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GENERALIZED CARLESON INEQUALITY ON SPACES OF HOMOGENEOUS TYPE

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1. Introduction

The purpose of this paper is to generalize the Carleson inequality, which is known to play important roles in harmonic analysis. The result given here is a generalization of Coifmann, Meyer, Stein [CMS]. A similar result is shown by Deng [D].

2. Preliminaries

Let (X, d, μ) be a space of homogeneous type, i.e., X is a topological space, μ is a positive Borel measure on X, and d is a pseudo-metric on X; more precisely, we assume that there are constant A and K so that for all $x, y, z \in X$ and all $\delta > 0$,

i) $d(x,y) \geq 0; d(x,y) = 0$ if and only if $x = y$;
ii) $d(x,y) = d(y,x)$;
iii) $d(x,z) \leq K[d(x,y) + d(y,z)]$,
iv) $\mu(B(x, 2\delta) \leq A_\mu(B(x,\delta))$,

where $B(x,\delta) = \{y \in X : d(x,y) < \delta\}$. We also assume that the collection $\{B(x,\delta)\}$ forms a basis for the topology of X. Property iv) is referred to the doubling property of μ. For the details, see [CW].

Let $X^+ = X \times (0,\infty)$, which is an analogue of a generalized upper-half space over X. Let $\Gamma(x) = \{(y,s) \in X^+ : x \in B(y,s)\}$. This is an analogue of nontangential or conical region. For any set $E \subset X$, the tent over E is the set

$$T(E) = \{(x,r) \in X^+ : B(x,r) \subset E\}.$$

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It is then easy to verify that $T(E) = [\bigcup_{x\notin E}\Gamma(X)]^c$. For a measurable function f defined on X^+, for $x \in X, p \geq 1$ and $\alpha \in \mathbf{R}$, let

$$T_{p,\alpha}(f)(x) = \left[\sup_{x\in B(\delta)} \frac{1}{\mu(B(\delta))}\int_{T(B(\delta))} |f(y,t)|^p \frac{d\mu(y)dt}{t^\alpha}\right]^{\frac{1}{p}},$$

where $B(\delta)$ is a ball with radius $\delta > 0$. Let $\mathcal{M}(f)$ denote the Hardy-Littlewood's maximal function

$$\mathcal{M}(f)(x) = \sup_{x\in B(\delta)} \frac{1}{\mu(B(\delta))}\int_{B(\delta)} |f(y)|d\mu(y).$$

For a positive Borel measure λ on $\overline{X^+} = X \times [0,\infty)$, if there exists a constant $C > 0$ such that $\lambda(T(B(\delta))) \leq C\mu(B(\delta))$ for any $\delta > 0$, then λ is called a Carleson measure. For a measurable function f on X^+, we define a p-area function $A_{p,\alpha}$ as follows:

$$A_{p,\alpha}(f)(x) = \left[\int_{\Gamma(x)} |f(y,t)|^p \frac{d\mu(y)dt}{t^{\alpha+1}}\right]^{\frac{1}{p}}, \quad \text{if} \quad 1 \leq p < \infty, \quad \alpha \in \mathbf{R}$$

and

$$A_{\infty,\alpha}(f) = \sup_{(y,t)\in\Gamma(x)} |f(y,t)|, \quad \text{if} \quad p = \infty.$$

For a nonnegative measurable function f on X^+, if there exists a constant C such that

$$\frac{1}{\mu(B(\delta))}\int_{B(\delta)} f d\mu \leq C \inf_{x\in B(\delta)} f(x)$$

for all $\delta > 0$, then f is said to belong to the class A_1 [M].

3. Generalized Carleson inequality

We begin with lemmas.

LEMMA 1 (COVERING LEMMA OF VITALI-WIENER TYPE). *Let X be a space of homogeneous type. Let $E \subset X$ be a bounded set i.e., E is contained in some ball. Let $r(x)$ be a positive number for each $x \in E$. Then there is a sequence of balls $B(x_i, r(x_i))$, $x_i \in E$, such that the balls $B(x_i, 4Kr(x_i))$ cover E, where K is the constant in the triangle inequality. Furthermore, every $x \in E$ is contained in some ball $B(x_i, 4Kr(x_i))$ satisfying $r(x) \le 2r(x_i)$.*

(See Theorem 1.2 on p.69 of [S])

PROPOSITION 1. *Let $p \ge 1$. Then $T_{p,\alpha}$ is of weak type (p, p).*

Proof. Fix $\lambda > 0$. Set

$$\Omega_\lambda = \{x \in X : T_{p,\alpha}(f)(x) > \lambda\}.$$

Assume first that Ω_λ is bounded. Apply lemma 1 to obtain a sequence of disjoint ball $B(x_i, r_i)$ so that $\Omega_\lambda \subset \bigcup B(x_i, 4Kr_i)$ and

$$\frac{1}{\mu(B(x_i, r_i))} \int_{T(B(x_i,r_i))} |f(y,t)|^p \frac{d\mu(y)dt}{t^\alpha} > \lambda^p.$$

Then from the doubling property of μ it follows that

$$\begin{aligned}
\mu(\Omega_\lambda) &\le \sum \mu(B(x_i, 4Kr_i)) \\
&\le C \sum \mu(B(x_i, r_i)) \\
&\le \frac{C}{\lambda^p} \int_{T(B(x_i,r_i))} |f(y,t)|^p \frac{d\mu(y)dt}{t^\alpha} \\
&\le \frac{C}{\lambda^p} \int_{X^+} |f(y,t)|^p \frac{d\mu(y)dt}{t^\alpha}.
\end{aligned}$$

If Ω_λ is not bounded, fix $a \in X$ and $R > 0$. Then $\Omega_\lambda \cap B(a, R)$ is a bounded set, so we can apply the above argument to obtain

$$\mu(\Omega_\lambda \cap B(a, R)) \le \frac{C}{\lambda^p} \int_{X^+} |f(y,t)|^p \frac{d\mu(y)dt}{t^\alpha}.$$

Letting $R \longrightarrow \infty$ we obtain the same estimate. This completes the proof.

LEMMA 2. *Let (X_1, μ_1) and (X_2, μ_2) be measure spaces and T be a sublinear operator that transforms measurable functions in (X_1, μ_1) into measurable functions in (X_2, μ_2). If T is of weak type (p, q), then*

$$\left[\int_K |Tf(y)|^r d\mu_2\right]^{\frac{1}{r}} \leq C\mu_2(K)^{\frac{1}{r}-\frac{1}{q}}\|f\|_p$$

holds for all $r, 0 < r < q$, and for any subset K of finite measure.

THEOREM 1. *Let $f \in L^1(\frac{d\mu dt}{t^\alpha}), 0 \leq \epsilon < 1$. Then $T_{1,\alpha}(f)^\epsilon \in A_1$ for any $\alpha \in \mathbf{R}$.*

Proof. Fix $B(\delta_0)$ and $x_0 \in B(\delta_0)$. Put

$$\Lambda_1 = \{B(\delta) : x_0 \in B(\delta), \delta \leq \delta_0\}$$

and

$$\Lambda_2 = \{B(\delta) : x_0 \in B(\delta), \delta > \delta_0\}.$$

Then we have

$$\begin{aligned} (1) \qquad T_{1,\alpha}(f)(x_0) \leq & \sup_{B(\delta)\in\Lambda_1} \frac{1}{\mu(B(\delta))}\int_{T(B(\delta))} |f(y,t)|\frac{d\mu(y)dt}{t^\alpha} \\ & + \sup_{B(\delta)\in\Lambda_2} \frac{1}{\mu(B(\delta))}\int_{T(B(\delta))} |f(y,t)|\frac{d\mu(y)dt}{t^\alpha} \\ \equiv & A(x_0) + B(x_0), \end{aligned}$$

where

$$A(x_0) = \sup_{B(\delta)\in\Lambda_1} \frac{1}{\mu(B(\delta))}\int_{T(B(\delta))} |f(y,t)|\frac{d\mu(y)dt}{t^\alpha},$$

and

$$B(x_0) = \sup_{B(\delta)\in\Lambda_2} \frac{1}{\mu(B(\delta))}\int_{T(B(\delta))} |f(y,t)|\frac{d\mu(y)dt}{t^\alpha}.$$

To estimate $B(x_0)$, suppose $B(\delta) \in \Lambda_2$. Since $B(\delta) \cap B(\delta_0) \neq \phi$ and $\delta > \delta_0$, there exists a constant $C \geq 1$ such that $B(C\delta) \supset B(\delta_0)$. Hence

$$\begin{aligned}
&\frac{1}{\mu(B(\delta))} \int_{T(B(\delta))} |f(y,t)| \frac{d\mu(y)dt}{t^\alpha} \\
&\leq \frac{\mu(B(C\delta))}{\mu(B(\delta))} \frac{1}{\mu(B(\delta))} \int_{T(B(\delta))} |f(y,t)| \frac{d\mu(y)dt}{t^\alpha} \\
&\leq C \inf_{x \in B(C\delta)} T_{1,\alpha}(f)(x) \\
&\leq C \inf_{x \in B(\delta_0)} T_{1,\alpha}(f)(x),
\end{aligned}$$

and so

$$B(x_0) \leq C \inf_{x \in B(\delta_0)} T_{1,\alpha}(f)(x). \tag{2}$$

As for $A(x_0)$, note that $B(C\delta_0) \supset B(\delta)$ for some constant C. Indeed, since $B(\delta_0) \cap B(\delta) \neq \phi$ and $\delta \leq \delta_0$, $B(\delta) \subset B(C\delta_0)$ for some constant C by the property of the pseudo-metric. Thus $T(B(\delta)) \subset T(B(C\delta_0))$, and so if we put $f_1(y,t) = f(y,t)\chi_{T(B(C\delta_0))}(y,t)$, then we have

$$\begin{aligned}
A(x_0) &= \sup_{B(\delta)\in\Lambda_2} \frac{1}{\mu(B(\delta))} \int_{T(B(\delta))} |f(y,t)| \frac{d\mu(y)dt}{t^\alpha} \\
&\leq \sup_{B(\delta)\in\Lambda_2} \frac{1}{\mu(B(\delta))} \int_{T(B(\delta))} |f_1(y,t)| \frac{d\mu(y)dt}{t^\alpha} \\
&\leq T_{1,\alpha}(f_1)(x).
\end{aligned} \tag{3}$$

Since $0 \leq \epsilon < 1$,

$$T_{1,\alpha}(f)(x_0)^\epsilon \leq A(x_0)^\epsilon + B(x_0)^\epsilon. \tag{4}$$

Hence it follows from (1),(2),(3), and (4) that

$$\begin{aligned}
&\frac{1}{\mu(B(\delta))} \int_{(B(\delta_0))} T_{1,\alpha}(f)(x_0)^\epsilon d\mu(x_0) \\
&\leq \frac{C}{\mu(B(\delta_0))} \int_{B(\delta_0)} A(x_0)^\epsilon d\mu(x_0) + C \inf_{x\in B(\delta_0)} T_{1,\alpha}(f)(x)^\epsilon \\
&\leq \frac{C}{\mu(B(\delta_0))} \int_{B(\delta_0)} T_{1,\alpha}(f_1)(x)^\epsilon d\mu(x) + C \inf_{x\in B(\delta_0)} T_{1,\alpha}(f)(x)^\epsilon
\end{aligned} \tag{5}$$

By lemma 2, the first term of the right-hand side of (5) is less than

$$C\left[\frac{1}{\mu(B(\delta_0))}\int_{B(C\delta_0)}|f(y,t)|\frac{d\mu(y)dt}{t^\alpha}\right]^\epsilon$$

and so (5) is reduced as

$$\frac{1}{\mu(B(\delta))}\int_{(B(\delta_0))}T_{1,\alpha}(f)(x_0)^\epsilon d\mu(x_0)\le C\inf_{x\in B(\delta_0)}T_{1,\alpha}(f)(x)^\epsilon,$$

which proves theorem.

LEMMA 3(WHITNEY DECOMPOSITION). *Let X be a space of homogeneous type. Let $O\subset X$ be an open set. Then there are positive constants M, C_1, C_2, C_3, and a sequence $\{B(x_i,\delta_i)\}$ of balls such that*

(i) $\bigcup_i B(x_i,\delta_i)=O$
(ii) $B(x_i,C_1\delta_i)\subset O$ and $B(x_i,C_2\delta_i)\cap O^c\neq\phi$,
(iii) *the balls $B(x_i,C_3\delta_i)$ are pairwise disjoint,*
(iv) *no point in O lies in more than M of the ball $B(x_i,\delta_i)$.*

(See [CMS].)

THEOREM 2. *Let $\frac{1}{p}+\frac{1}{q}=1$, $1\le p\le\infty$. Suppose there exist constants $A>0, B>0$ such that $A\delta^n\le\mu(B(x,\delta))\le B\delta^n$ for all $\delta>0$. Then there exists a constant C such that*

$$\int_{X+}f(x,t)v(x,t)\frac{d\mu(y)dt}{t^\alpha}\le C\int_X A_{p,n-1+\alpha}(f)(x)T_{q,\alpha}(v)(x)d\mu(x)$$

for all f, v.

Proof. Assume first that $1\le p<\infty$. For each integer k, set

$$\Omega_k=\{x\in X: A_{p,n-1+\alpha}(f)(x)>2^k\}$$

and

$$\Omega_k=\{x\in X:\mathcal{M}(\chi_{\Omega_k})(x)>\frac{1}{2}\}.$$

Observe that $\Omega_k\supset\Omega_{k+1}$ and $\Omega_k^*\supset\Omega_{k+1}^*$ for each k. Further $\cup_{k=-\infty}^{\infty}$ $T(\Omega_k)$ contains the support of f. By Lemma 3, there exist collections of disjoint balls $B(\delta_{k,j})$ and $B(\delta^*_{k,j})$ such that $\Omega_k \subset \cup_j B(c\delta_{k,j})$ and $\Omega^*_k \subset \cup_j B(c\delta_{k,j})$. Also we can choose a constant c so that $T(\Omega_k) \subset \cup T(B(c\delta_{k,j}))$ and that $T(\Omega^*_k) \subset \cup_j T(B(c\delta^*_{k,j}))$. Put $I_{k,j} = T(B(c\delta_{k,j}))$ and $J_{k,j} = T(B(c\delta^*_{k,j}))$ for simplicity. Then it follows from Hölder's inequality that

$$\begin{aligned}
&\left|\int_{X+} f(y,t)v(y,t)\frac{d\mu(y)dt}{t^\alpha}\right| \\
&\leq \left|\sum_{k=-\infty}^{\infty}\int_{T(\Omega^*_k)\setminus T(\Omega^*_{k+1})} f(y,t)v(y,t)\frac{d\mu(y)dt}{t^\alpha}\right| \\
&\leq \left|\sum_{k=-\infty}^{\infty}\sum_{j=1}^{\infty}\int_{J_{k,j}\setminus\cup_s J_{k+1,s}} f(y,t)v(y,t)\frac{d\mu(y)dt}{t^\alpha}\right| \\
&\leq \sum_{k=-\infty}^{\infty}\sum_{j=1}^{\infty}\left[\int_{J_{k,j}\setminus\cup_s J_{k+1,s}} |f(y,t)|^p\frac{d\mu(y)dt}{t^\alpha}\right]^{\frac{1}{p}} \\
&\quad\left[\int_{J_{k,j}\setminus\cup_s J_{k+1,s}} |v(y,t)|^q\frac{d\mu(y)dt}{t^\alpha}\right]^{\frac{1}{q}}.
\end{aligned}\tag{1}$$

Put

$$A_{k,j} = \int_{J_{k,j}\setminus\cup_s J_{k+1,s}} |t(y,t)|^p\frac{d\mu(y)dt}{t^\alpha}$$

and

$$B_{k,j} = \left[\int_{J_{k,j}\setminus\cup_s J_{k+1,s}} |v(y,t)|^q\frac{d\mu(y)dt}{t^\alpha}\right]^{\frac{1}{q}}.$$

If we show

$$A_{k,j} \leq c\int_{B(c\delta^*_{k,j})\setminus\cup_s B(c\delta_{k+1,s})} A_{p,n-1+\alpha}(f)(x)^p d\mu(x)\tag{2}$$

and

$$B_{k,j} \leq \mu(B(c\delta^*_{k,j}))^{-\frac{1}{p}}\int_{B(c\delta^*_{k,j})} T_{q,\alpha}(v)(x)d\mu(x),\tag{3}$$

then we have

$$
\begin{aligned}
&\left|\int_{X+} f(y,t)v(y,t)\frac{d\mu(y)dt}{t^{\alpha}}\right| \\
&\leq A_{k,j}^{\frac{1}{p}} B_{k,j} \\
&\leq C \sum_{k=-\infty}^{\infty}\sum_{j=1}^{\infty} \mu(B(c\delta^*_{k,j})^{\frac{1}{p}} 2^k \mu B(c\delta^*_{k,j})^{-\frac{1}{p}} \int_{B(c\delta^*_{k,j})} T_{q,\alpha}(v)d\mu \\
&\leq C \sum_{k=-\infty}^{\infty}\sum_{j=1}^{\infty} 2^k \int_{B(c\delta^*_{k,j})} T_{q,\alpha}(v)d\mu \\
&\leq C \sum_{k=-\infty}^{\infty} 2^k \int_{\Omega^*_k} T_{q,\alpha}(v)d\mu.
\end{aligned} \tag{4}
$$

Since $T_{q,\alpha}(v) \in A_1$, it follows from proposition 1 that

$$
\int_X \mathcal{M}(\chi_{\Omega_k})(x)^2 T_{q,\alpha}(v)(x)d\mu(x) \leq C \int_X \chi_{\Omega_k}(x)^2 T_{q,\alpha}(v)(x)d\mu(x),
$$

and so

$$
\begin{aligned}
\int_{\Omega^*_k} T_{q,\alpha}(v)d\mu &\leq 4 \int_X \mathcal{M}(\chi_{\Omega_k})(x)^2 T_{q,\alpha}(v)(x)d\mu(x) \\
&\leq C \int_X \chi_{\Omega_k}(x)^2 T_{q,\alpha}(v)(x)d\mu(x) \\
&= C \int_{\Omega_k} T_{q,\alpha}(v)(x)d\mu(x).
\end{aligned} \tag{5}
$$

Therefore by (5), (4) can be rewritten as

$$
\begin{aligned}
\left|\int_{X+} f(y,t)v(y,t)\frac{d\mu(y)dt}{t^{\alpha}}\right| &\leq C \sum_{k=-\infty}^{\infty} 2^k \int_{\Omega^*_k} T_{q,\alpha}(v)d\mu \\
&\leq C \sum_{k=-\infty}^{\infty} 2^k \int_{\Omega^*_k} T_{q,\alpha}(v)d\mu \\
&= C \int_X A_{p,n-1+\alpha}(f) T_{q,\alpha}(v)d\mu(x).
\end{aligned} \tag{6}
$$

So if we show (2) and (3), then (6) holds for $1 \le p < \infty$.

To see (2), observe that

(7)
$$\begin{aligned}
&\int_{B(c\delta^*_{k,j}\setminus\cup_s B(c\delta_{k+1,s})} A_{p,n-1+\alpha}(f)(x)^p d\mu(x)\\
&= \int_{B(c\delta^*_{k,j}\setminus\cup_s B(c\delta_{k+1,s})} d\mu(x) \int_{\Gamma(x)} |f(y,t)|^p \frac{d\mu(y)dt}{t^{n+\alpha}}\\
&= \int_{[B(c\delta^*_{k,j})\setminus\cup_s B(c\delta_{k+1,s})]\times X+} |f(y,t)|^p \chi_{\Gamma(x)}(y,t)\frac{d\mu(x)d\mu(y)dt}{t^{n+\alpha}}\\
&\ge \int_{J_{k,j}\setminus\cup_s J_{k+1,s}} |f(y,t)|^p \frac{d\mu(y)dt}{t^{n+\alpha}} \int_{B(c\delta^*_{k,j})\setminus\cup_s B(c\delta_{k+1,s})} \chi_{\Gamma(x)}(y,t)d\mu(x).
\end{aligned}$$

For any fixed $(y,t) \in J_{k,j}\setminus \cup_s J_{k+1,s}$, we have

(8)
$$B(y,t)\cap(\Omega^*_{k+1})^c \ne \varphi.$$

In fact, suppose $B(y,t) \subset \Omega^*_{k+1} \equiv \cup_j B(c\delta_{k+1,j}{}^*)$. Then $(y,t) \in T(B(c\delta^*_{k+1,j}))$, for some j, that is, $(y,t) \in J_{k+1,j}$, which leads to a contradiction. Hence (8) holds. So we can choose a point $p \in B(y,t)$ so that $\mathcal{M}(\chi_{\Omega_{k+1}})(x) \le \frac{1}{2}$, which implies

(9)
$$\frac{1}{\mu(B(y,t))}\int_{B(y,t)} \chi_{\Omega_{k+1}}(x)d\mu(x) \le \frac{1}{2},$$

and so by (9), we have

(10)
$$\begin{aligned}
&\frac{1}{\mu(B(y,t))}\int_{B(c\delta^*_{k,j})\setminus\cup_s B(c\delta_{k+1,s})} \chi_{\Gamma(x)}(y,t)d\mu(x)\\
&\ge \frac{1}{\mu(B(y,t))}\int_{B(c\delta^*_{k,j})\setminus\cup_s B(c\delta_{k+1,s})} \chi_{B(y,t)}(x)d\mu(x)\\
&\ge \frac{1}{\mu(B(y,t))}\int_{B(c\delta^*_{k,j})\setminus\cup_s B(c\delta_{k+1,s})} \chi_{B(y,t)}(x)d\mu(x)\\
&= \frac{1}{\mu(B(y,t))}\int_{B(y,t)} [1-\chi_{B(y,t)\cap\cup_s B(c\delta_{k+1,s})}(x)]d\mu(x)\\
&= \frac{1}{\mu(B(y,t))}\int_{B(y,t)} [1-\chi_{B(y,t)\cap\Omega_{k+1}}(x)]d\mu(x)\\
&\ge \frac{1}{2}.
\end{aligned}$$

In the second inequality of (10), we used the fact $B(y,t) \subset B(c\delta^*_{k,j})$. Thus

$$\begin{aligned}\int_{B(c\delta^*_{k,j})\setminus\cup_s B(c\delta_{k+1,s})} \chi_{\Gamma(x)}(y,t)d\mu(x) &\geq C\mu(B(y,t)) \\ &\geq Ct^n,\end{aligned}$$

which implies (2). To estimate (3), note that

$$\begin{aligned}&\int_{B(c\delta^*_{k,j})} T_{q,\alpha}(v)(x)d\mu(x) \\ &\geq \int_{B(c\delta^*_{k,j})} \left[\frac{1}{\mu(B(c\delta^*_{k,j}))}\int_{J_{k,j}} |v(y,t)|^q \frac{d\mu(y)dt}{t^\alpha}\right]^{\frac{1}{q}} d\mu(x) \\ &= \left[\int_{J_{k,j}} |v(y,t)|^q \frac{d\mu(y)dt}{t^\alpha}\right]^{\frac{1}{q}} \mu(B(c\delta_{k,j}{}^*))^{\frac{1}{p}},\end{aligned}$$

This shows (3).

Finally, assume $p = \infty$. Then

$$\begin{aligned}&\left|\int_{X+} f(y,t)v(y,t)\frac{d\mu(y)dt}{t^\alpha}\right| \\ &\leq \sum_{k=-\infty}^{\infty} \int_{T(\Omega_k)\setminus T(\Omega_{k+1})} |f(y,t)v(y,t)|\frac{d\mu(y)dt}{t^\alpha} \\ &\sum_{k=-\infty}^{\infty}\sum_{j=1}^{\infty} 2^{k+1}\int_{I_{k,j}} v(y,t)\frac{d\mu(y)dt}{t^\alpha} \\ &\leq C\sum_{k=-\infty}^{\infty} 2^k \sum_{j=1}^{\infty}\int_{B(c\delta_{k,j})} T_{1,\alpha}(v)(x)d\mu(x) \\ &\leq C\int_X A_\infty(f)(x)T_{1,\alpha}(v)(x)d\mu(x).\end{aligned}$$

This completes the proof.

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