2$$

$$\leq\|P_{N-2}e_t(t)\|_{L^2_w(-1,1)}^2 + \|e_x(t)\|^2 \leq\|P_{N-2}e_t(0)\|_{L^2_w(-1,1)}^2 + \|e_x(0)\|^2$$

$$+ \int_0^t M^2 ds + \int \|P_{N-2}e_t(x, s)\|_{L^2_w(-1,1)}^2 ds$$

We know that $\|e_x(0)\|^2 \leq c\|e_0\|_{H^\frac{1}{2}_\theta(\Omega)}^2$ and applying Gronwall's inequality we conclude the proof.
4. Numerical results

Set $u^N(x,t) = \sum_{i=0}^{N} a_i(t)l_i(x)$ is a $N$-degree Lagrange polynomial with $N + 1$ nodes as $-1 = x_0 < x_1 < x_2 \cdots < x_n = 1$. We substitute $u^N(x,t)$ into (2), we get

$$\frac{d^2 a_i(t)}{dt^2} - l''_1(x_i)a_1(t) + \cdots + l''_N(x_i)a_n(t) + |a^3_i(t)|^\alpha a_i(t) = f(x_i, t)$$
$$i = 0, 1, 2, \cdots, N$$

Applying the boundary condition and the difference equation with

$$\frac{d^2 a_i(t_j)}{dt^2} = \frac{a_i(t_{j+1}) - 2a_i(t_j) + a_i(t_{j-1})}{h^2}$$
$$a_i(t_0) = 0$$
$$a_i(t_1) = hu_1(x_i)$$

where $h$ is a mesh size and $t_j = jh$, we obtain a system of $N - 1$ equations. For one example, let

$$f(x,t) = -2 \sin(\pi x) + |(t - t^2) \sin(\pi x)|^\alpha (t - t^2) \sin(\pi x)$$
$$+ \pi^2(t - t^2) \sin(\pi x)$$
$$\frac{\partial u}{\partial t}(x,0) = \sin(\pi x),$$

then we obtain the numerical solution as table 1.

Practically, this example has exact solution such that $(t-t^2) \sin(\pi x)$. We can calculate errors. These errors are approximately less than $C \left( \frac{1}{N} \right)^{(N-1)} hN(N-1) \left( \frac{N}{2} \right)^{(N-1)}/\left( \frac{N}{2} \right)!$ in that $C \left( \frac{1}{N} \right)^{(N-1)}$ is estimated from $M$ which is in theorem 2., and $N(N-1) \left( \frac{N}{2} \right)^{(N-1)}/\left( \frac{N}{2} \right)!$ is calculated by a second order differentiation of $N$-degree Lagrange polynomial. Briefly, the errors are less than $(\frac{1}{2})^{(N-1)} hN(N-1)/\left( \frac{N}{2} \right)!$ and are independent of $\alpha$. 

\[ K = \left( \frac{1}{2} \right)^{(N-1)} hN(N - 1) / \binom{N}{2} \]

<table>
<thead>
<tr>
<th>numerical</th>
<th>exact</th>
<th>error</th>
<th>( k )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( N = 8 )</td>
<td>( 9.91306E - 3 )</td>
<td>( 9.9E - 3 )</td>
<td>( 1.3096E - 5 )</td>
</tr>
<tr>
<td>( N = 12 )</td>
<td>( 9.900053E - 3 )</td>
<td>( 9.9E - 3 )</td>
<td>( 5.3E - 8 )</td>
</tr>
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Table 1. The numerical estimation of \( u(1/2, 0.01) \),
error and error bound. For time value \( t \)
we get results by 10th iteration.

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