WEAK TYPE INEQUALITY FOR CERTAIN MAXIMAL FUNCTIONS ON SPACES OF HOMOGENEOUS TYPE

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1. Introduction

Classical Fatou's theorem asserts that the non-tangential limit of the Poisson integral of an integrals function on \([-\pi, \pi]\) exists a.e. In [NS] Nagel and Stein generalized this theorem by showing that there are certain approach region \(\Omega \subset R^{n+1}_+\), which are not contained in any non-tangential region, but for which the limit

\[
\lim_{\Omega \ni (x,r) \to (x_0,0)} u(x,r)
\]

exists a.e. on \(R^n\). For the complex case Korányi showed a similar result in [K].

In [Su] Sueiro studied a certain maximal operator \(M_\Omega\) which generalizes the classical Hardy-Littlewood maximal operator to study Poisson-Szego integral on the unit ball in \(C^n\) or on the Siegel half-space by showing that \(M_\Omega\) is of weak type \((1,1)\). In this paper we are going to characterize condition to be \(M_\Omega\) of weak type \((p,q), p,q \geq 1\). In this direction see also [HMS, MW, R, W, We].

2. Preliminaries

Let \((X, d, \mu)\) be a space of homogeneous type, i.e., \(X\) is a topological space, \(\mu\) is a positive Borel measure on \(X\), and \(d\) is a pseudo-metric on \(X\); more precisely, we assume that there are constants \(A\) and \(K\) so that for all \(x, y, z \in X\) and all \(\delta > 0\),

i) \(d(x, y) \geq 0; d(x, y) = 0\) if and only if \(x = y\);
ii) \( d(x, y) = d(y, x); \)

iii) \( d(x, z) \leq K[d(x, y) + d(y, z)]; \)

iv) \( \mu(B(x, 2\delta)) \leq A\mu(B(x, \delta)), \quad \delta > 0, \)

where \( B(x, \delta) = \{ y \in X : d(x, y) < \delta \} \) form a basis for the topology of \( X. \) Property iv) is referred to the doubling property of \( \mu. \) For details, see [CW1].

Suppose that there is a given set \( \Omega_{x_0} \subset X \times (0, \infty) \equiv X^+ \) for each \( x_0 \in X. \) Let \( \Omega \) be the family \( \{ \Omega_{x_0} : x_0 \in X \}. \) We define a maximal function associated with \( \Omega \) as follows. For \( f \in L^1_{loc}(X, d\mu) \) and \( x_0 \in X \) set

\[
\mathcal{M}_\Omega f(x_0) = \sup_{(x,r) \in \Omega_{x_0}} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|d\mu,
\]

where \( |B(x,\delta)| = \mu(B(x,\delta)) \) for simplicity.

For \( (x, r) \in X^+, \) and \( \alpha > 0, \) define

\[
S_{\alpha}(x, r) = \{ x_0 \in X : \Omega_{x_0}(r) \cap B(x, \alpha r) \neq \emptyset \},
\]

where

\[
\Omega_{x_0}(r) = \{ x \in X : (x, r) \in \Omega_{x_0} \}
\]

is the cross section of \( \Omega_{x_0} \) of height \( r > 0. \)

Let \( (u, w) \) be a pair of non-negative measurable functions on \( X. \) Let \( T \) be an operator. If there exists a constant \( C \) so that

\[
u(\{ |Tf| > \lambda \}) \leq C\left\{ \frac{\|f\|_{p,w}}{\lambda} \right\}^q, \quad 1 \leq p \leq \infty, \quad 1 \leq q < \infty,
\]

then \( T \) is said to be of weak type \((p, q)\) with respect to weight \((u, w)\).

For \( q = \infty, \) if \( \|Tf\|_{q,u} \leq C\|f\|_{p,w}, \) then \( T \) is also said to be weak type \((p, q) = (p, \infty)\) with respect to weight \((u, w).\) Denote

\[
u(A) = \int_A \nu d\mu \quad \text{and} \quad \|f\|_{p,w} = \left( \int |f|^p w d\mu \right)^{1/p}
\]
as usual.

A pair of weights \((u, w)\) is of the class \( A_{p,q}(\Omega) \) if

\[
u(S_{\alpha}(x, r)) \left( \int_{B(x,b(\alpha)r)} w^{-\frac{1}{p-1}} d\mu \right)^{q(1-\frac{1}{p})} \leq C(\alpha, K),
\]
if $1 < p < \infty, 1 \leq q < \infty$ and

$$\frac{u(S_\alpha(x, r))}{w(B(x, b(\alpha) r))^q} \leq C(\alpha, K), \quad \text{if} \quad p = 1, 1 \leq q < \infty,$$

for some constants $C(\alpha, K)$ and $b(\alpha) = b(\alpha, K)$.

Note that the constants need not be same at each occurance.

3. Weak type inequality of $M_\Omega$

The following lemma is given in [CW2]. See also [Su] for slightly improved one.

**Lemma 1.** Let $E$ be a bounded subset of $X$. Suppose that $r(x)$ is a positive number for each $x \in E$. Then there is a sequence of disjoint balls \( \{B(x_i, r(x_i))\} \), $x_i \in E$, such that

$$E \subset \bigcup_i B(x_i, 4Kr_i), \quad r_i = r(x_i),$$

where $K$ is the constant in the triangle inequality (iii). Furthermore, every $x \in E$ is contained in some ball $B(x_i, 4Kr_i)$ satisfying $r(x) \leq 2r_i$.

**Theorem 1.** Suppose $M_\Omega$ is of weak type $(p, q)$ with respect to a weight $(u, w)$, $1 < p, q < \infty$, with respect to a pair of weights $(u, w)$. Then $(u, w) \in A_{p, q}(\Omega)$.

**Proof.** Assume that $f$ is nonnegative without loss of generality. Let $M_\Omega$ is of weak type $(p, q)$. Let $x_0 \in S_\alpha(x, r)$. Then there exists $y \in X$ such that

$$y \in \Omega_{x_0}(r) \cap B(x, \alpha r).$$

If $d(y, z) < r$, then

$$d(z, x) \leq K(d(z, y) + d(y, x)) \leq K(\alpha r + r) = rK(\alpha + 1),$$

and thus

$$B(y, r) \subseteq B(x, K(\alpha + 1)r).$$
In the same fashion,

\[ B(x, \delta) \subseteq B(y, K(\delta + \alpha r)) \]

for all \( \delta > 0 \). If we choose \( \delta \) so that \( K(\delta + \alpha r) = r \), then

\[ B(x, b(\alpha) r) \subseteq B(y, r) \]

for some \( b(\alpha) \). In fact, \( b(\alpha) = \frac{1}{K} - \alpha \). Now let \( \chi_A \) be the characteristic function of a set \( A \). Then from (1) it follows that

\[
\mathcal{M}_\Omega(f \chi_{B(x, K(\alpha+1)r)})(x_0) \geq \frac{1}{|B(y, r)|} \int_{B(y, r)} f \chi_{B(x, K(\alpha+1)r)} d\mu \\
\geq \frac{1}{|B(y, r)|} \int_{B(y, r)} f d\mu.
\]

Put

\[ f_{B(y, r)} = \frac{1}{|B(y, r)|} \int_{B(y, r)} f d\mu \]

for simplicity. If

\[ 0 < t < f_{B(y, r)}, \]

then

\[ S_\alpha(x, r) \subseteq \{ \mathcal{M}_\Omega(f \chi_{B(x, K(\alpha+1)r)}) > t \} \]

and so

\[
u(S_\alpha(x, r)) \leq \nu(\{ \mathcal{M}_\Omega(f \chi_{B(x, K(\alpha+1)r)}) > t \}) \\
\leq \frac{C}{t^q} \left( \int_{B(x, K(\alpha+1)r)} f^p w d\mu \right)^{\frac{q}{p}}.
\]

Thus

\[ t^q \nu(S_\alpha(x, r)) \leq C \left( \int_{B(x, K(\alpha+1)r)} f^p w d\mu \right)^{\frac{q}{p}} \]

and therefore

\[ f_{B(y, r)}^q \nu(S_\alpha(x, r)) \leq C \left( \int_{B(x, K(\alpha+1)r)} f^p w d\mu \right)^{\frac{q}{p}}. \]
or
\[
(2) \quad \frac{u(S_\alpha(x,r))}{|B(y,r)|^q} \left( \int_{B(y,r)} f \, d\mu \right)^q \leq C \left( \int_{B(x,K(\alpha+1)r)} f^p w \, d\mu \right)^{\frac{2}{p}}.
\]

Since \( B(y,r) \subset B(x,K(\alpha+1)r) \) by (1), if we replace \( f \) by \( f \chi_{B(y,r)} \) in (2), then
\[
(3) \quad \frac{u(S_\alpha(x,r))}{|B(y,r)|^q} \left( \int_{B(y,r)} f \, d\mu \right)^q \leq C \left( \int_{B(y,r)} f^p w \, d\mu \right)^{\frac{2}{p}}.
\]

Replace again \( f \) by \( f \chi_{B(x,b(\alpha)r)} \) in (3) to obtain
\[
(4) \quad \frac{u(S_\alpha(x,r))}{|B(y,r)|^q} \left( \int_{B(x,b(\alpha)r)} f \, d\mu \right)^q \leq C \left( \int_{B(x,b(\alpha)r)} f^p w \, d\mu \right)^{\frac{2}{p}}.
\]

Since
\[
|B(y,r)| \leq |B(x,K(\alpha+1)r)| \leq C|B(x,r)|
\]
by the doubling property, we have
\[
(5) \quad \frac{u(S_\alpha(x,r))}{|B(x,r)|^q} \left( \int_{B(x,b(\alpha)r)} f \, d\mu \right)^q \leq C \left( \int_{B(x,b(\alpha)r)} f^p w \, d\mu \right)^{\frac{2}{p}}.
\]

Suppose \( p > 1 \). Replacing \( f \) by \( w^{\frac{1}{1-p}} \) so that \( f = f^p w \), we obtain
\[
\frac{u(S_\alpha(x,r))}{|B(x,r)|^q} \left( \int_{B(x,b(\alpha)r)} w^{-\frac{1}{p-1}} \, d\mu \right)^{q(1-\frac{1}{p})} \leq C.
\]

Suppose \( p = 1 \). If \( q \geq 1 \), then set \( f \equiv 1 \) to obtain
\[
\frac{u(S_\alpha(x,r))}{w(B(x,b(\alpha)r))^q} \leq C,
\]
and this completes the proof. ||||
Theorem 2. Assume that $\Omega$ satisfies the following conditions i) and ii):

i) If $x_0 \in X$, $(x, r) \in \Omega_{x_0}$ and $s \geq r$, then $(x, s) \in \Omega_{x_0}$

ii) A weight $(u, w) \in A_{p,q}(\Omega), p, q > 1$.

Then $M_\Omega$ is of weak type $(p, q)$.

Proof. The proof is motivated in the argument in [Su]. Assume $f \geq 0$ without loss of generality. Suppose $(u, w) \in A_{p,q}$.

Let

$$E_\lambda = \left\{ x \in X : \sup_{r > 0} \frac{1}{|B(x, r)|} \int_{B(x, r)} f \, d\mu > \lambda \right\}$$

for any $\lambda > 0$. For each $x \in E_\lambda$ let

$$r(x) = \sup \left\{ r > 0 : \frac{1}{|B(x, r)|} \int_{B(x, r)} f \, d\mu > \lambda \right\}$$

Then following the argument in the proof of theorem 1.5 [Su], we have

\begin{equation}
\{ M_\Omega f > \lambda \} \subset \cup_i S_\alpha \left( x_i, \frac{4K}{\alpha} r_i \right).
\end{equation}

Thus

$$u(\{ M_\Omega f > \lambda \}) \leq \sum_i u \left( S_\alpha \left( x_i, \frac{4K}{\alpha} r_i \right) \right)$$

\begin{equation}
\leq C \sum_i \left| B \left( x_i, \frac{4K}{\alpha} r_i \right) \right|^q \left( \int_{B(x_i, b(\alpha) \frac{4K}{\alpha} r_i)} w^{-\frac{1}{p'-1}} \, d\mu \right)^{-q \left( 1 - \frac{1}{p} \right)}.
\end{equation}

Since

$$b(\alpha) \frac{4K}{\alpha} = \left( \frac{1}{K} - \alpha \right) \frac{4K}{\alpha} = 4 \left( \frac{1}{\alpha} - K \right) \geq 1,$$

we have

\begin{equation}
B \left( x_i, b(\alpha) \frac{4K}{\alpha} r_i \right) \supset B(x_i, r_i).
\end{equation}
Then it follows from (3) and Hölder inequality that

\begin{equation}
\begin{aligned}
\mu(\{M_\Omega f > \lambda\}) &
\leq C \sum_i |B(x_i, r_i)|^q \left( \int_{B(x_i, r_i)} w^{-\frac{1}{p-1}} \, d\mu \right)^{-q(1-\frac{1}{p})} \\
&\leq \frac{C}{\lambda^q} \sum_i \left( \int_{B(x_i, r_i)} f^p w \, d\mu \right)^{\frac{p}{p'}} \left( \int_{B(x_i, r_i)} w^{-\frac{1}{p-1}} \, d\mu \right)^{q\left(\frac{p-1}{p'}\right)-q\left(\frac{p-1}{p}\right)} \\
&\leq \frac{C}{\lambda^q} \left( \int_X f^p w \, d\mu \right)^{\frac{p}{p'}}. \\
&= \frac{C}{\lambda^q} \|f\|_{L^p(w \, d\mu)}^q.
\end{aligned}
\end{equation}

Note that the last inequality in (4) is used by the fact that \( \{B(x_i, r_i)\} \) is a disjoint family. This proves that \( M_\Omega \) is of weak type \((p, q)\) with respect to the weight \((u, w)\).

For the unbounded case a similar argument is valid and this completes the proof. |||||

For \( x_0 \in X \), set

\[ \hat{\Omega}_{x_0} = \{(x, r) \in X^+: (x, s) \in \Omega_{x_0} \text{ for some } s \leq r\} \]

and set

\[ \hat{S}_\alpha(x, r) = \{x_0 \in X: \hat{\Omega}_{x_0}(r) \cap B(x, \alpha r) \neq \emptyset\}. \]

In this context we define a maximal function \( M_{\hat{\Omega}} f \) by

\[ M_{\hat{\Omega}} f(x_0) = \sup_{(x, r) \in \hat{\Omega}_{x_0}} \frac{1}{|B(x, r)|} \int_{B(x, r)} |f| \, d\mu. \]

Also we can define \( \hat{A}_{p, q}(\hat{\Omega}) \) condition as follows. A weight \((u, w)\) is of the class \( A_{p, q}(\hat{\Omega}) \) if

\[ \frac{u(\hat{S}_\alpha(x, r))}{|B(x, r)|^q} \left( \int_{B(x, b(\alpha) r)} w^{-\frac{1}{p-1}} \, d\mu \right)^{q(1-\frac{1}{p})} \leq C(\alpha, K), \]
if $1 < p < \infty$, $1 \leq q < \infty$, and

$$\frac{u(\hat{S}_\alpha(x,r))}{(w(B(x,b(\alpha)r)))^q} \leq C(\alpha,K),$$

if $p = 1, 1 \leq q < \infty$ for some constant $C(\alpha,K)$ and constant $b(\alpha) = b(\alpha,K)$.

**Theorem 3.** If $\mathcal{M}_\Omega$ is of weak type $(p,q)$ with respect to weight $(u,w)$ then $(u,w) \in A_{p,q}(\hat{\Omega})$.

**Proof.** As in the proof of theorem 1, if $x_0 \in \hat{S}_\alpha(x,r)$, then $B(y,r) \subset B(x,K(\alpha+1)r)$ for some $(y,r) \in \hat{\Omega}_{x_0}$. But then $(y,s) \in \Omega_{x_0}$ for $s \leq r$. Hence $B(y,s) \subset B(x,K(\alpha+1)r)$. Also $B(x,b(\alpha)r) \subset B(y,r)$ for some $b(\alpha)$. The remaining part of the proof follows those arguments in the proof of theorem 1.

**Theorem 4.** If $(u,w) \in A_{p,q}(\hat{\Omega})$, then $\mathcal{M}_\Omega$ is of weak type $(p,q)$ with respect to $(u,w)$.

**Proof.** Suppose $(u,w) \in A_{p,q}(\hat{\Omega})$. Since $S_\alpha(x,r) \subset \hat{S}_\alpha(x,r),(u,w)$ satisfies the condition (ii) of theorem 2. Hence $\mathcal{M}_\Omega$ is of weak type $(p,q)$. Since $\mathcal{M}_\Omega f \leq \mathcal{M}_\hat{\Omega} f$ for all $f$, $\mathcal{M}_\Omega$ is of weak type $(p,q)$. This completes the proof.

**References**


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