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ON CONVERGENCE OF FINITE DIFFERENCE SCHEMES FOR GENERALIZED SOLUTIONS OF SOBOLEV EQUATIONS

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1. Introduction

Let Ω be a rectangular domain in $\mathbb{R}^2$ with boundary $\partial\Omega$, and T be a positive real number such that $0 < T < \infty$. We consider finite difference approximations for the generalized solutions of Sobolev equations of the form

$$(A(t)u(x,t))_{tt} + B(t)u(x,t) = f(x,t), \quad (x,t) \in \Omega \times [0,T], \tag{1.1a}$$

with initial conditions

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \tag{1.1b}$$

and a boundary conditon

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times [0,T]. \tag{1.1c}$$

Here, $A(t)$ and $B(t)$ are second order, self-adjoint, and uniformly elliptic operators with smooth coefficients which have the following forms, respectively,

$$A(t)v = -\sum_{l,q=1}^{2} \frac{\partial}{\partial x_l}\Big(a_{lq}(x,t)\frac{\partial v}{\partial x_q}\Big) + a(x,t)v \tag{1.2}$$

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and

$$B(t)v = -\sum_{l,q=1}^{2} \frac{\partial}{\partial x_l}\Big(b_{lq}(x,t)\frac{\partial v}{\partial x_q}\Big) + b(x,t)v. \tag{1.3}$$

Difference schemes have been investigated for both linear and non-linear Sobolev equations when $A(t)v = -v$ by Douglas [2] and others. And error estimates have been obtained using energy arguments in association with Taylor's Theorem. This traditional approach requires a high degree of smoothness for the solutions. This means that Taylor's Theorem is not the appropriate framework in which to establish improved orders of convergence in weaker norm for finite difference approximations of non-smooth solutions.

For the generalized solutions of two point boundary value problems, Tikhonov and Samarskii[10] have estabilished rates of convergence which are compatible with the natural smoothness of their solutions. Subsequently, Makarov and Samarskii[9] have extended this analysis to the method of lines in case of one dimensional space. By applying a bilinear version of Bramble-Hilbert Lemma, Lazarov *et al.* [6]–[7] have generalized these earlier results to include elliptic and parabolic equations. More recently, Jovanović[4] used these techniques to study the convergence of finite difference approximations for the generalized solutions of second order linear parabolic and equations with variable coefficients. In the above mentioned papers, the error estimates are derived in terms of discrete $H^{s,\frac{s}{2}}$-norms ($s = 1, 2$).

The above results are extende to yield rates of converegence in the discrete $L^\infty(0,T;L^2)$- and $L^\infty(0,T;H^1)$- norms. Following the essential idea which underlies Wheeler's [13] convergence esimates for finite element approximations, Pani *et al.*[10] introduced a discrete elliptic projection technique to obtain orders of convergence which are compatible with the spatial smoothness of generalized solutions. The same technique is also applied to problem (1.1) to obtain optimal error estimates.

For the appropriate background theory on existence, uniqueness and regularity of the solutions of problem (1.1), we use Vityuk[12].

The paper is structured in the following manner. The preliminary materials are discussed in Section 2. In Section 3, a semidiscrete scheme and the stability results for a modified schemes are derived which establish the error estimates in L^2 and H^1 norms. In Section 4, a discrete projection technique is applied to the semidiscrete scheme, and related error estimates are derived. This section concludes by establishing $O(h^\alpha)$-converegence, $1 < \alpha \leq 2$, in the discrete L^2-norm, when the generalized solution is in $H^1(0,T;H^\alpha)$. In Section 5, a fully discrete scheme is analyzed, and the corresponding stability and error estimates are derived with reduced and without reduced regularity.

2. Preliminaries

We may assume that the domain Ω is the unit square in $\mathbb{R}^2$. For any positive integer M, let $h = \frac{1}{M}$ denote the spatial mesh size and $x_{ij} = (ih, jh)$, for $i,j = 0,1,\ldots,M$. Let $\Omega_h = \{x_{ij} : x_{ij} \in \Omega\}$ and $\partial\Omega_h = \{x_{ij} : x_{ij} \in \partial\Omega\}$. We can cover the whole of $\mathbb{R}^2$ with such a square grid, and will denote it by $\mathbb{R}^2_h$.

For any function v defined on Ω_h, we adopt the following notation: for $x \in \Omega_h$ and $l = 1,2$,

$$v^{\pm}(x) = v(x \pm h\mathbf{e}_l), \quad v^{+l,-q}(x) = v(x + h\mathbf{e}_l - h\mathbf{e}_q),$$

and

$$\nabla_l v(x) = \frac{v(x + h\mathbf{e}_l) - v(x)}{h}, \quad \bar{\nabla}_l v(x) = \frac{v(x) - v(x - h\mathbf{e}_l)}{h},$$

where $\mathbf{e}_l$ is the l-th unit vector in $\mathbb{R}^2$.

The Steklov mollifiers are defined in the following manner:

$$S = S_1^2 S_2^2 \text{ with } S_l^2 = S_l^+ S_l^-, \qquad l = 1,2,$$

where

$$S_l^+\phi(x) = \int_0^1 \phi(x + sh\mathbf{e}_l)\,ds, \quad S_l^-\phi(x) = \int_{-1}^0 \phi(x + sh\mathbf{e}_l)\,ds.$$

The operators $S_l^\pm$ commute, and the follwing relations hold

$$S_l^2\phi(x) = \int_{-1}^0 (1+s)\phi(x + she_l)\,ds + \int_0^1 (1-s)\phi(x + she_l)\,ds,$$

and

$$S_l^+ \frac{\partial \phi}{\partial x_l} = \nabla_l \phi, \quad S_l^- \frac{\partial \phi}{\partial x_l} = \bar{\nabla}_l \phi. \tag{2.1}$$

Let $\mathcal{D}_h$ denote the mesh functions defined on $\mathbb{R}_h^2$ which vanish outside of Ω_h. For $u, v \in \mathcal{D}_h$, we now introduce the discrete L^2 space, denoted by $L_h^2(\Omega_h)$, with inner product and norm given by

$$\langle w, v \rangle = h^2 \sum_{x \in \mathbb{R}_h^2} w(x) v(x),$$

and

$$\|w\|_{0,h} = \langle w, w \rangle^{\frac{1}{2}},$$

respectively. Further, for $w \in \mathcal{D}_h$, let $H_h^1 = H_h^1(\Omega_h)$ denote the discrete analogue of the H_0^1–Sobolev space with norm $\|w\|_{1,h}^2 = \|w\|_{0,h}^2 + \sum_{l=1}^2 \|\nabla_l w\|_0^2$. We also introduce a discerete $H^2 \cap H_0^1$–Sobolev space with norm

$$\|w\|_{2,h}^2 = \|w\|_{1,h}^2 + \sum_{l,q=1}^{2} \|\nabla_l \bar{\nabla}_q w\|_{0,h}^2, \quad w \in \mathcal{D}_h$$

and denote it by $H_h^2 = H_h^2(\Omega_h)$.

Whenever there is no confusion, we write $\|w\|$ and $\|w\|_j$, for $j = 1, 2$, in place of $\|w\|_{0,h}$ and $\|w\|_{j,h}$, respectively. Throughout the paper, $\|\cdot\|_{L^2}$ and $\|\cdot\|_{W^{m,p}(\Omega)}$ will denote the norm in the L^2 and the Sobolev space $W^{m,p}(\Omega)$, respectively. Further, $|\cdot|_{W^{m,p}(\Omega)}$ denotes the seminorm on $W^{m,p}(\Omega)$. In particular, for $p = 2$, we denote $W^{m,p}(\Omega)$ by $H^m(\Omega)$.

For functions v and w in $\mathcal{D}_h$, the following identities are easy consequences of summation by parts:

$$\langle \nabla_l v, w \rangle = -\langle v, \bar{\nabla}_l w \rangle, \qquad l = 1, 2. \tag{2.2}$$

The basic lemmas, which will be used in the following sections, are given below. Along with the Bramble-Hilbert Lemma (see, Bramble and Hilbert[1] and Dupon t and Scott[3]), the following bilinear version of it will be needed for our convergence analysis.

LEMMA 2.1. *Let $P_{[r]}$ be the set of all polynomials of degree $\leq [r]$, where $[r]$ denotes the largest integer less than $r > 0$. If η is a bounded linear functional on $W^{\alpha,p}(\Omega) \times W^{\beta,q}(\Omega)$, with $\alpha, \beta \in (0, \infty)$ and $p, q \in [1, \infty]$ such that*

$$\eta(U, v) = 0, \qquad \forall U \in P_{[\alpha]}(\Omega), \quad \forall v \in W^{\beta,q}(\Omega),$$

$$\eta(u, V) = 0, \qquad \forall u \in W^{\alpha,p}(\Omega), \quad \forall V \in P_{[\beta]}(\Omega),$$

then there exists a positive constant C such that

$$| \eta(u,) | \leq C \mid u \mid_{W^{\alpha,p}(\Omega)} \mid v \mid_{W^{\beta,q}(\Omega)}, \quad \forall u \in W^{\alpha,p}(\Omega), \quad \forall v \in W^{\beta,q}(\Omega).$$

We state the following discrete version of Gronwall's lemma for subsequent use. For a proof, see Lees[8].

LEMMA 2.2. *Let $\{w^n\}$ be a sequence of nonnegative real numbers satisfying*

$$w_n \leq \alpha_n + \sum_{p=0}^{n-1} \beta_p w_p, \quad n \geq 0, \quad \alpha_n, \beta_n \geq 0.$$

Then there is a constant C such that

$$w_n \leq \alpha_n + C \sum_{p=0}^{n-1} \beta_p \alpha_p.$$

We shall frequently use the following inequality

$$ab \leq \varepsilon a^2 + \frac{1}{4\varepsilon} b^2, \qquad a, b \in R, \quad \varepsilon > 0.$$

We write C as a generic positive constant independent of the discretizing parameters h and k.

3. Semidiscrete Schemes for Sobolev Equations

In this section, the stability and error analysis for a semdiscrete scheme of (1.1) will be given. Let $A_h(t)$ and $B_h(t)$ be the finite difference approximations for the operator $A(t)$ and $B(t)$, respectively, defined by

$$A_h(t)V = -\frac{1}{2}\sum_{l,q=1}^{2}[\nabla_l(a_{lq}(t)\bar{\nabla}_q V) + \bar{\nabla}_l(a_{lq}(t)\nabla_q V)] + S(a(t))V,$$

and

$$B_h(t)V = -\frac{1}{2}\sum_{l,q=1}^{2}[\nabla_l(b_{lq}(t)\bar{\nabla}_q V) + \bar{\nabla}_l(b_{lq}(t)\nabla_q V)] + S(b(t))V,$$

for $x \in \Omega_h$ and $t \in [0,T]$. In order to minimize the regularity imposed on $a(t)$ and $b(t)$ in the subsequent analysis, it is necessary to work with the Steklov mollification $S(a(t))$ and $S(b(t))$ in the above approximations. The semidiscrete approxiamtion $u_h(t)$ of (1.1) is defined to be the solution of

$$(3.1) \qquad \frac{d^2\{A_h u_h\}}{dt^2} + B_h(t)u_h(t) = Sf, \quad (x,t) \in \Omega_h \times (0,T],$$

$$u_h(x,t) = 0, \quad (x,t) \in \partial\Omega_h \times (0,T],$$

$$u_h(x,0) = u_0(x), \quad h_{h,t}(x,0) = 0, \quad x \in \Omega_h.$$

Let $H^1_{0,h} = \{v \in H^1_h : v = 0 \,\text{on}\, \partial\Omega_h\}$. On the basis of the assumptions imposed on $A(t)$ and $B(t)$, the following lemma can be verified using summation by parts.

LEMMA 3.1. *For* $V, W \in H^1_{0,h}$,

(1) $\|V\|^2 \leq C\sum_{l=1}^{2}\|\nabla_l V\|^2$,
(2) $\langle A_h(t)V, V\rangle \geq C_0\|V\|_1^2$,
(3) $\langle A_h(t)V, W\rangle \leq C\|V\|_1\|W\|_1$,

where C_0 and C are positive constants. The same properties hold for $B_h(t)$.

For the subsequent error estimates, we derive a stability result for a modified semidiscrete version of (3.1); namely,

$$\{A_h(t)u_h\}_{tt} + B_h(t)u_h = Sf(t) + \sum_{l=1}^{2} \bar{\nabla}_l F(t), \tag{3.2}$$

where F is zero on $\partial\Omega_h$ and $F(t) = F(0) + \int_0^t F_t(s)\,ds$.

We derive the stability estimate in terms of the following discrete norm

$$|||\phi(t)|||_1^2 = \|\phi_t(t)\|_1^2 + \|\phi(t)\|_1^2, \quad t \in [0,T].$$

THEOREM 3.1. *Let u_h be a solution of (3.2). Then there exists a constant C such that*

$$|||u_h(t)|||_1 \le C(|||u_h(0)|||_1 + \|F(0)\| + \int_0^t \|Sf(s)\|\,ds + \int_0^t \|F_t(s)\|\,ds).$$

Proof. Form the discrete L^2 inner product between (3.2) and $u_{h,t}$ to yield

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\langle A_h(t)u_{h,t}, u_{h,t}\rangle &+ \langle B_h(t)u_h, u_h\rangle)\\ &= \langle Sf, u_{h,t}\rangle + \sum_{l=1}^{2}(\langle F_t, \nabla_l u_h\rangle - \frac{d}{dt}\langle F, \nabla_l u_h\rangle)\\ &\quad + \frac{1}{2}\langle A_{h,t}(t)u_{h,t}, u_h\rangle + \frac{1}{2}\langle B_{h,t}(t)u_h, u_h\rangle.\end{aligned}$$

Integrate with respect to t and apply Lemma 3.1 to obtain

$$\begin{aligned}|||u_h|||_1^2 \le C\Big[&|||u_h(0)|||_1^2 + \int_0^t \|Sf(s)\|\,|||u_h(s)|||_1\,ds + \|F(0)\|\,|||u_h(0)|||_1\\ &+ \int_0^t \|F_t(s)\|\,|||u_h(s)|||_1\,ds + \int_0^t |||u_h(s)|||_1^2\,ds\Big].\end{aligned}$$

Setting

$$|||u_h(t^*)|||_1 = \sup_{t\in[0,T]} |||u_h(t)|||_1,$$

we establish the required result using Gronwall's inequality. □

THEOREM 3.2. *Let u be a generalized solution of (1.1) and u_h a solution of (3.1). Then, for $e(t) = u(t) - u_h(t)$ with $e(0) = 0$, we have*

$$|||e(t)|||_1 \leq Ch^{\alpha-1}, \qquad 1 \leq \alpha \leq 3, \quad t \in [0,T].$$

Proof. From (1.1) and (3.1), $e(t)$ satisfies

$$\begin{aligned}\frac{d^2}{dt^2}(A_h(t)e(t)) + B_h(t)e(t) &= [A_h(t)u - SA(t)u]_{tt} + [B_h(t)u - SB(t)u] \\ &= G_1(t) + G_2(t).\end{aligned}$$

Following Pani *et al.* [10], the integrand $G_2(t)$ is rewritten as

$$G_2(t) = \sum_{l,q=1}^{2} \bar{\nabla}_l \xi_{lq}(t) + \xi(t),$$

where

$$\xi_{lq} = \xi_{lq}^{(1)} + \xi_{lq}^{(2)} + \xi_{lq}^{(3)} + \xi_{lq}^{(4)}$$

with

$$\begin{aligned}\xi_{lq}^{(1)} &= S_l^+ S_{3-l}^2\Big(b_{lq}\frac{\partial u}{\partial x_q}\Big) - (S_l^+ S_{3-l}^2 b_{lq})\Big(S_l^+ S_{3-l}^2\frac{\partial u}{\partial x_q}\Big), \\ \xi_{lq}^{(2)} &= \left[S_l^+ S_{3-l}^2 b_{lq} - \frac{1}{2}(b_{lq} + b_{lq}^{+l})\right]\Big(S_l^+ S_{3-l}^2\frac{\partial u}{\partial x_q}\Big), \\ \xi_{lq}^{(3)} &= \frac{1}{2}(b_{lq} + b_{lq}^{+l})\left[S_l^+ S_{3-l}^2\frac{\partial u}{\partial x_q} - \frac{1}{2}(\nabla_q u + \bar{\nabla}_q u^{+l})\right], \\ \xi_{lq}^{(4)} &= -\frac{1}{4}(b_{lq} - b_{lq}^{+l})(\nabla_q u - \bar{\nabla}_q u^{+l}),\end{aligned}$$

and

$$\xi = (Sb)u - S(bu).$$

We also rewrite G_1 as

$$G_1(t) = \Big(\sum_{l,q=1}^{2} \bar{\nabla}_l \eta_{lq}(t) + \eta(t)\Big)_{tt},$$

where $\eta_{lq}(t)$ and $\eta(t)$ are the same as $\xi_{lq}(t)$ and $\xi(t)$ except that $b_{lq}(t)$ and $b(t)$ are replaced by $a_{lq}(t)$ and $a(t)$, respectively.

From Theorem 3.1 with $F(t) = \sum_{q=1}^{2} \eta_{lq,tt}(t) + \sum_{q=1}^{2} \xi_{lq}(t)$,

$$\begin{aligned}|||e(t)|||_1 \leq & C\Big[\int_0^t (\|\eta_{tt}(s)\| + \|\xi(s)\|)ds \\ & + \sum_{l,q=1}^{2} (\|\eta_{lq,ttt}(0)\| + \int_0^t \|\eta_{lq,tt}(s)\|\, ds) \\ & + \sum_{l,q=1}^{2} (\|\xi_{lq}(0)\| + \int_0^t \|\xi_{lq,t}(s)\|\, ds)\Big].\end{aligned}$$

Since $\xi = (Sb)(u - Su) + (Sb)(Su) - S(bu)$, we obtain, applying Lemma 2.1, the following estimate as in Pani *et al.* [10]

$$\|\xi(t)\| \leq Ch^{\alpha}\|u(t)\|_{H^{\alpha}(\Omega)}, \quad 1 < \alpha \leq 2.$$

Further, we estimates $\xi_{lq,t}$ as in [10] to find that

$$\sum_{l,q=1}^{2} \int_0^t \|\xi_{lq,t}(s)\|\, ds \leq h^{\alpha-1}\|u\|_{H^{\alpha}}, \quad 1 < \alpha \leq 3.$$

Since η and η_{lq} are like the ξ and ξ_{lq} terms respectively, we obtain

$$\|\eta_{tt}(t)\| \leq Ch^{\alpha}\|u(t)\|_{H^{\alpha}(\Omega)}, \quad 1 < \alpha \leq 2,$$

and

$$\sum_{l,q=1}^{2} \int_0^t \|\eta_{lq,ttt}(s)\|\, ds \leq Ch^{\alpha-1}, \quad 1 < \alpha \leq 3.$$

Hence the proof is completed. □

4. L^2-error estimates with reduced regularity

In this section, as in Wheeler [13], we also examine the error estimate under reduced regularity for the generalized solution u. Accordingly, we define the discrete projection as

$$B_h\tilde{u} = S(f - (Au)_{tt}), \quad (x,t) \in \Omega_h \times [0,T]. \tag{4.1}$$

For given u, there exists a unique $\tilde{u}$ on $\Omega_h \times [0,T]$ since B_h satisfies (ii) in Lemma 3.1. Let $\rho = u - \tilde{u}$ be the error associated with (1.10) and (4.1). We rewrite (4.1) as

$$B_h\rho = -[A_hu - S(Au)]_{tt}. \tag{4.2}$$

LEMMA 4.1. *There exists a constant C, which depends on u, such that*

$$\|\rho(t)\| + h\|\rho(t)\|_1 \leq Ch^{\alpha}, \quad 1 < \alpha \leq 2.$$

Proof. For the estimation of $\|\rho\|_1$, we take the discrete inner product of (4.2) with ρ and obtain

$$\langle B_h\rho, \rho\rangle = -\langle (A_hu - SAu)_{tt}, \rho\rangle = \langle G_1, \rho\rangle.$$

An application of (2.2) yields

$$\langle G_1, \rho\rangle = \sum_{l,q=1}^{2} \langle \eta_{lq,tt}, \nabla_l\rho\rangle + \langle \eta_{tt}, \rho\rangle.$$

Using the estimate for G_1 from Theorem 3.2, we obtain

$$| \langle G_1, \rho\rangle | \leq Ch^{\alpha-1}\|\rho\|_{H^1}, \quad 1 < \alpha \leq 2.$$

From the coercivity of A_h, the required estimation of $\|\rho\|_1$ follows.

For the discerete L^2 estimate, we define Φ as the solution of

$$\begin{aligned} B_h\Phi &= \rho, \qquad x \in \Omega_h, \\ \Phi &= 0, \qquad x \in \partial\Omega_h. \end{aligned} \tag{4.3}$$

Because of the coercivity of A_h, Φ is a unique solution of (4.3) with appropriate regulrity

$$\|\Phi\|_2 \leq C\|\rho\|. \tag{4.4}$$

Forming the discrete inner product between (4.3) and ρ, we find that

$$\langle \rho, \rho\rangle = \langle B_h\rho, \Phi\rangle = \langle G_1, \Phi\rangle. \tag{4.5}$$

For the estimate of G_1, we obtain from Lemma 2.1 that

$$\Big|\sum_{l,q=1}^{2}\langle \eta^{(1)}_{lq,tt}, \nabla_l\Phi\rangle\Big| \leq Ch^\alpha |u|_{H^\alpha}\|\Phi\|_1, \quad 1 \leq \alpha \leq 2,$$

where C depends on $|a_{lq,tt}|_{W^{1,\infty}(\Omega)}$.
For the estimate of $\sum_{l,q=1}^{2}\langle \eta^{(3)}_{lq,tt}, \nabla_l\Phi\rangle$, we consider it term by term.
For $l = 1, q = 2$,

$$\begin{aligned}\langle \eta^{(3)}_{12,tt}, \nabla_1\Phi\rangle = & \langle \nabla_2[S_1^+S_2^-u - \frac{1}{2}(u + u^{+1,-2})], \frac{1}{2}(a_{12} + a_{12}^{+1})_{tt}\nabla_1\Phi\rangle \\ & + \langle \nabla_2[S_1^+S_2^-u - \frac{1}{2}(u + u^{+1,-2})]_t, (a_{12} + a_{12}^{+1})_t\nabla_1\Phi\rangle \\ & + \langle \nabla_2[S_1^+S_2^-u - \frac{1}{2}(u + u^{+1,-2})]_{tt}, \frac{1}{2}(a_{12} + a12^{+1})\nabla_1\Phi\rangle.\end{aligned}$$

The application of (2.2) yields

$$\begin{aligned}\langle \eta^{(3)}_{12,tt}, \nabla_1\Phi\rangle \leq & C\{\|S_1^+S_2^-u_{tt} - \frac{1}{2}(u + u^{+1,-2})\| \\ & + \|(S_1^+S_2^-u - \frac{1}{2}(u + u^{+1,-2}))_t\| \\ & + \|(S_1^+S_2^-u - \frac{1}{2}(u + u^{+1,-2}))_{tt}\|\}\|\Phi\|_2.\end{aligned}$$

Since $S_1^+S_2^-u - \frac{1}{2}(u + u^{+1,-2})$ vanishes for all $u \in P_1$, an application of the Bramble–Hilbert Lemma yields

$$\|S_1^+S_2^-u - \frac{1}{2}(u + u^{+1,-2})\| \leq Ch^\alpha|u|_{H^\alpha(\Omega)}, \quad 1 < \alpha \leq 2.$$

The estimate for $\langle \eta^{(3)}_{21,tt}, \nabla_2 \Phi\rangle$ is obtained in a similar manner.

For $l = 1, q = 1$,

$$
\begin{aligned}
\langle \eta^{(3)}_{11,tt}, \nabla_1 \Phi\rangle =& \langle \nabla_1 (S_2^2 u - u), \frac{1}{2}(a_{11} + a_{11}^{+1})_{tt} \nabla_1 \Phi\rangle \\
&+ \langle \nabla_1 (S_2^2 u - u)_t, (a_{11} + a_{11}^{+1})_t \nabla_1 \Phi\rangle \\
&+ \langle \nabla_1 (S_2^2 u - u)_{tt}, \frac{1}{2}(a_{11} + a_{11}^{+1}) \nabla_1 \Phi\rangle \\
=& - \langle (S_2^2 u - u), \frac{1}{2}\bar{\nabla}_1((a_{11} + a_{11}^{+})_{tt} \nabla_1 \Phi)\rangle \\
&- \langle (S_2^2 u - u)_t, \bar{\nabla}_1((a_{11} + a_{11}^{+})_t \nabla_1 \Phi)\rangle \\
&- \langle (S_2^2 u - u)_{tt}, \frac{1}{2}\bar{\nabla}_1((a_{11} + a_{11}^{+})_{tt} \nabla_1 \Phi)\rangle.
\end{aligned}
$$

Since $S_2^2 u - u$ vanishes for all $u \in P_1$, the Bramble-Hilbert Lemma again yields

$$
|\langle \eta^{(3)}_{11,tt}, \nabla_1 \Phi\rangle| \le C h^{\alpha} |u|_{H^{\alpha}(\Omega)} \|\Phi\|_2, \quad 1 \le \alpha \le 2.
$$

The estimate for $\langle \eta^{(3)}_{22,tt}, \nabla_2 \Phi\rangle$ is obtained in a similar manner.

Using Lemma 2.1, we can derive

$$
\left| \sum_{l,q=1}^{2} \langle \eta^{(2)}_{lq,tt}, \nabla_l \Phi\rangle \right| \le C h^{\alpha} |u|_{H^{\alpha-1}(\Omega)} \|\Phi\|_2,
$$

$$
\left| \sum_{l,q=1}^{2} \langle \eta^{(4)}_{lq,tt}, \nabla_l \Phi\rangle \right| \le C h^{\alpha} |u|_{H^{\alpha}(\Omega)} \|\Phi\|_2,
$$

and

$$
\begin{aligned}
|\langle \eta_{tt}, \Phi\rangle| &\le |\langle (Sa)(u - Su) + (Sa)(Su) - S(au), \Phi\rangle| \\
&\le C h^{\alpha} [|u|_{H^{\alpha}} + |u|_{H^{\alpha-1}}] \|\Phi\|, \qquad 1 < \alpha \le 2.
\end{aligned}
$$

It follows that

$$
|\langle G_2, \Phi\rangle| \le C h^{\alpha} \|\Phi\|_2.
$$

Using the regularity condition (4.4), we obtain

$$
\|\rho\| \le C h^{\alpha} \|u\|_{H^{\alpha}(\Omega)}, \quad 1 \le \alpha \le 2.
$$

This completes the proof. □

LEMMA 4.2. *For $t \in [0,T]$, there exists a constant C such that*

$$\|\rho_t(t)\| + h^{\alpha-1}\|\rho_t(t)\|_1 \le Ch^\alpha, \quad 1 \le \alpha \le 2.$$

Proof. Differentiate (4.2) with respect to t to obtain

$$B_h\rho_t = -B_{h,t}\rho(t) - [A_h u - SAu]_{tt}. \tag{4.6}$$

Form the inner product between (4.6) and ρ_t, and then proceed as in Lemma 4.1 to establish

$$\|\rho_t\|_1 \le C[\|\rho\|_1 + Ch^{\alpha-1}\left[\|u_t\|_{H^\alpha(\Omega)} + \|u\|_{H^\alpha(\Omega)}\right].$$

We repeat the duality argument using ρ_t instead of ρ in equation (4.5) to obtain

$$\|\rho_t\| \le C[\|\rho(t)\| + Ch^\alpha[\|u(t)\|_{H^\alpha(\Omega)} + \|u_t\text{-}H^\alpha(\Omega)], \quad 1 < \alpha \le 2.$$

The required estimate for ρ_t follows from the estimate for ρ in Lemma 4.1. □

We differentiate (4.6) with respect to t to find that

$$B_h\rho_{tt} = -2B_{h,t}\rho_t - B_{h,tt}\rho - [A_h u - SAu]_{ttt}.$$

Then we obtain the error of ρ_{tt}. Its proof follows from Lemmas 4.1–4.2.

LEMMA 4.3. *For $t \in [0,T]$, there exists a constant C such that*

$$\|\rho_{tt}(t)\| + h\|\rho_{tt}(t)\|_1 \le Ch^\alpha, \quad 1 < \alpha \le 2.$$

We turn to a bound for $\|\tilde{u}_t\|_2$.

LEMMA 4.4. *For all $t \in [0,T]$, there exists a constant C such that*

$$\|\tilde{u}_{tt}(t)\|_2 \le C.$$

Proof. From equation (4.1) and (4.4), it follows that

$$\|\tilde{u}(t)\|_2 \le C, \qquad t \in [0,T].$$

Differentiate (4.1) with respect t, and apply the regularity condition (4.4) along with Lemma 3.1 to obtain

$$\|\tilde{u}_t\|_2 \le C[\|\tilde{u}(t)\|_2 + \|S(f-(Au)_{tt})_t\|].$$

The above bound for $\|\tilde{u}(t)\|_2$ establishes the corresponding result for $\|\tilde{u}_t(t)\|_2$.

Finally, differentiate (4.1) twice with respect to t to obtain

$$B_h\tilde{u}_{tt} = S(f-(Au)_{tt})_{tt} - B_{h,tt}\tilde{u} - 2B_{h,t}\tilde{u}_t.$$

The above argument is now repeated with obvious modifications to establish the required result. □

It is here that we exploit the full potential of the Steklov mollifications. Let $\theta(t) = u_h(t)\tilde{u}(t)$, then the error $e(t) = u(t) - u_h(t) = \rho(t) - \theta(t)$.

THEOREM 4.1. *Let u and u_h be the solutions of (1.1) and (3.1), respectively. Then there exists a constant C such that*

$$\|e(t)\| + h\|e(t)\|_1 \le C[\|\theta(0)\| + h\|\theta(0)\|_1 + h^\alpha],\ 1 < \alpha \le 2.$$

Proof. Since the estimate for $\rho(t)$ is given in Lemma 4.1, it is sufficient to estimate $\theta(t)$. From (1.1), (3.1), and (4.1), it follows that

$$\frac{d^2}{dt^2}(A_\theta) + B_h(t)\theta = -(A_h(\tilde{u}-u))_{tt} - (A_hu - SAu)_{tt}. \tag{4.7}$$

From Theorem 3.1 with $F = 0$, we obtain

$$|||\theta|||_1 \le C[|||\theta(0)|||_1 + \int_0^t \{\|(A_h(\tilde{u}-u))_{tt}\| + \|(A_hu - SAu)_{tt}\|\}\, ds].$$

From Lemma 4.3, it follows that

$$|||\theta|||_1 \le C[|||\theta(0)|||_1 + h^\alpha \int_0^t \|u_{tt}(s)\|\, ds].$$

Since $\|\theta\| \le |||\theta|||_1$, we now use the triangle inequality to complete the proof. □

5. Fully discrete Schemes for Sobolev equations

We now discuss a fully discrete scheme for (1.1) which is symmetric around $t = t_n$. Let us introduce

$$\partial_t U^n = \frac{U^{n+1} - U^n}{k}, \quad \hat{U}^n = \frac{U^{n+\frac{1}{2}} + U^{n-\frac{1}{2}}}{2} = \frac{U^{n+1} + 2U^n + U^{n-1}}{4}.$$

Then a fully discrete scheme is

$$(5.1a) \qquad \partial_t \bar{\partial}_t (A_h(t_n) U^n) + B_h(t_n) \hat{U}^n = S f^n,$$

$$(5.1b) \qquad \partial_t A_h(t_1) U^1 = (A_h(0) u(0))_t + \frac{1}{2} k(-B_h(0) u_0 + S f^0),$$

$$U^0 = u_0, \quad x \in \Omega_h,$$

$$U = 0, \quad x \in \partial\Omega_h.$$

As in the case of semidiscrete schem, we first derive the stability analysis for a modified version; namely,

$$(5.2) \qquad \partial_t \bar{\partial}_t (A_h(t_n) U^n) + B_h(t_n) \hat{U}^n = S f^n + \sum_{l=1}^{2} \bar{\nabla} l F^n,$$

where $F(t_n)$ in $\mathcal{D}_h$, for each $0 \leq t_n \leq T$.

We establish the stability of (5.2) in the following discrete norm

$$|||\phi^{n+\frac{1}{2}}|||_1 = \|\partial_t \phi^n\|_1 + \|\phi^{n+\frac{1}{2}}\|_1.$$

THEOREM 5.1. *Let U be a solution of (5.2). Then there exists a constant C such that*

$$|||U^{n+\frac{1}{2}}|||_1 \leq C\{|||U^{\frac{1}{2}}|||_1 + \|F^0\| + k \sum_{m=1}^{n} \|S f^m\| + k \sum_{m=1}^{n} \|\partial_t F^m\|\}.$$

Proof. Note that

$$\langle \partial_t \bar{\partial}_t (A_h(t_m) U^m), \bar{\partial}_t U^{m+\frac{1}{2}} \rangle = \frac{1}{2} \bar{\partial}_t \|\partial_t (A_h(t_m) U^m)\|^2,$$

and

$$\begin{aligned}&\langle B_h(t_m)\hat{U}^m, \bar{\partial}_t U^{m+\frac{1}{2}}\rangle \\ =&\frac{1}{2}\left[\bar{\partial}_t\langle B_h(t_m)U^{m+\frac{1}{2}}, U^{m+\frac{1}{2}}\rangle - \langle\bar{\partial}_t(B_h(t_m))U^{m-\frac{1}{2}}, U^{m-\frac{1}{2}}\rangle\right].\end{aligned}$$

Form the discrete L^2 inner product between (5.2) (ith $n = m$) and $\bar{\partial}_t U^{m+\frac{1}{2}} = (\partial_t U^m + \bar{\partial}_t U^m)/2$ to obtain

$$\begin{aligned}(5.3)\qquad &\langle\partial_t\bar{\partial}_t(A_h(t_m)U^m), \bar{\partial}_U^{m+\frac{1}{2}}\rangle + \langle B_h(t_m)\hat{U}^m, \bar{\partial}_t U^{m+\frac{1}{2}}\rangle \\ &= \langle Sf^m, \bar{\partial}_t U^{m+\frac{1}{2}}\rangle + \sum_{l=1}^{2}\langle\bar{\nabla}_l F^m, \bar{\partial}_t U^{m+\frac{1}{2}}\rangle.\end{aligned}$$

Multiplying both side of (5.3) by $2k$ nd summing from $m = 1$ to n, we have

$$\begin{aligned}|||U^{m+\frac{1}{2}}|||_1^2 \le C[|||U^{\frac{1}{2}}|||_1^2 &+ k\sum_{m=1}^{n}\|Sf^m\|\,|||U^{m+\frac{1}{2}}|||_1 + \|F^0\|\,|||U^{\frac{1}{2}}|||_1 \\ &+ k\sum_{m=1}^{n}\|\bar{\partial}_t F^m\|\,|||U^{m+\frac{1}{2}}|||_1 + k\sum_{m=0}^{n-1}|||U^{m+\frac{1}{2}}|||_1^2].\end{aligned}$$

Set $|||U^{J+1}|||_1 = \max_{0\le m\le n}|||U^{m+\frac{1}{2}}|||_1$, for fixed J with $0 \le J \le n$. Then

$$\begin{aligned}|||U^{n+\frac{1}{2}}|||_1 \le& C[|||U^{\frac{1}{2}}|||_1 + \|F^0\| + k\sum_{m=1}^{n}(\|Sf^m\| + \|\bar{\partial}_t F^m\|) \\ &+ k\sum_{m=0}^{n-1}|||U^{m+\frac{1}{2}}|||_1].\end{aligned}$$

An application of Lemma 2.2 now completes the proof.

Let $e^n = u(t_n) - U^n$. We present an error analysis for (5.1).

THEOREM 5.2. *There exists a constant C such that*

$$|||e^{n+\frac{1}{2}}|||_1 \le C(h^{\alpha-1} + k^2), \quad 2 < \alpha \le 3.$$

Proof. From (1.1) and (5.1), it follows that

$$\begin{aligned}
&\partial_t\bar{\partial}_t A_h(t_m)e^m + B_h(t_m)\hat{e}^m \\
=&\partial_t\bar{\partial}_t A_h(t)u^m + B_h(t_m)\hat{u}^m - [S(Au)_{tt}(t_m) + SB(t_m)u(t_m)] \\
=&[S(A(t_m)u)_{tt}(t_m)_S(A(t_m)u)_{tt}(t_m)] \\
&+ [\partial_t\bar{\partial}_t(A(t_m)u(t_m)) - (A(t_m)u)_{tt}(t_m)] \\
&+ [B_h(t_m)\hat{u}^m - B_h(t_m)u(t_m)] + [B_h(t_m)u(t_m) - SB(t_m)u(t_m)] \\
=&G_3^m + G_4^m + G_5^m + \eta^m + \sum_{l,q=1}^{2} \bar{\nabla}\eta_{lq}^m.
\end{aligned}$$

Apply Theorem 5.1 to obtain

$$\begin{aligned}
|||e^{n+\frac{1}{2}}|||_1 \le& C[|||e^{\frac{1}{2}}|||_1 + k\sum_{m=1}^{n}(\|G_3^m\| + \|G_4^m\| + \|G_5^m\| + \|\eta^m\|) \\
&+ \|G_6^0\| + k\sum_{m=1}^{n}\|\bar{\partial}_t G_6^m\|],
\end{aligned}$$

where $G_6^m = \sum_{l,q=1}^{2}\eta_{lq}^m$. Note that

$$k\sum_{m=1}^{n}\|G_4^n\| \le Ck^2\int_0^{t_n}\|u_{tttt}(s)\|_{H^2}\,ds$$

and

$$k\sum_{m=1}^{n}\|G_5^m\| \le Ck^2\int_0^{t_n}\|u_{tt}(s)\|_{H^2}\,ds.$$

Further, for the estimation of $|||e^{\frac{1}{2}}|||_1$, we recall that

$$|||e^{\frac{1}{2}}|||_1^2 = \|\partial_t e^0\|_1^2 + \|e^{\frac{1}{2}}\|_1^2,$$

where

$$\begin{aligned}
\partial_t(A_h(0)e^0) =& \bar{\partial}_t(A_h(t_1)u(t_1)) - (A_h(0)u(0))_t - \frac{k}{2}(-B_h(0)u(0) + Sf^0) \\
=&\frac{1}{k}[A_h(t_1)u(t_1) - A_h(0)u(0) - k(A_h(0)u(0))_t - \frac{k^2}{2}(A_h(0)u(0))_{tt}] \\
&+ \frac{k}{2}\{[(A_h(0)u(0))_{tt} - S(A(0)u(0))_{tt}] + [B_h(0)u(0) - SB(0)u(0)]\}.
\end{aligned}$$

Hence,

$$\|\partial_t e^0\|_1^2 \le C[k^4\|u_{ttt}\|^2_{L^\infty(0,k;H^1)} + k^2h^{2(\alpha-1)}(\|u_{tt}(0)\|^2_{H^\alpha} + \|u(0)\|_{H^{\alpha+1}}).$$

Finally, for $\|e^{\frac{1}{2}}\|_1^2$, we observe that

$$\begin{aligned} A_h(t_1)e^1 &= A_h(0)e^0 \\ &+ [A_h(t_1)u(t_1) - A_h(0)u(0) - k(A_h(0)u(0))_t - \frac{k^2}{2}(A_h(0)u(0))_{tt}] \\ &+ \frac{k^2}{2}\{[(A_h(0)u(0))_{tt} - S(A(0)u(0))_{tt}] + [B_h(0)u(0) - SB(0)u(0)]\}. \end{aligned}$$

Since $e(0) = 0$, we thus have

$$\|e^{\frac{1}{2}}\|_1 \le Ck^4[\|u_0\|^2_{H^2} + \|u_{tt}(0)\|^2_{H^2} + \int_0^k \|u_{ttt}\|^2_{H^1}\, ds].$$

the result follows from Theorems 3.2 and 4.1. □

We now ready to consider the error analysis for (5.1) with reduced regularity.

THEOREM 5.3. *There exists a constant C such that*

$$\|e^{n+\frac{1}{2}}\| + h\|e^{n+\frac{1}{2}}\|_1 \le C(|||\theta^{\frac{1}{2}}|||_1 + h^\alpha + k^2), \quad 1 < \alpha \le 2.$$

Proof. Since the estimates of ρ are known, we need only to find the estimates for θ. From (5.1) and (4.1), it follows that

$$\begin{aligned} &\partial_t\bar{\partial}_t(A_h(t_m)u(t_m)) \\ =&\partial_t\bar{\partial}_t(A_h(t_m)\rho^m) + [S(A(t_m)u(t_m))_{tt} - (A(t_m)u(t_m))_{tt}] \\ &+ [(A(t_m)u(t_m))_{tt} - \partial_t\bar{\partial}_t(A_h(t_m)u(t_m))] + B_h(t_m)[\tilde{u}^m - \hat{\tilde{u}}^m] \\ =&\partial_t\bar{\partial}_t(A_h(t_m)\rho^m) - J_1^m - J_2^m + J_5^m. \end{aligned}$$

Apply Theorem 5.1 to obtain

$$|||\theta^{n+\frac{1}{2}}|||_1 \le C[|||\theta^{\frac{1}{2}}|||_1 + k\sum_{m=1}^n(\|\partial_t\bar{\partial}_t\rho^m\| + \|J_1^m\| + \|J_2^m\| + \|J_5^m\|)].$$

Note that

$$k \sum_{m=1}^{n} \|J_5^m\| \le Ck^2 \int_0^{t_n} \|\tilde{u}_{tt}(s)\|_{H^2}\, ds$$
$$\le Ck^2 \int_0^{t_n} \|u_{tttt}(s)\|_{L^2}\, ds,$$

where the last inequality can be easily deduced from Lemma 4.4 using the discre te projection. Hence, we obtain

$$|||\theta^{n+\frac{1}{2}}|||_1 \le C[|||\theta^{\frac{1}{2}}|||_1 + h^\alpha \int_0^{t_n} \|u_{tt}(s)\|_{H^\alpha}\, ds + k^2 \int_0^{t_n} \|u_{tttt}(s)\|_{L^2}\, ds],$$

and the required result follows from the triangle inequality. □

REMARK. In the above argument, higher regularity has been assumed for u with respect to t. However, it may be possible to modify this regularity condition by using Steklov mollifcation in time.

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