ESTIMATES OF INVARIANT METRICS ON SOME PSEUDOCONVEX DOMAINS IN $\mathbb{C}^n$

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1. Introduction

In this paper we will estimate from above and below the values of the Bergman, Caratheodory and Kobayashi metrics for a vector $X$ at $z$, where $z$ is any point near a given point $z_0$ in the boundary of pseudoconvex domains in $\mathbb{C}^n$. Throughout this paper, $\Omega$ will be a smoothly bounded pseudoconvex domain in $\mathbb{C}^n$ with smooth defining function $r$ and $z_0 \in b\Omega$ is a point of finite type $m$ in the sense of D'Angelo [7], and the Levi-form $\overline{\partial} \partial r(z)$ of $b\Omega$ has $(n-2)$-positive eigenvalues at $z_0$. Note that the type $m$ at $z_0$ is an even integer in this case. We first give the definition of each of the above metrics. Let $X$ be a holomorphic tangent vector at a point $z$ in $\Omega$. Denote the set of holomorphic functions on $\Omega$ by $A(\Omega)$. Then the Bergman metric $B_\Omega(z; X)$, the Caratheodory metric $C_\Omega(z; X)$ and the Kobayashi metric $K_\Omega(z; X)$ are defined by

$$C_\Omega(z; X) = \sup \{|Xf(z)| : f \in A(\Omega), \|f\|_{L^\infty(\Omega)} \leq 1\}$$

$$K_\Omega(z; X) = \inf \{1/r : \exists f : D_r \subset \mathbb{C}^1 \to \mathbb{C}^n \text{ such that } f_*(\frac{\partial}{\partial z}|_0) = X\}$$

$$B_\Omega(z : X) = b_\Omega(z; X)/(K_\Omega(z, \bar{z}))^{\frac{1}{4}},$$

where $D_r$ denotes the disc of radius $r$ in $\mathbb{C}^1$, and

$$K_\Omega(z, \bar{z}) = \sup \{|f(z)|^2 : f \in A(\Omega), \|f\|_{L^2(\Omega)} \leq 1\}$$

$$b_\Omega(z; X) = \sup \{|Xf(z)| : f \in A(\Omega), f(z) = 0, \|f\|_{L^2(\Omega)} \leq 1\}.$$
We may assume that \( \partial r / \partial z_1 \neq 0 \) in a small neighborhood \( U \) of \( z_0 \). After a linear change of coordinates, we can find coordinate functions \( z_1, z_2, \ldots, z_n \) defined on \( U \) such that

\[
L_1 = \frac{\partial}{\partial z_1},
\]

\[
L_j = \frac{\partial}{\partial z_j} + b_j \frac{\partial}{\partial z_1}, L_j r = 0, \quad b_j(z_0) = 0, \; j = 2, \ldots, n,
\]

which form a basis of \( \mathcal{C}T(U) \) and satisfy

\[
\partial \bar{\partial} r(z_0)(L_i, \bar{L}_j) = \delta_{ij}, \quad 2 \leq i, j \leq n - 1,
\]

where \( \delta_{ij} = 1 \) if \( i = j \) and \( \delta_{ij} = 0 \) otherwise. For any integers \( j, k > 0 \), set

\[
\mathcal{L}_{j,k} \partial \bar{\partial} r(z) = \frac{L_n \ldots L_n L_n \ldots L_n \partial \bar{\partial} r(z)(L_n, \bar{L}_n),}{(j-1) \text{times} \; (k-1) \text{times}}
\]

and define

\[
C_l(z) = \max\{|\mathcal{L}_{j,k} \partial \bar{\partial} r(z)| : j + k = l\},
\]

\[
\eta(z, \delta) = \min\{\delta / C_l(z) : l = 2, \ldots, m\}.
\]

Let \( X = b_1 L_1 + b_2 L_2 + \ldots + b_n L_n \) be a holomorphic tangent vector at \( z \) and set

\[
M_m(z; X) = |b_1||r(z)|^{-1} + \sum_{k=2}^{n-1} |b_k||r(z)|^{-1/2}
\]

\[
+ |b_n| \sum_{l=2}^{m} |C_l(z)|^{1/l}|r(z)|^{-1/l}.
\]

Then we can state the main result as follows

**Theorem 1.** Let \( \Omega \) be a smoothly bounded pseudoconvex domain in \( \mathbb{C}^n \) and let \( z_0 \in b\Omega \) be a point of finite type \( m \) in the sense of D’Angelo. Also assume that the Levi-form \( \partial \bar{\partial} r(z) \) of \( b\Omega \) has \( (n-2) \)-positive eigenvalues at \( z_0 \). Then there exist a neighborhood \( U \) about \( z_0 \) and positive constants \( c \) and \( C \) such that for all \( X = b_1 L_1 + \ldots + b_n L_n \) at \( z \in U \cap \Omega \),

\[
cM_m(z; X) \leq B_\Omega(z; X), \quad C_\Omega(z; X), \quad K_\Omega(z; X) \leq CM_m(z; X).
\]
Remark. Because $|C_m(z)| \geq c' > 0$ for all $z \in U \cap \Omega$, (1.2) says, in particular, that

$$B_\Omega(z; X), C_\Omega(z; X), K_\Omega(z; X)$$

$$\geq (|b_1| |r(z)|^{-1} + \sum_{k=2}^{n-1} |b_k| |r(z)|^{-1/2} + |b_n| |r(z)|^{-1/m})$$

for a holomorphic vector field $X = b_1 L_1 + \ldots + b_n L_n$ at $z$.

Several authors found some results about these metrics for some pseudoconvex domains in $\mathbb{C}^n$, but in each case the lower bounds are different from the upper bounds [1,5,8,9,12]. In [2], Catlin got a result similar to above theorem in $\mathbb{C}^2$, and this has motivated the author to investigate the above theorem. To prove the theorem, we must get a complete geometric analysis near $z_0$ and this will be done by using the "maximal plurisubharmonic functions" constructed in [6]. In [10], K.T. Hahn got the following inequalities

(1.4) $C_\Omega(z; X) \leq B_\Omega(z; X), \quad K_\Omega(z; X)$.

Therefore the estimates for the lower bounds of $C_\Omega(z; X)$ will suffice for the lower bounds of $B_\Omega(z; X)$ and $K_\Omega(z; X)$. For upper bounds of $B_\Omega(z; X)$, we will use the following estimates for the Bergman kernel function $K_\Omega(z, \bar{z})$

(1.5) $K_\Omega(z, \bar{z}) \approx \sum_{l=2}^{m} |C_l(z)|^{2/l} |r(z)|^{-n-2/l}$,

which was shown by the author in [6].

Although we are employing some of the methods similar to those of Catlin in $\mathbb{C}^2$-case, where he used estimates for the $\bar{\partial}$-Neumann operator. we will show some technical theorems in detail to clarify the difference between $\mathbb{C}^2$ and $\mathbb{C}^n$ case.
2. Pushing out the boundary and Bumping theorem

For each \( z' \in U \), we take the biholomorphism \( \Phi_{z'}^{-1} : \mathbb{C}^n \to \mathbb{C}^n \) which straightens \( b\Omega \) near \( z_0 \) [6, Proposition 2.2]. That is, \( \Phi_{z'} \) satisfies \( \Phi_{z'}^{-1}(z') = 0, \Phi_{z'}^{-1}(z) = z' \), and

\[
\tau(\Phi_{z'}(\zeta)) = \tau(z') + \Re \zeta + \sum_{\alpha = 2}^{n-1} \sum_{j,k < m/2 \atop j,k > 0} \Re \left( b_{j,k}^{\alpha} \zeta_k \bar{\zeta}_n \zeta_{\alpha} \right) \\
+ \sum_{j+k \leq m \atop j,k > 0} a_{j,k}(z') \zeta_j \bar{\zeta}_n + \sum_{\alpha = 2}^{n-1} |\zeta_{\alpha}|^2 \\
+ \mathcal{O} \left( |\zeta_1||\zeta| + |\zeta''|^2|\zeta| + |z''||\zeta_n|^m + |\zeta_n|^m \right).
\]

(2.1)

Set \( \rho(\zeta) = r \circ \Phi_{z'}(\zeta) \), and set

\[
A_l(z') = \max \{|a_{j,k}(z')|; j+k = l\}, \quad 2 \leq l \leq m,
\]

\[
B_{l'}(z') = \max \{|b_{j,k}^{\alpha}(z')|; j+k = l'\}, \quad 2 \leq \alpha \leq n-1, \quad 2 \leq l' \leq m/2.
\]

(2.2)

For each \( \delta > 0 \), we define \( \tau(z', \delta) \) as follows;

\[
\tau(z', \delta) = \min_{2 \leq l \leq m \atop 2 \leq l' \leq m/2} \left\{ (\delta/A_l(z'))^{1/l}, (\delta^{1/2}/B_{l'}(z'))^{1/l'} \right\}.
\]

(2.3)

Since \( A_m(z_0) \geq c > 0 \), it follows that \( A_m(z') \geq c' > 0 \) for all \( z' \in U \) if \( U \) is sufficiently small. This gives the inequality,

\[
\delta^{1/2} \lesssim \tau(z', \delta) \lesssim \delta^{1/m}, \quad z' \in U.
\]

Remark 2.1. It was shown in [6, section 2] that \( (\delta^{1/2}z''(B_{l'}(z')) \gg \tau(z', \delta) \) whenever \( \delta > 0 \) is sufficiently small. Hence the terms mixed with \( \zeta_n \) and \( \zeta_{\alpha}, \alpha = 2, \ldots, n-1 \), would not be and important ones in (2.1) and (2.3) and hence \( \tau(z', \delta) = \min \{(\delta/A_l(z'))^{1/l}; 2 \leq l \leq m\} \) for sufficiently small \( \delta \).

The definition of \( \tau(z', \delta) \) easily implies that if \( \delta' < \delta'' \), then

\[
(\delta'/\delta'')^{1/2} \tau(z', \delta'') \leq \tau(z', \delta') \leq (\delta'/\delta'')^{1/m} \tau(z', \delta'').
\]
Because we are fixing \( z' \) in this section, we set \( \tau_1 = \delta, \tau_2 = \ldots = \tau_{n-1} = \delta^{1/2}, \tau_n = \tau(z', \delta) = \tau \) and define

\[
R_\delta(z') = \{ \zeta \in \mathbb{C}^n; |\zeta_k| < \tau_k, k = 1, 2, \ldots, n \}, \text{ and}
\]

\[
Q_\delta(z') = \{ \Phi_{z'}(\zeta); \zeta \in R_\delta(z') \}.
\]

Then for \( z \in Q_\delta(z') \), the author showed in [6, Proposition 2.7] that

\[
\tau(z', \delta) \lesssim \eta(z, \delta) \lesssim \tau(z', \delta) \quad \text{and}
\]

\[
\eta(z, \delta) \approx \tau(z, \delta).
\]

For \( \epsilon > 0 \), we let \( \Omega_\epsilon = \{ z : r(z) < \epsilon \} \) and set \( S(\epsilon) = \{ z : -\epsilon < r(z) < \epsilon \} \). In [6, Proposition 3.2], the author proved the following theorem which shows the existence of smooth plurisubharmonic functions on \( \overline{\Omega} \) with "maximal Hessian" near \( b\Omega \).

**Theorem 2.1.** For all small \( \delta > 0 \), there is a plurisubharmonic function \( \lambda_\delta \in C^\infty(\Omega_\delta) \) with the following properties

(i) \( |\lambda_\delta(z)| \leq 1, z \in U \cap \Omega_\delta \).

(ii) For all \( L = \sum_{j=1}^n b_j L_j \) at \( z \in U \cap S(\delta), \)

\[
\partial \bar{\partial} \lambda_\delta(z)(L, \bar{L}) \approx \delta^{-2}|b_1|^2 + \delta^{-1} \sum_{k=2}^{n-1} |b_k|^2 + \tau^{-2}|b_n|^2,
\]

(iii) If \( \Phi_{z'} \) is the map associated with a given \( z' \in U \cap S(\delta) \), then for all \( \zeta \in R_\delta(z') \) with \( |\rho(\zeta)| < \delta \),

\[
|D^\alpha(\lambda_\delta \circ \Phi_{z'})(\zeta)| \lesssim C \alpha \delta^{-\alpha_n} \delta^{-1/2(\alpha_2 + \ldots + \alpha_{n-1})} \tau^{-\alpha_n}
\]

where \( \alpha = (\alpha_1, \ldots, \alpha_n) \).

With this family of functions \( \lambda_\delta \), we shall construct for each \( z' \in U \cap b\Omega \) and each small \( \delta > 0 \), a domain (locally defined in \( U \)) \( \Omega_{z', \delta} \) which contains \( \Omega \) such that the boundary of \( \Omega_{z', \delta} \) is pushed out as far as possible, given the constraints that \( d(z', b\Omega_{z', \delta}) < \delta \) and that \( b\Omega_{z', \delta} \) is pseudoconvex. Since \( z' \) will be fixed in this section, we will work in \( \zeta \)-coordinates defined by \( \Phi_{z'}(\zeta) = z \).
Set \( \rho(\zeta) = r(\Phi_{z'}(\zeta)) \) and set \( U' = \{ \zeta : \Phi_{z'}(\zeta) \in U \} \). For all small \( s \) and \( \delta > 0 \), define

\[
J_\delta(z', \zeta) = \left[ \delta^2 + |\zeta_1|^2 + \sum_{\alpha=2}^{n-1} |\zeta_\alpha|^4 + \sum_{l=2}^{m} A_l(z')^2 |\zeta_n|^{2l} \right]^{1/2}
\]

and

\[
W_{s, \delta}(z') = \{ \zeta \in U' : |\rho(\zeta)| < s J_\delta(\zeta) \}.
\]

**NOTE.** From Remark 2.1, and (2.3), the terms \( B_{l'}(z')^2 |\zeta_n|^{2l'} |\zeta_\alpha|^2 \), \( 2 \leq l' \leq m/2 \) will be absorbed into \( \sum_{l=2}^{m} A_l(z')^2 |\zeta_n|^{2l} \) in the definition of \( J_\delta(z', \zeta) \).

Set \( J_\delta(z', \zeta) = J_\delta(\zeta) \) for the convenience.

**PROPOSITION 2.2.** For each \( z' \in U \cap b\Omega \) and each small \( \delta > 0 \), there exists a small real-valued function \( H_{z', \delta}(\zeta) \) defined in \( W_{s, \delta}(z') \) (where \( s \) is a small constant independent of \( z' \) and \( \delta \) ) such that

- (i) \(-J_\delta(\zeta) \approx H_{z', \delta}(\zeta)\),
- (ii) for any \( L = b_1 L'_1 + b_2 L'_2 + \ldots + b_n L'_n \),

\[
\partial \bar{\partial} H_{z', \delta}(L, \bar{L})(\zeta) \approx J_\delta(\zeta) \left[ \frac{|b_1|^2}{(J_\delta(\zeta))^2} + \sum_{k=2}^{n-1} \frac{|b_k|^2}{(J_\delta(\zeta))^2} + \frac{|b_n|^2}{\tau(z', J_\delta(\zeta))^2} \right],
\]

- (iii) for any \( L = b_1 L'_1 + \ldots + b_n L'_n \) at \( \zeta \),

\[
|LH_{z', \delta}| \lesssim J_\delta(\zeta) \left( \frac{|b_1|}{J_\delta(\zeta)} + \sum_{k=2}^{n-1} \frac{|b_k|}{(J_\delta(\zeta))^{1/2}} + \frac{|b_n|}{\tau(z', J_\delta(\zeta))} \right)
\]

where \( L'_k = (\Phi_{z'}^{-1}) L_k \), \( k = 1, 2, \ldots, n \).

**Proof.** Set \( N_1 = [\log_2(1/\delta)] \). Let \( D_R = \{ \zeta \in \mathbb{C}^n : |\zeta_i| < R, \ i = 1, 2, \ldots, n \} \), and let \( \psi \in C_0^\infty(D_2 - D_{1/4}) \) be a function that satisfies \( \psi(\zeta) = 1 \) for \( \zeta \in D_1 - D_{1/2} \). For all \( k, 1 \leq k < N_1 \), set

\[
\psi_k(\zeta) = \psi \left( 2^k \zeta_1, 2^{k/2} \zeta_2, \ldots, 2^k \zeta_{n-1}, \tau(z', 2^{-k})^{-1} \zeta_n \right),
\]
and for \( k = N_1 \), set
\[
\psi_{N_1}(\zeta) = \phi \left( 2^{N_1} \zeta_1, 2^{N_1/2} \zeta_2, \ldots, 2^{N_1/2} \zeta_{n-1}, \tau(z', 2^{-N_1})^{-1} \zeta_n \right),
\]
where \( \phi \in C^\infty_0(D_2) \) satisfies \( \phi(\zeta) = 1 \) for \( \zeta \in D_1 \). If one combines (2.3), (2.5) and the fact that \( (\delta^{1/2} / B_l(z'))^{1/l'} \gg \tau(z', \delta) \) for \( l' = 2, \ldots, m/2 \), one obtains that
\[
J_\delta(\zeta) \approx 2^{-k}, \quad \zeta \in \text{supp} \, \psi_k.
\]

For each \( \delta > 0 \), set \( \lambda'_\delta = \lambda_\delta \circ \Phi_{z'} \), where \( \lambda_\delta \) is the plurisubharmonic function as in Theorem 2.1. Choose \( N_0 \) so that \( \lambda_{2^{-k}t} \) is well-defined for all \( \zeta \in \text{supp} \, \psi_k \) whenever \( k \geq N_0 \), and set
\[
H_{z', \delta}(\zeta) = \sum_{k=N_0}^{N_1} 2^{-k} \psi_k(\zeta) (\lambda'_{2^{-k}t}(\zeta) - 2).
\]

Then \( H_{z', \delta} \) is well-defined (fixed finite sum independent of \( z' \) and \( \delta \)). From (2.5), (2.7) and from the fact that \( H_{z', \delta}(\zeta) \approx -2^{-k} \) for \( \zeta \in \text{supp} \, \psi_k \), property (i) follows. Also the major part of the Hessian of \( H_{z', \delta} \) will be \( \partial \overline{\partial} \lambda_{2^{-k}t}(\zeta) \) and other error terms will be absorbed into \( \partial \overline{\partial} \lambda_{2^{-k}t}(\zeta) \) for sufficiently small \( t \). This fact together property (i) proves properties (ii) and (iii). \( \Box \)

Set \( \Omega_{z'} = \Phi_{z'}^{-1}(\Omega) \) and set \( \Omega_{z', \epsilon} = \{ \zeta \in \mathbb{C}^n : \rho(\zeta) < \epsilon \} \).

**Proposition 2.3.** For all small \( \delta > 0 \), there exist a function \( g_\delta(\zeta) \) and constants \( b > 0 \) and \( C > 0 \) so that

(i) \( g_\delta \) is defined and plurisubharmonic on \( \Omega_{z', b\delta} \),

(ii) \( \text{supp} \, g_\delta \subset R_{C\delta}(z') \cap \Omega_{z', b\delta} \),

(iii) \( |g_\delta(\zeta)| \leq 1, \zeta \in \Omega_{z', b\delta} \),

(iv) if \( L = b_1 L'_1 + \ldots + b_n L'_n \) at \( \zeta \in R_{b\delta}(z') \), then
\[
\partial \overline{\partial} g_\delta(L, \overline{L})(\zeta) \geq \delta^{-2} |b_1|^2 + \delta^{-1} \sum_{k=1}^{n-1} |k|^2 + \tau(z', \delta)^{-2} |b_n|^2, \text{ and}
\]

(v) \( |D^\alpha g_\delta(\zeta)| \leq C \delta^{-\alpha_1} \delta^{-1/2(\alpha_2 + \ldots + \alpha_{n-1})} \tau^{-\alpha_n}, \zeta \in R_{C\delta}(z') \),
where \( \alpha = (\alpha_1, \ldots, \alpha_n) \).

**Proof.** If we use Theorem 2.1, the proof of this theorem will be very close to that of \( \mathcal{C}^2 \)-case of Proposition 4.2 in [2] and hence it will be omitted here.

With Proposition 2.2, we can prove the following theorem which shows that the boundary of \( \Omega_{\zeta'} \) can be pushed out essentially as far as possible.

**Theorem 2.4.** For each sufficiently small \( \delta > 0 \), there is one parameter family of "maximal pushed-out" pseudoconvex domains \( \{ \Omega_{\zeta', \delta} \}_{\epsilon > 0} \) which contain \( \Omega_{\zeta'} \) near the origin.

**Proof.** Let \( U'_1 \) be a small neighborhood of the origin with \( U'_1 \subset U' = \Phi_{\zeta'}^{-1}(U) \). Then one has \( |dH_{\zeta', \delta}(\zeta)| \lesssim 1 \) for \( \zeta \in W_{s, \delta}(z') \) by the property (iii) of Theorem 2.2. Hence for all small \( \epsilon > 0 \), the function

\[
\rho_{\zeta', \delta}(\zeta) = \rho(\zeta) + \epsilon H_{\zeta', \delta}(\zeta)
\]
satisfies \( \frac{\partial \rho_{\zeta', \delta}}{\partial \xi_1} \neq 0 \) in \( U'_1 \) and therefore form a family of defining functions of hypersurfaces \( \{ \zeta : \rho_{\zeta', \delta}(\zeta) = 0 \} \) in \( W_{s, \delta}(z') \). For \( \zeta' \in b\Omega_{\zeta'} \cap U'_1 \), let \( \zeta'' \) be the unique projection of \( \zeta' \) onto \( \{ \zeta : \rho_{\zeta', \delta}(\zeta) = 0 \} = b\Omega_{\zeta', \delta} \). Suppose \( L'' \rho_{\zeta', \delta}(\zeta'') = 0 \) with \( |L''| = 1 \). If one writes \( L'' = \epsilon L'_1 + s_2 L'_2 + \ldots + s_n L'_n = \epsilon L'_1 + T' \), then \( L'' \rho_{\zeta', \delta}(\zeta'') = 0 \) implies that

\[
-\epsilon L'_1(\rho + \epsilon H_{\zeta', \delta})(\zeta'') + T'(\rho + \epsilon H_{\zeta', \delta})(\zeta'') = \epsilon (L'_1 \rho + \epsilon L'_1 H_{\zeta', \delta})(\zeta'') + \epsilon T' H_{\zeta', \delta}(\zeta'') = 0,
\]

which shows that

\[
|\epsilon| \approx |\epsilon T' H_{\zeta', \delta}(\zeta'')| \\
\lesssim \epsilon J_{\delta}(\zeta'') \left( \sum_{k=2}^{n-1} |s_k| J_{\delta}(\zeta'')^{-1/2} + \tau(z', J_{\delta}(\zeta''))^{-1} |s_n| \right).
\]

Therefore we may assume that \( |T'| \geq \epsilon \) provided that \( |\zeta''| \) is sufficiently small (i.e., \( U' \) is sufficiently small). Because \( \partial \bar{\partial} \rho(T', \bar{T'})(\zeta'') \geq \)
0, one has
\[
\partial \bar{\partial} \rho(L''', \bar{L''})(\zeta''') = \partial \bar{\partial} \rho(T' + \epsilon L_1, \bar{T'} + \epsilon \bar{L}_1)(\zeta''') \\
= \partial \bar{\partial} \rho(T', \bar{T}')(\zeta''') + O(\epsilon) \\
\geq -\epsilon J_{\delta}(\zeta''') \left( \tau(z', J_{\delta}(\zeta'''))^{-1} |s_n| + \sum_{k=2}^{n-1} |s_k| J_{\delta}(\zeta''')^{-1/2} \right).
\]
Since \(|\zeta''' - \zeta'| \leq J_{\delta}(\zeta')\), one sees that \(J_{\delta}(\zeta') \approx J_{\delta}(\zeta''')\). If one combines this fact and \(|T'| \geq 1/2\), and the property (ii) of Proposition 2.2, one gets,
\[
\partial \bar{\partial} \rho^e_{z', \delta}(L''', \bar{L'''}) \\
\geq -C \epsilon J_{\delta}(\zeta') \left( \tau(z', J_{\delta}(\zeta'))^{-1} |s_n| + \sum_{k=2}^{n-1} |s_k| J_{\delta}(\zeta')^{-1/2} \right) \\
+ \epsilon c J_{\delta}(\zeta') \left( J_{\delta}(\zeta')^{-2} |\epsilon|^2 + \sum_{k=2}^{n-1} |s_k|^2 J_{\delta}(\zeta')^{-1} + \tau^{-2}|s_n|^2 \right) \geq 0
\]
provided that \(J_{\delta}(\zeta')\) is sufficiently small (or equally if \(|\zeta'|\) is sufficiently small). This completes the proof. \(\square\)

Now we choose \(\epsilon_0 > 0\) so that
\[
\sup\{\rho(\zeta) : \zeta \in R_{C, \delta}(z') \text{ and } \rho^e_{z'}(\zeta \leq 0) < b \delta, \}
\]
where \(b\) is the small number as in Proposition 2.3. This \(\epsilon_0 > 0\) can be chosen independently of \(z'\) and \(\delta\), and we set \(\rho_{z'}(\zeta) = \rho^e_{z'}(\zeta)\). Then the function \(g_{\delta}(\zeta)\) (as in Proposition 2.3) is well defined on the set \(\{\zeta : \rho_{z'}(\zeta) < 0\} = \Omega_{z', \delta}\). For \(\zeta'\) near 0, define a polydisc \(P_a(\zeta')\) by
\[
P_a(\zeta') = \{\zeta \in \mathbb{C}^n : |\zeta_1 - \zeta'_1| < a J_{\delta}(\zeta'), |\zeta_n - \zeta'_n| < \tau(z', a J_{\delta}(\zeta')), |\zeta_k - \zeta'_k| < (a J_{\delta}(\zeta'))^{1/2}, \ k = 2, \ldots, n - 1\}.
\]
Proposition 2.5. There exist constants $a > 0$ and $d_1 > 0$ (independent of $z', \zeta'$ and $\delta$) so that if $\zeta' \in \Omega_{z'}$ and $|\zeta'| < d_1$, then $\rho_{z'}(\zeta) < 0$ for $\zeta \in P_a(\zeta')$.

Proof. We may assume that $\zeta' \in b\Omega_{z'}$ (this will be the worst case). If $a$ is sufficiently small (independent of $z'$ and $\delta$), then

\begin{equation}
J_\delta(\zeta) \approx J_\delta(\zeta'), \quad \zeta \in P_a(\zeta').
\end{equation}

From the property (i) of Proposition 2.2, and with (2.9), there exists a small constant $c > 0$, such that

\begin{equation}
H_{z', \delta}(\zeta) \leq -cJ_\delta(\zeta'), \quad \zeta \in P_a(\zeta').
\end{equation}

By a simple Taylor's theorem argument, one can show that

\begin{equation}
|\rho(\zeta)| \leq CaJ_\delta(\zeta'), \quad \zeta \in P_a(\zeta').
\end{equation}

Since $\rho_{z'}(\zeta) = \rho(\zeta) + \epsilon_0 H_{z', \delta}(\zeta)$, using (2.10) and (2.11) we have $\rho_{z'}(\zeta) < 0$ if $a$ is chosen so that $a < c\epsilon_0/C$. This completes the proof. \(\square\)

The existence of the following two-sided bumping family of pseudoconvex domains was shown by the author in [4].

Theorem 2.6. Let $\Omega$ be a smoothly bounded pseudoconvex domain and let $z_0 \in b\Omega$ be a point of finite type. Then there is a neighborhood $V$ of $z_0$ and a family of smoothly bounded pseudoconvex domains $\{\Omega_t\}_{-1 \leq t \leq 1}$ satisfying the following properties:

(i) $\Omega_0 = \Omega$,
(ii) $\Omega_{t_1} \subset \Omega_{t_2}$ if $t_1 < t_2$,
(iii) $\{\partial\Omega_t\}_{-1 \leq t \leq 1}$ is a $C^\infty$ family of real hypersurfaces in $\mathbb{C}^n$ and the points of $\partial\Omega_t \cap V$ are finite type,
(iv) $D_t - D_{-t} \subset V$ for all $t$.

Remark 2.2.

(1) Property (iii) means that $\partial\Omega_t \longrightarrow \partial\Omega_{t_0}$ in $C^\infty$-topology as $t$ goes to $t_0$.

(2) There is a neighborhood $V$ of $z_0 \in b\Omega$ such that the types of the points of $V \cap b\Omega$ are bounded and Theorem 2.6 holds on that neighborhood [3].
(3) From the construction of $\Phi_{z'}$ and $\rho_z'(\zeta)$, we can choose $d_1 > 0$ and a neighborhood $U \subset \subset V$ of $z_0$ (independent of $z'$) so that $\rho_z'$ is defined in $\{\zeta : |\zeta| < d_1\}$ and satisfies all the properties in this section for all $z' \in b\Omega \cap U$.

Set $\Omega_{t,z'} = \{\zeta \in \mathbb{C}^n : \Phi_{z'}(\zeta) \in \Omega_t\}$, where $\{\Omega_t\}$ is the family of domains as in Theorem 2.6. Let us denote $J_\delta(\zeta) = J_\delta(\zeta, z')$ to clarify the dependence of $z'$. Set

$$\Omega_{z',\delta} = \{\zeta : |\zeta| < d_1\text{ and } \rho_z'(\zeta) < 0\} \text{ and }$$

$$b\Omega_{z',\delta} = \{\zeta : |\zeta| < d_1\text{ and } \rho_z'(\zeta) = 0\}.$$  

The construction of $\rho_z'$ in this section shows that if $\zeta \in \overline{\Omega}_{z'}$ and if $d_1/2 < |\zeta| < d_1$, then

$$d(\zeta, b\Omega_{z',\delta}) \geq J_\delta(\zeta, z').$$

Since $A_m(z') \gtrsim 1$ for all $z' \in U$, it follows that $J_\delta(\zeta, z') \gtrsim 1$ when $d_1/2 < |\zeta| < d_1$. Therefore there is a constant $c_1 > 0$ so that

$$d(\zeta, b\Omega_{z',\delta}) \geq c_1,$$

for $\zeta \in U \cap b\Omega$ and $d_1/2 < |\zeta| < d_1$. Choose $t = t_0$ sufficiently small so that

$$d(\zeta, b\Omega_{t_0,z'}) < c_1/2 \text{ if } d_1/2 < |\zeta| < d_1.$$

Now define a domain $\tilde{\Omega}_{z',\delta}$ by

$$\tilde{\Omega}_{z',\delta} = \{\zeta \in \Omega_{t_0,z'} : |\zeta| \geq d_1\} \cup \{\Omega_{t_0,z'} \cap \Omega_{z',\delta}\}.$$  

Since pseudoconvexity is a local condition, $\tilde{\Omega}_{z',\delta}$ is a pseudoconvex domain. By combining the properties of $\Omega_{z',\delta}$ and $\Omega_{t_0,z'}$, we obtain

**Proposition 2.7.** For all $z'$ near $z_0$ and all $\delta$, $0 < \delta < \delta_0$, the domain $\tilde{\Omega}_{z',\delta}$ has the following properties;

(i) $\tilde{\Omega}_{z',\delta}$ is a bounded pseudoconvex domain that contains $\Omega_{z'}$.

(ii) the function $g_\delta$ of Proposition 2.3 is defined on $\tilde{\Omega}_{z',\delta}$.

(iii) there is a constant $a > 0$ so that for all $\zeta' \in \Omega_{z'}$ with $|\zeta'| < d_1$, $P_a(\zeta') \subset \tilde{\Omega}_{z',\delta}$.

(iv) in the region $|\zeta| > d_1/2$, the boundaries $b\tilde{\Omega}_{z',\delta}$ are independent of $\delta$ and depend smoothly on $z'$.

(v) in the region $\{\zeta : d_1/2 < |\zeta| < d_1\}$, the boundaries $b\tilde{\Omega}_{z',\delta}$ are of finite type, uniformly in $z'$, and $\delta$. 

3. Metric Estimates

For the lower bounds, it is enough to find lower bounds for $C_{\Omega}(z; X)$ because of (1.4). Assume that $r(z) = -b\delta/2$ and let $z'$ be the projection of $z$ onto $b\Omega$, and $\Phi_{z'}$ be its associated map. Here $b > 0$ is the number as in Proposition 2.3. Set $\zeta^\delta = (-b\delta/2, 0, \ldots, 0) = (\zeta_1^\delta, \zeta_2^\delta, \ldots, \zeta_n^\delta)$. Then by (2.1), (2.2) and (2.3), there is a small constant $c \leq b$ such that the polydisc
\begin{equation}
B = \{ \zeta : |\zeta_1 + b\delta/2| < c\delta, |\zeta_n| < c\tau(z', \delta), |\zeta_k| < c\delta^{1/2}, 2 \leq k \leq n - 1 \},
\end{equation}
is contained in $\Omega_{z'}$ and hence the properties (iv) and (v) of Proposition 2.3 hold on $B$. Let $Y = (\Phi_{z'}^{-1})_* X = b_1 L'_1 + \ldots + b_n L'_n$ be a vector at $\zeta$, where $L'_i = (\Phi_{z'}^{-1})_*$, for $i = 1, 2, \ldots, n$. From the coordinate changes as in Proposition 2.2 in [6], one has
\begin{align}
L'_1 &= \frac{\partial}{\partial \zeta_1}, \\
L'_k &= \sum_{j=2}^{n-1} P_{kj} \lambda_j^{-1/2} \frac{\partial}{\partial \zeta_j} - \left( \frac{\partial \rho}{\partial \zeta_1} \right)^{-1} \\
&\quad \sum_{j=2}^{n-1} P_{kj} \lambda_j^{-1/2} \frac{\partial \rho}{\partial \zeta_j} \frac{\partial}{\partial \zeta_1} \quad 2 \leq k \leq n - 1, \\
L'_n &= \frac{\partial}{\partial \zeta_n} + b(\zeta) \frac{\partial}{\partial \zeta_1},
\end{align}
where $b(\zeta) = -\left( \frac{\partial \rho}{\partial \rho_1} \right)^{-1} \left( \frac{\partial \rho}{\partial \zeta_n} \right)$ and $P = (P_{kj})$ is a unitary matrix, and $\lambda_j$'s are positive eigenvalues of $\partial \bar{\partial} r(z')$, $j = 2, \ldots, n - 1$. We may assume that $\lambda_j \geq c > 0$ on $U$ for $j = 2, \ldots, n - 1$. Set $\tau_1 = \delta$, $\tau_n = \tau(z', \delta)$, $\tau_k = \delta^{1/2}$, $k = 2, \ldots, n - 1$. Let $k_0$ be the minimum number such that
\begin{equation}
|b_{k_0}|^{-1} \tau_{k_0}^{-1} = \max \{|b_k|^{-1} \tau_k^{-1} : k = 1, 2, \ldots, n \}.
\end{equation}
Set $\nu(\zeta) = \delta^{-1}(c + b\delta/2)$ if $k_0 = 1$, $\nu(\zeta) = \tau^{-1} \zeta_n$ if $k_0 = n$, and set $\nu(\zeta) = \delta^{-1/2} \sum_{j=2}^{n-1} P_{k0j} \lambda_j^{1/2} \zeta_j$ for $2 \leq k_0 \leq n - 1$. Since we may assume
that \( c \leq 1 \) and \( \lambda_j \leq 1 \), we have the inequality \( \sup_B |v| \leq 1 \). From the expansion in (2.1), one can see that \( \frac{\partial \phi}{\partial \zeta_j} (\zeta^\delta) = 0, j = 2, \ldots, n \), and hence from (3.2),

\[
(3.4) \quad |Y^* v(\zeta^\delta)| = \max \{|b_j \tau_k^{-1} : k = 1, 2, \ldots, n\},
\]

provided that \( \delta \) is sufficiently small. Set \( \phi(\zeta) = g_\delta(\zeta) + |\zeta|^2 \) and set \( \lambda(\zeta) = \chi(\phi(\zeta)) \), where \( \chi(t) \) is a smooth convex increasing function with \( \chi'(t) \geq 1 \) and \( g_\delta \) is the function as in Proposition 2.3. Using the standard \( \overline{\partial} \)-estimates on \( \tilde{\Omega}_{z', \delta} \) with weight \( e^{-\lambda(\zeta)} \), and from the estimate \( \sum_{i,j=1}^n \frac{\partial^2 \phi}{\partial \zeta_i \partial \zeta_j} (\zeta) t_i \overline{t}_j \geq \sum_{j=1}^n |t_j|^2 \tau_j^{-2} \) for \( \zeta \in B \), one has for any \( g = \sum_{i=1}^n g_i \overline{d \zeta_i} \in D(T^*) \cap D(S) \) with \( Sg = 0 \),

\[
(3.5) \quad \int_{\tilde{\Omega}_{z', \delta} - B} |g|^2 e^{-\lambda} dV + \int_B \sum_{i=1}^n \tau_i^{-2} |g_i|^2 e^{-\lambda} dV \lesssim \|T^* g\|_\lambda^2.
\]

where \( T^* \) and \( S \) are densely defined operators induced from \( \overline{\partial}^* \) and \( \overline{\partial} \). Suppose \( f = \sum_{i=1}^n f_i d \zeta_i \) satisfies \( Sf = 0 \). Then from standard theory of \( \overline{\partial} \) and (3.5), there is \( u \in L^2(\tilde{\Omega}_{z', \delta}, \lambda) \) (=weighted \( L^2 \)-space in \( \tilde{\Omega}_{z', \delta} \)) such that \( \overline{\partial} u = f \) in weak sense) and,

\[
(3.6) \quad \|u\|_\lambda^2 \lesssim \int_{\tilde{\Omega}_{z', \delta} - B} |f|^2 e^{-\lambda} + \int_B \sum_{i=1}^n \tau_i^2 |f_i|^2 e^{-\lambda} dV.
\]

For \( c \geq d > 0 \), set \( B_d = \{ \zeta : |\zeta_i - \zeta^\delta_i| < d \tau_i, i = 1, 2, \ldots, n \} \). Since \( \sum_{i,j=1}^n \frac{\partial^2 \phi}{\partial \zeta_i \partial \zeta_j} (\zeta) t_i \overline{t}_j \geq \sum_{j=1}^n \tau_j^{-2} |t_j|^2 \) on \( B \), there is a small constant \( d > 0 \) (independent of \( \tau_1, \ldots, \tau_n \)) so that

\[
(3.7) \quad \phi(\zeta) \geq Re h(\zeta) + d \sum_{i=1}^n \tau_i^{-2} |\zeta_i - \zeta_i^\delta|^2, \quad \zeta \in B_d,
\]

where

\[
h(\zeta) = 2 \sum_{i=1}^n \frac{\partial \phi}{\partial \zeta_i} (\zeta^\delta)(\zeta_i - \zeta_i^\delta) \sum_{i,j=1}^n \frac{\partial^2 \phi}{\partial \zeta_i \partial \zeta_j} (\zeta^\delta)(\zeta_i - \zeta_i^\delta)(\zeta_j - \zeta_j^\delta).
\]
Let $\psi \in C_0^\infty(U)$, where $U$ is the unit polydisc in $\mathbb{C}^n$ such that $\psi(\zeta) = 1$ if $|\zeta_i| \leq 1/2$, $i = 1, 2, \ldots, n$. From (3.7), we conclude that if

$$
\psi_d(\zeta) = \psi \left( \frac{\zeta_1 + b\delta/2}{d\tau_1}, \frac{\zeta_2}{d\tau_2}, \ldots, \frac{\zeta_n}{d\tau_n} \right),
$$

and if $a = nd^3/8$, then

(3.8) \hspace{1cm} \text{Re } h(\zeta) \leq -a, \text{ for } \zeta \in \{ \zeta : \phi(\zeta) \leq a \} \cap \text{supp } \overline{\partial} \psi_d.

Let $\chi$ be a smooth convex increasing function that satisfies $\chi(t) = 0$ for $t \leq a/2$ and $\chi''(t) > 0$ for $t > a/2$. Now define

$$
\lambda_s(\zeta) = \phi(\zeta) + s^2 \chi(\phi(\zeta))
$$

and set

$$
\alpha_s = \overline{\partial}(\psi_d v e^{sh}) = ve^{sh} \overline{\partial} \psi_d = \sum_{i=1}^n \alpha_{s,i} d\overline{\zeta}_i.
$$

Since $|\overline{\partial} \psi_d| \lesssim \tau_i^{-1}$, it follows that $\alpha_s = \sum \alpha_{s,i} d\overline{\zeta}$ satisfies

(3.9) \hspace{1cm} \int_B \sum_{i=1}^n \tau_i^2 |\alpha_{s,i}|^2 e^{-\lambda_s} dV \lesssim \int_{\text{supp } \overline{\partial} \psi_d} e^{2s \text{Re } h - \phi - s^2 \chi(\phi)} dV.

Suppose $\phi(\zeta) \geq 0$. Then $\chi(\phi(\zeta)) \geq \chi(a) > 0$, so the $s^2$-term is predominant. If $\phi(\zeta) \leq a$ and $\zeta \in \text{supp } \overline{\partial} \psi_d$, then $\text{Re } h(\zeta) \leq -a$ by (3.8). So the integrand in the integral on the right-hand side of (3.9) approaches zero uniformly as $s$ converges to infinity. Hence from (3.6) and (3.9), we conclude that for any $\epsilon_0 > 0$, there exists $s_0 > 0$ (independent of $\tau_1, \ldots, \tau_n$) and a function $u_{s_0}$ so that $\overline{\partial} u_{s_0} = \alpha_{s_0}$ and

(3.10) \hspace{1cm} \int_{\tilde{\Omega}_{s', \delta}} |u_{s_0}|^2 e^{-\lambda_{s_0}} dV \lesssim \int_B \sum_{i=1}^n \tau_i^2 |\alpha_{s_0,i}|^2 e^{-\lambda_{s_0}} dV

\hspace{1cm} \lesssim \int_{\text{supp } \overline{\partial} \psi_d} \epsilon_0 dV \lesssim \epsilon_0 \prod_{i=1}^n \tau_i^2.$
From the property (v) of Proposition 2.3, there is \( \epsilon > 0 \) (independent of \( z' \) and \( \delta \)) so that \( \phi(\zeta) < a/2 \) for all \( z \in B_{e} = \{ \zeta : |\zeta_{i} - \zeta_{i}^{\delta}| < e\tau_{i}, \ i = 1, 2, \ldots, n \} \). Therefore \( \lambda_{s}(\zeta) \) is independent of \( s \) for \( \zeta \in B_{e} \) and hence \( u_{s_{0}} \) is holomorphic on \( B_{e} \) and satisfy

\[
\left| \frac{\partial u_{s_{0}}(\zeta^{\delta})}{\partial z_{k}} \right|^{2} \lesssim \tau_{k}^{-2} \left( \prod_{j=1}^{n} \tau_{j}^{-2} \right) \int_{B_{e}} \left| u_{s_{0}} \right| e^{-\lambda_{s_{0}}} dV \lesssim \tau_{k}^{-2} \left( \prod_{j=1}^{n} \tau_{j}^{-2} \right) \left( \epsilon_{0} \prod_{j=1}^{n} \tau_{j}^{2} \right) = \epsilon_{0} \tau_{k}^{-2},
\]

for \( k = 1, 2, \ldots, n \). Therefore it follows from (3.2) that

\[
|X u_{s_{0}}(\zeta^{\delta})| \lesssim \sqrt{\epsilon_{0}} \sum_{k=1}^{n} |b_{k}| \tau_{k}^{-1} \leq n \sqrt{\epsilon_{0}} \max \{ |b_{k}| \tau_{k}^{-1} : k = 1, 2, \ldots, n \}.
\]

Set \( f = v \psi_{d} e^{s_{0}h} - u_{s_{0}} \). Then \( f \) is holomorphic and from (3.4), it follows that

(3.11) \[
|Y f(\zeta^{\delta})| \gtrsim \max \{ |b_{k}| \tau_{k}^{-1} : k = 1, 2, \ldots, n \},
\]

provided that \( \epsilon_{0} \) is sufficiently small.

Let us assume, for a moment, that \( \sup_{\Omega_{z'}} |f| \leq C \), where \( C \) is independent of \( z' \) and \( \delta \). Then (3.11) and the definition of Caratheodory metric shows that

(3.12) \[
C_{\Omega_{z'}}(Y; \zeta^{\delta}) \geq C_{\Omega_{z'}, s}(Y; \zeta^{\delta}) \gtrsim \max \{ |b_{k}| \tau_{k}^{-1} : k = 1, 2, \ldots, n \}.
\]

On the other hand, the polydisc \( B \) about \( \zeta^{\delta} \) lies in \( \Omega_{z'} \). So one can easily obtain that

(3.13) \[
C_{\Omega_{z'}}(\zeta^{\delta}; Y) \leq C_{B}(\zeta^{\delta}; Y) = \max \{ |b_{k}| \tau_{k}^{-1} : k = 1, 2, \ldots, n \}.
\]

From (1.1), (1.2), (2.3) and (2.4), we have

\[
\max \{ |b_{k}| \tau_{k}^{-1} : k = 1, 2, \ldots, n \} \approx M_{m}(z; X)
\]
and hence from the invariant property of Caratheodory metric, and with (3.12), (3.13), one has

\begin{equation}
(3.14) \quad C_\Omega(z; X) = C_{\Omega_{z,\delta}}(\zeta^\delta; Y) \approx M_m(z; X).
\end{equation}

To show that \( \sup_{\Omega_{z,\delta}} |f| \leq C \), we use the fact that \( f \) is holomorphic in a larger domain \( \hat{\Omega}_{z,\delta} \). Assume \( \zeta \in \overline{\Omega}_{z,\delta} \) and \( |\zeta| < d_1 \). Then from Proposition 2.7, one can see that \( P_a \subset \hat{\Omega}_{z,\delta} \). Since \( |\nu \psi_\delta e^{\psi_\delta} \zeta| \lesssim 1 \) and from the estimate (3.10), it follows that \( \int_{P_a(\zeta)} |f|^2 dV \lesssim \prod_{j=1}^n \tau_j^2 \), and hence

\[
|f(\zeta)| \lesssim (Vol(P_a(\zeta)))^{-1} \int_{P_a(\zeta)} |f|^2 dV \lesssim 1,
\]

because \( Vol(P_a(\zeta)) \gtrsim \prod_{j=1}^n \tau_j^2 \). When \( |\zeta| \geq d_1 \), we use the Kohn's global regularity theory and some cut-off functions as Catlin did in [2]. Therefore we proved that \( \sup_{\Omega_{z,\delta}} |f| \lesssim 1 \) and hence (3.14) has been proved.

To obtain an upper bound for the Bergman metric, we note that \( \Omega_{z,\delta} \) contains the polydisc \( B \) about \( \zeta^\delta \). Thus by elementary estimates, one has for any \( f \in L^2(\Omega_{z,\delta}) \cap A(\Omega_{z,\delta}) \),

\[
\left| \frac{\partial f}{\partial \zeta_k}(\zeta^\delta) \right| \lesssim \tau_k^{-1} \prod_{j=1}^n \tau_j^{-1} \|f\|_{L^2(\Omega_{z,\delta})},
\]

for \( k = 1, 2, \ldots, n \). From (2.1) and (3.2), it follows that the coefficient \( b(\zeta) \) of \( \frac{\partial}{\partial \zeta_i} \) in \( L'_n \) satisfies \( |b(\zeta^\delta)| \lesssim \delta \) and \( |\frac{\partial b}{\partial \zeta_j}(\zeta^\delta)| \lesssim \delta^{1/2} \), for \( j = 2, \ldots, n - 1 \). Therefore, if \( Y = \sum_{k=1}^n b'_{k,k} \) is a vector at \( \zeta^\delta \), then

\begin{equation}
(3.15) \quad b_{\Omega_{z,\delta}}(\zeta^\delta; Y) \lesssim \left( \sum_{k=1}^n |b_k| \tau_k^{-1} \right) \prod_{j=1}^n \tau_j^{-1}.
\end{equation}

In [6], the author showed that

\begin{equation}
(3.16) \quad K_{\Omega_{z,\delta}}(\zeta^\delta, \zeta^\delta) \approx \prod_{j=1}^n \tau_j^{-2}.
\end{equation}
Combining (3.15), (3.16) and from the definition of $B_\Omega(z; X)$, it follows that

$$B_\Omega(z; X) = B_{\Omega_z}(\zeta^\delta; Y) \lesssim \sum_{k=1}^n |b_k|\tau_k^{-1}.$$ 

and hence one has

(3.17) \hspace{1cm} C_{\Omega}(z; X) \approx B_\Omega(z; Y) \approx M_m(z; X).$

To show $K_\Omega(z; X) \approx M_m(z; X)$, we set

$$a_k = -\left(\frac{\partial \rho}{\partial \zeta_1}(\zeta^\delta)\right)^{-1}\sum_{j=2}^{n-1} \bar{P}_{kj}\lambda_j^{-1/2} \frac{\partial \rho}{\partial \rho_j}(\zeta^\delta), \quad k = 2, \ldots, n-1,$$

and set

$$b_0 = -\left(\frac{\partial \rho}{\partial \zeta_1}(\zeta^\delta)\right)^{-1}\left(\frac{\partial \rho}{\partial \zeta_n}(\zeta^\delta)\right).$$

Therefore we have $|a_k|, |b_0| \lesssim \delta$ on $B$. Set

$$R = \min\{d_2c\tau_k|b_k|^{-1} : k = 1, 2, \ldots, n\}.$$

Then

$$f(t) = (-b\delta/2 + (b_1 + \sum_{k=2}^{n-1} a_kb_k + b_nd_0)t, \lambda_2^{-1/2}\sum_{k=2}^{n-1} b_k\bar{P}_{k2}t,$$

$$\ldots, \lambda_{n-1}^{-1/2}\sum_{k=2}^{n-1} b_k\bar{P}_{k,n-1}t, b_nt)$$

defines a map $f : D_R \rightarrow B$ with $f_*(\frac{\partial}{\partial \tau}|_0) = X$ provided that $d_2$ is sufficiently small. Hence

$$K_{\Omega_z}(\zeta^\delta; Y) \leq K_B(\zeta^\delta; Y) \leq R^{-1} \leq \max\{|b_k|(cd_2\tau_k)^{-1} : 1 \leq k \leq n\}$$

$$\lesssim \max\{|b_k|\tau_k^{-1} : k = 1, 2, \ldots, n\}$$

$$\lesssim \sum_{k=1}^n |b_k|\tau_k^{-1} \lesssim C_{\Omega_z}(\zeta^\delta; Y).$$
Again from the invariant property of $K_\Omega(z; X)$ and (1.4), it follows that

\begin{equation}
K_\Omega(z; X) = K_{\Omega_\zeta}(\zeta^\delta; Y) \approx C_\Omega(z; X)
\end{equation}

If one combines (3.17) and (3.18), one will get

$$C_\Omega(z; X) \approx B_\Omega(z; X) \approx K_\Omega(z; X) \approx M_m(z; X),$$

and this proves our main theorem. \qed

Remark 2.3. It seems that Kohn's ideal type and D'Angelo's finite 1-type are same in our case. This will be discussed in a forthcoming article.

References


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