CLASS FUNCTION TABLE
MATRIX OF FINITE GROUPS

Won-Sun Park

1. Introduction

Let \( G \) be a finite group with \( k \) distinct conjugacy classes \( C_1, C_2, \cdots, C_k \) and \( F \) an algebraically closed field such that \( \text{char}(F) \nmid |G| \). We denote by \( \text{Irr}_F(G) \) the set of all irreducible \( F \)-characters of \( G \) and \( \text{Cf}_F(G) \) the set of all class functions of \( G \) into \( F \). Then \( \text{Cf}_F(G) \) is a commutative \( F \)-algebra with an \( F \)-basis \( \text{Irr}_F(G) = \{ \chi_1, \chi_2, \cdots, \chi_k \} \). Thus the map

\[
(\ , \ ) : \text{Cf}_F(G) \times \text{Cf}_F(G) \to F
\]
defined by

\[
(\theta, \eta) = \frac{1}{|G|} \sum_{x \in G} \theta(x) \eta(x^{-1})
\]
is a nondegenerate symmetric bilinear form. For \( \theta \in \text{Cf}_F(G) \), define \( \overline{\theta} : G \to F \) by \( \overline{\theta}(x) = \theta(x^{-1}) \). Then \( \overline{\theta} \in \text{Cf}_F(G) \) and \( \overline{\theta} = \theta \), \( \overline{\theta + \eta} = \overline{\theta} + \overline{\eta}, \overline{\theta \eta} = \overline{\theta} \overline{\eta}, \overline{1_G} = 1_G \) where \( \theta, \eta \in \text{Cf}_F(G) \) and \( 1_G \) is the principal character.

Define \( T : \text{Cf}_F(G) \to \text{End}_F(\text{Cf}_F(G)) \) by

\[
T(\theta)(\eta) = \overline{\theta} \eta
\]
for \( \theta, \eta \in \text{Cf}_F(G) \). Then \( T \) is a faithful representation of \( F \)-algebra \( \text{Cf}_F(G) \).

Let \( M : \text{Cf}_F(G) \to M_k(F) \) be a matrix representation of \( \text{Cf}_F(G) \) afforded by \( T \) relative to the ordered \( F \)-basis \( \text{Irr}_F(G) = \{ \chi_1, \chi_2, \cdots, \chi_k \} \).

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Since \( T(\theta)(\chi_i) = \bar{\theta}_i \chi_i = \sum_{t=1}^{k} (\bar{\theta}_i, \chi_t) \chi_t = \sum_{t=1}^{k} (\theta \chi_t, \chi_i) \chi_t \), we have

\[
M(\theta) = (m_{ij}),
\]

where \( m_{ij} = (\theta \chi_i, \chi_j) \).

For a linear transformation \( f \) of an \( F \)-vector space \( V \), let \([f]_\alpha^\beta \) be a matrix of \( f \) relative to the ordered \( F \)-basis \( \alpha \) and \( \beta \) of \( V \). Then \([g]_\alpha^\beta [f]_\beta^\gamma = [g \circ f]_\gamma^\gamma \). Of course, \( M(\theta) = \left[ T(\theta) \right]_{Irr_F(G)}^{Irr_F(G)} \).

Define \( T^* : Cf_F(G) \to \text{End}_F(Cf_F(G)) \) by

\[
T^*(\theta)(\eta) = \theta \eta.
\]

Then \( T^* \) is a faithful representation of \( Cf_F(G) \). Let \( M^* \) be a matrix representation afforded by \( T^* \) relative to the ordered \( F \)-basis \( Irr_F(G) = \{\chi_1, \chi_2, \cdots, \chi_k\} \). Then since \( T^*(\theta)(\chi_i) = \theta \chi_i = \sum_{t=1}^{k} (\theta \chi_i, \chi_t) \chi_t \), we have

\[
M^*(\theta) = \left[ T^*(\theta) \right]_{Irr_F(G)}^{Irr_F(G)} = M(\theta)^t = M(\bar{\theta}) = \left[ T(\bar{\theta}) \right]_{Irr_F(G)}^{Irr_F(G)}.
\]

Since \( M \) is a monomorphism of \( F \)-algebras, we obtain the following.

**Lemma 1.1.** For \( \theta, \eta \in Cf_F(G) \),

1. \( M(\theta + \eta) = M(\theta) + M(\eta) \), and so \( M(n\theta) = nM(\theta) \) for positive integer \( n \)
2. \( M(\theta \eta) = M(\theta)M(\eta) \), and so \( M(\theta^n) = M(\eta^n) \) for positive integer \( n \)
3. \( M(a\theta) = aM(\theta) \) for \( a \in F \)
4. \( M(1_G) = 1 \) where \( 1_G \) is a principal character and 1 is an identity matrix
5. \( \text{Im} M = \{ M(\theta) \mid \theta \in Cf_F(G) \} \) is a commutative \( F \)-algebra with an \( F \)-basis \( \{ M(\chi_1), M(\chi_2), \cdots, M(\chi_k) \} \).

**Remark.** In the case that \( F \) is the complex field \( \mathbb{C} \), for each \( \mathbb{C} \)-character \( \theta \) we have \( \bar{\theta}(x) = \theta(x^{-1}) = \bar{\theta}(\overline{x}) \). Hence \( M^*(\theta) = \overline{M(\theta)^*} \).
2. The class function table matrix

Let $F$ be an algebraically closed field with $\text{char}(F) = 0$. The class function table of the ordered $F$-basis $\alpha = \{\alpha_1, \alpha_2, \cdots, \alpha_k\}$ of $Cf_F(G)$ is given by the following form

<table>
<thead>
<tr>
<th></th>
<th>$C_1$</th>
<th>$C_2$</th>
<th>$\cdots$</th>
<th>$C_k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_1$</td>
<td>$\alpha_1(C_1)$</td>
<td>$\alpha_1(C_2)$</td>
<td>$\cdots$</td>
<td>$\alpha_1(C_k)$</td>
</tr>
<tr>
<td>$\alpha_2$</td>
<td>$\alpha_2(C_1)$</td>
<td>$\alpha_2(C_2)$</td>
<td>$\cdots$</td>
<td>$\alpha_2(C_k)$</td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
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<td>$\ddots$</td>
<td>$\vdots$</td>
</tr>
<tr>
<td>$\alpha_k$</td>
<td>$\alpha_k(C_1)$</td>
<td>$\alpha_k(C_2)$</td>
<td>$\cdots$</td>
<td>$\alpha_k(C_k)$</td>
</tr>
</tbody>
</table>

**Definition.** The class function table matrix $X_\alpha$ of $G$ with respect to the ordered $F$-basis $\alpha = \{\alpha_1, \alpha_2, \cdots, \alpha_k\}$ of $Cf_F(G)$ is the form

$$X_\alpha = (\alpha_i(C_j))_{k \times k}.$$

The character table matrix $X = (\chi_i(C_j))_{k \times k}$ of $G$ is the class function table matrix with respect to the ordered $F$-basis $\text{Irr}_F(G) = \{\chi_1, \chi_2, \cdots, \chi_k\}$.

Let $$\overline{X} = (\overline{\chi}_i(C_j))_{k \times k}.$$ Then $X$ and $\overline{X}$ are invertible. Define $f_i : G \to F$ by

$$f_i(x_j) = \begin{cases} 1 & (x_j \in C_i) \\ 0 & (x_j \notin C_i). \end{cases}$$

Then $\alpha = \{f_1, f_2, \cdots, f_k\}$ is an $F$-basis of $Cf_F(G)$ and $\theta = \sum_{i=1}^{k} \theta(C_i)f_i$ for $\theta \in Cf_F(G)$.

Define $h_i : G \to F$ by

$$h_i(x_j) = \delta_{ij} \frac{|G|}{|C_i|} (x_j \in C_j).$$

Then $\beta = \{h_1, h_2, \cdots, h_k\}$ is an $F$-basis of $Cf_F(G)$ and $h_i = \sum_{t=1}^{k} \overline{X}_i(C_t)\chi_t$ and $f_i = \frac{|C_i|}{|G|} h_i$. 

Since \( T(\theta)(\chi_i) = \overline{\theta}\chi_i = \sum_{t=1}^{k} (\overline{\theta}\chi_i, \chi_t)\chi_t = \sum_{t=1}^{k} (\theta\chi_t, \chi_i)\chi_t \), we have

\[
M(\theta) = (m_{ij}),
\]

where \( m_{ij} = (\theta\chi_i, \chi_j) \).

For a linear transformation \( f \) of an \( F \)-vector space \( V \), let \([f]_\alpha^\beta \) be a matrix of \( f \) relative to the ordered \( F \)-basis \( \alpha \) and \( \beta \) of \( V \). Then \([g]_\alpha^\gamma \cdot [f]_\gamma^\beta = [g \circ f]_\alpha^\beta \). Of course, \( M(\theta) = [T(\theta)]_{\text{Irr}_F(G)}^{\text{Irr}_F(G)} \).

Define \( T^* : C\text{f}_F(G) \to \text{End}_F(C_f F(G)) \) by

\[
T^*(\theta)(\eta) = \theta\eta.
\]

Then \( T^* \) is a faithful representation of \( C\text{f}_F(G) \). Let \( M^* \) be a matrix representation afforded by \( T^* \) relative to the ordered \( F \)-basis \( \text{Irr}_F(G) = \{\chi_1, \chi_2, \cdots, \chi_k\} \). Then since \( T^*(\theta)(\chi_i) = \theta\chi_i = \sum_{t=1}^{k} (\theta\chi_i, \chi_t)\chi_t \), we have

\[
M^*(\theta) = [T^*(\theta)]_{\text{Irr}_F(G)}^{\text{Irr}_F(G)} = M(\theta)^t = M(\overline{\theta}) = [T(\overline{\theta})]_{\text{Irr}_F(G)}^{\text{Irr}_F(G)}.
\]

Since \( M \) is a monomorphism of \( F \)-algebras, we obtain the following.

**Lemma 1.1.** For \( \theta, \eta \in C\text{f}_F(G) \),

1. \( M(\theta + \eta) = M(\theta) + M(\eta) \), and so \( M(n\theta) = nM(\theta) \) for positive integer \( n \)
2. \( M(\theta\eta) = M(\theta)M(\eta) \), and so \( M(\theta^n) = M(\eta)^n \) for positive integer \( n \)
3. \( M(a\theta) = aM(\theta) \) for \( a \in F \)
4. \( M(1_G) = 1 \) where \( 1_G \) is a principal character and \( 1 \) is an identity matrix
5. \( \text{Im}M = \{M(\theta) \mid \theta \in C\text{f}_F(G)\} \) is a commutative \( F \)-algebra with an \( F \)-basis \( \{M(\chi_1), M(\chi_2), \cdots, M(\chi_k)\} \).

**Remark.** In the case that \( F \) is the complex field \( \mathbb{C} \), for each \( \mathbb{C} \)-character \( \theta \) we have \( \overline{\theta}(x) = \theta(x^{-1}) = \overline{\theta(x)} \). Hence \( M^*(\theta) = \overline{M(\theta)^*} \).
it follows that
\[ \overline{X}^{-1}M(\overline{\theta})\overline{X} = \text{diag}(\theta(C_1), \theta(C_2), \cdots, \theta(C_k)) \] and
\[ \overline{X}^{-1}M(\theta)\overline{X} = \text{diag}(\overline{\theta}(C_1), \overline{\theta}(C_2), \cdots, \overline{\theta}(C_k)). \]

Thus we have the following theorem.

**Theorem 2.1.** Let \( M \) be a matrix representation of \( \text{Cf}_F(G) \) afforded by the representation \( T : \text{Cf}_F(G) \to \text{End}_F(\text{Cf}_F(G)) \) defined by \( T(\theta)(\eta) = \overline{\theta}\eta \) for \( \theta, \eta \in \text{Cf}_F(G) \). Then
\[ X^{-1}M(\theta)X = \overline{X}^{-1}M(\overline{\theta})\overline{X} = \text{diag}(\theta(C_1), \theta(C_2), \cdots, \theta(C_k)) \]
and
\[ X^{-1}M(\overline{\theta})X = \overline{X}^{-1}M(\theta)\overline{X} = \text{diag}(\overline{\theta}(C_1), \overline{\theta}(C_2), \cdots, \overline{\theta}(C_k)). \]

**Corollary 2.2.** Let \( M \) be a matrix representation of \( \text{Cf}_F(G) \) afforded by the representation \( T : \text{Cf}_F(G) \to \text{End}_F(\text{Cf}_F(G)) \) defined by \( T(\theta)(\eta) = \overline{\theta}\eta \) for \( \theta, \eta \in \text{Cf}_F(G) \). Let \( D = \frac{1}{|G|}\text{diag}(|C_1|, |C_2|, \cdots, |C_k|) \). Then \( XD\overline{X}^t = I \).

**Proof.** Since \( \chi_j = \sum_{i=1}^k \chi_j(C_i)f_i \), we have \( [T(1_G)]_{\text{Ir}_F(G)}^\alpha = X^t \).

From \( \{[T(1_G)]_{\text{Ir}_F(G)}^\alpha[T(1_G)]_{\text{Ir}_F(G)}^\beta[T(1_G)]_{\text{Ir}_F(G)}^\alpha\}^t = [T(1_G)]_{\text{Ir}_F(G)}^\alpha \), it follows that \( XD\overline{X}^t = I \).

In Theorem 2.1, we have
\[ \{\theta(C_1), \theta(C_2), \cdots, \theta(C_k)\} = \{\overline{\theta}(C_1), \overline{\theta}(C_2), \cdots, \overline{\theta}(C_k)\}. \]
Hence, in Theorem 2.1, \( \theta(C_1), \theta(C_2), \cdots, \theta(C_k) \) are eigenvalues of both \( M(\theta) \) and \( M(\overline{\theta}) \), and so the polynomial \( \prod_{i=1}^m (x - \theta(C_i)) \) is the characteristic polynomial of both \( M(\theta) \) and \( M(\overline{\theta}) \).

Let \( D_k \) be the set of all \( k \times k \) diagonal matrices over \( F \) and \( XD_kX^{-1} = \{X \Delta X^{-1} | \Delta \in D_k\} \). Then \( XD_kX^{-1} \) is an \( F \)-vector space of a dimension \( k \). For \( M(\theta) \in \text{Im}M \), we have \( M(\theta) \in XD_kX^{-1} \) by Theorem 2.1. Hence \( \text{Im}M \subseteq XD_kX^{-1} \). Since \( \text{Im}M \) is an \( F \)-vector space of dimension \( k \), it follows that \( \text{Im}M = XD_kX^{-1} \).
For conjugacy classes \( C_1, C_2, \cdots, C_k \) of \( G \), there is a permutation \( \sigma \) such that \( \sigma(i) = j \) if \( C_i^{-1} = C_j \). Of course, \( \nu = \{h_{\sigma(1)}, h_{\sigma(2)}, \cdots, h_{\sigma(k)}\} \) is a basis of \( CF_F(G) \) and \( h_{\sigma(i)} = \sum_{i=1}^{k} \chi_i(C_i)\chi_i \).

Therefore,

\[
[T(1_G)]^\beta_\alpha = \frac{1}{|G|} \text{diag}(|C_1|, |C_2|, \cdots, |C_k|),
\]

\[
[T(1_G)]^{Irr_F(G)}_\nu = X^{-1} [T(1_G)]^{Irr_F(G)}_\beta = \overline{X}.
\]

Of course, \( [T(1_G)]^{Irr_F(G)}_\nu = X^{-1} \) and \( [T(1_G)]^{Irr_F(G)}_\nu = \overline{X}^{-1} \). And for any ordered \( F \)-basis \( \Gamma \) of \( CF_F(G) \), we have \( [T(1_G)]^\Gamma_\Gamma = I = M(1_G) \).

From

\[
T(\theta)(f_i) = \overline{\theta}f_i = \overline{\theta}(C_i)f_i, \quad T(\overline{\theta})f_i = \theta(C_i)f_i,
\]

\[
T(\theta)(h_i) = \overline{\theta}(C_i)h_i, \quad T(\overline{\theta})h_i = \theta(C_i)h_i,
\]

\[
T(\theta)(h_{\sigma(i)}) = \overline{\theta}(C_{\sigma(i)})h_{\sigma(i)} = \theta(C_i)h_{\sigma(i)} \quad \text{and} \quad T(\overline{\theta})(h_i) = \theta(C_i)h_i,
\]

it follows that

\[
[T(\theta)]^\alpha_\alpha = [T(\theta)]^\beta_\beta = \text{diag}(\overline{\theta}(C_1), \overline{\theta}(C_2), \cdots, \overline{\theta}(C_k)) \quad \text{and}
\]

\[
[T(\theta)]^\nu_\nu = \text{diag}(\theta(C_1), \theta(C_2), \cdots, \theta(C_k)) = [T(\overline{\theta})]^\alpha_\alpha = [T(\theta)]^\beta_\beta.
\]

Since

\[
[T(1_G)]^{Irr_F(G)}_{Irr_F(G)}[T(\theta)]^{Irr_F(G)}_{Irr_F(G)}[T(1_G)]^{Irr_F(G)}_{Irr_F(G)} = [T(\theta)]^\nu_\nu \quad \text{and}
\]

\[
[T(1_G)]^{Irr_F(G)}_{Irr_F(G)}[T(\overline{\theta})]^{Irr_F(G)}_{Irr_F(G)}[T(1_G)]^{Irr_F(G)}_{Irr_F(G)} = [T(\overline{\theta})]^\nu_\nu,
\]

we have

\[
X^{-1}M(\theta)X = \text{diag}(\theta(C_1), \theta(C_2), \cdots, \theta(C_k)) \quad \text{and}
\]

\[
X^{-1}M(\overline{\theta})X = \text{diag}(\overline{\theta}(C_1), \overline{\theta}(C_2), \cdots, \overline{\theta}(C_k)).
\]

From

\[
[T(1_G)]^{Irr_F(G)}_{Irr_F(G)}[T(\overline{\theta})]^{Irr_F(G)}_{Irr_F(G)}[T(1_G)]^{Irr_F(G)}_{Irr_F(G)} = [T(\overline{\theta})]^\beta_\beta \quad \text{and}
\]

\[
[T(1_G)]^{Irr_F(G)}_{Irr_F(G)}[T(\theta)]^{Irr_F(G)}_{Irr_F(G)}[T(1_G)]^{Irr_F(G)}_{Irr_F(G)} = [T(\theta)]^\beta_\beta,
\]

\[
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\]
\[ \cdots, + a_0 \text{ is the minimal polynomial of } T^*(\theta) \text{ iff } q(x) \text{ is the minimal polynomial of } \theta \in Cf_F(G). \]

Let \( q(x) = x^r + a_{r-1}x^{r-1} + \cdots, + a_0 \) be the minimal polynomial of \( \theta \in Cf_F(G). \) If \( b_01_G + b_1\theta + \cdots, + b_{r-1}\theta^{r-1} = 0, \) then \( f(x) = b_0 + b_1x + \cdots, + b_{r-1}x^{r-1} \) is a polynomial having \( \theta \) as a root. Therefore, \( q(x)|f(x). \) Since \( \deg f(x) < \deg q(x), \) this yields that \( f(x) = 0. \)

That is, \( b_0 = b_1 = \cdots, = b_{r-1} = 0. \) Hence \( \{1_G, \theta^1, \theta^2, \cdots, \theta^{r-1}\} \) is independent. For any \( b_01_G + b_1\theta + \cdots, + b_m\theta^m \in Cf_F(G), \) let \( g(x) = b_0 + b_1x + \cdots, + b_mx^m. \) Then \( g(x) = q(x)h(x) + s(x) \) with \( s(x) = c_0 + c_1x + \cdots, + c_tx^t, t \leq r - 1 \) by the division algorithm. And we have

\[
\begin{align*}
    b_01_G + b_1\theta + \cdots, + b_m\theta^m &= g(\theta) = s(\theta) \\
    &= c_01_G + c_1\theta + \cdots, + c_t\theta^t.
\end{align*}
\]

Therefore, \( Cf_F(G) \) is generated by \( \{1_G, \theta^1, \theta^2, \cdots, \theta^{r-1}\}. \)

Hence \( \{1_G, \theta^1, \theta^2, \cdots, \theta^{r-1}\} \) is an \( F \)-basis of \( Cf_F(G). \)

Since \( \dim_FCf_F(G) = k, \) we have \( r = k. \) Therefore the minimal polynomial \( q(x) = x^k + a_{k-1}x^{k-1} + \cdots, + a_0 \) of \( \theta \) is the characteristic polynomial of \( T^*(\theta). \) Hence \( \theta \) takes exactly \( k \) distinct values.

References


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**Theorem 2.3.** Let \( \alpha \) and \( \beta \) be any ordered \( F \)-basis of \( \text{Cf}_F(G) \). Then

\[
[T(\theta)]^\alpha_{\alpha} = ((X_{\alpha}X_{\beta}^{-1})^t)^{-1}[T(\theta)]^\beta_{\beta}(X_{\alpha}X_{\beta}^{-1})^t.
\]

**Proof.** Let \( \alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_k\} \) and \( \beta = \{\beta_1, \beta_2, \ldots, \beta_k\} \). Let \( \alpha_i = \sum_{i=1}^{k} a_{it}\beta_t \). Then

\[
[T(1_G)]^\beta_{\alpha} = \begin{pmatrix}
a_{11} & a_{21} & \ldots & a_{k1} \\
a_{21} & a_{22} & \ldots & a_{k2} \\
\ldots & \ldots & \ldots & \ldots \\
a_{1k} & a_{2k} & \ldots & a_{kk}
\end{pmatrix}
\]

Therefore, \( X_{\alpha} = \{[T(1_G)]^\beta_{\alpha}\}^t X_{\beta} \). Of course, \( X = \{[T(1_G)]^\beta_{\alpha}\}^t X_{\alpha} \) for any ordered \( F \)-basis \( \alpha \). Since \( X \) and \( [T(1_G)]^\alpha_{\alpha} \) are invertible, \( X_{\alpha} \) is invertible. Thus

\[
X^{-1}M(\theta)^tX = X_{\alpha}^{-1}\{[T(1_G)]^\beta_{\alpha}\}^t \{[T(\theta)]^\beta_{\alpha}\}^t \{[T(1_G)]^\alpha_{\alpha}\}^t X_{\alpha} = X_{\alpha}^{-1}\{[T(\theta)]^\alpha_{\alpha}\}^t X_{\alpha}.
\]

Since \( X^{-1}M(\theta)^tX = X_{\beta}^{-1}\{[T(\theta)]^\beta_{\beta}\}^t X_{\beta} \) for ordered \( F \)-basis \( \beta \),

\[
X_{\alpha}^{-1}\{[T(\theta)]^\beta_{\alpha}\}^t X_{\alpha} = X^{-1}M(\theta)^tX = X_{\beta}^{-1}\{[T(\theta)]^\beta_{\beta}\}^t X_{\beta}.
\]

Hence \( [T(\theta)]^\alpha_{\alpha} = ((X_{\alpha}X_{\beta}^{-1})^t)^{-1}[T(\theta)]^\beta_{\beta}(X_{\alpha}X_{\beta}^{-1})^t \).

**Theorem 2.4.** Assume that \( G \) has exactly \( k \) distinct conjugacy classes. If \( \text{Cf}_F(G) \) is generated by \( \{\theta^i | i \geq 0\} \) for some \( \theta \in \text{Cf}_F(G) \), then the followings hold.

1. \( \{1_G, \theta^1, \theta^2, \ldots, \theta^{k-1}\} \) is an \( F \)-basis of \( \text{Cf}_F(G) \).
2. \( \theta \) takes exactly \( k \) distinct values.

**Proof.** Let \( T \) and \( T^* \) be representation defined in the introduction. Let \( q(x) = x^r + a_{r-1}x^{r-1} + \cdots + a_0 \) be the minimal polynomial of \( T^*(\theta) \). Then \( T^*(\theta^r + a_{r-1}\theta^{r-1} + \cdots + a_01_G) = 0 \) iff \( \theta^r + a_{r-1}\theta^{r-1} + \cdots + a_01_G = 0 \). Therefore, \( q(x) = x^r + a_{r-1}x^{r-1} + \cdots + a_01_G = 0 \) is a polynomial of degree \( r \) with coefficients in \( \mathbb{C} \). Since \( \theta \) is an \( F \)-basis of \( \text{Cf}_F(G) \), \( \theta \) takes exactly \( k \) distinct values.