SPANNING COLUMN RANK 1 SPACES OF NONNEGATIVE MATRICES

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1. Introduction

There are some papers on structure theorems for the spaces of matrices over certain semirings. Beasley, Gregory and Pullman [1] obtained characterizations of semiring rank 1 matrices over certain semirings of the nonnegative reals. Beasley and Pullman [2] also obtained the structure theorems of Boolean rank 1 spaces. Since the semiring rank of a matrix differs from the column rank of it in general, we consider a structure theorem for semiring rank in [1] in view of column rank.

In this paper, we obtain a characterization of column rank 1 matrices and a structure theorem for the vector space of matrices whose nonzero members all have spanning column rank 1 over nonnegative part of a unique factorization domain that is not a field in the reals.

2. Definitions and preliminaries

Let $\mathbf{R}$ denote the field of reals and $\mathbf{S}$ denote an arbitrary semiring of nonnegative reals. Let $\mathbf{U}_+$ be the nonnegative part of a unique factorization domain which is not a field in $\mathbf{R}$. Such examples are $\mathbb{Z}_+, (\mathbb{Q}[\pi])_+$ etc., where $\mathbb{Z}, \mathbb{Q}$ denote the rings of integers and rationals, respectively, and $\pi$ is a transcendental number over $\mathbb{Q}$.

Let $\mathbf{A}$ be an $m \times n$ matrix over $\mathbf{S}$. If $\mathbf{A}$ is a nonzero matrix, then the semiring rank [3] of $\mathbf{A}$, $r(\mathbf{A})$, is the least $k$ for which there exist $m \times k$ and $k \times n$ matrices $\mathbf{F}$ and $\mathbf{G}$ over $\mathbf{S}$ such that $\mathbf{A} = \mathbf{F}\mathbf{G}$. The zero matrix is assigned the semiring rank 0. The set of $m \times n$ matrices
with entries in $S$ is denoted by $M_{m,n}(S)$. Addition, multiplication by scalars, and the product of matrices are defined as if $S$ were a field.

If $V$ is a nonempty subset of $S^k \equiv M_{k,1}(S)$ that is closed under addition and multiplication by scalars, then $V$ is called a vector space over $S$. The notions of subspace and of spanning sets are the same as if $S$ were a field. As with fields, a basis for a vector space $V$ is a spanning subset of least cardinality. That cardinality is the dimension, $\dim(V)$, of $V$.

For an $m \times n$ matrix $A$ over $S$, the column rank $[5], c(A)$, is the dimension of the vector space spanned by its columns, and the spanning column rank $[4], sc(A)$, is the minimum number of the columns of $A$ which span its column space.

It follows that

\[(2.1) \quad 0 \leq r(A) \leq c(A) \leq sc(A) \leq n\]

for all $m \times n$ matrices $A$ over $S$. But these rank functions may differ over certain semirings as shown in the following example.

**Example 2.1.** Consider a matrix $A = [3, 6 - 2\sqrt{7}, 2\sqrt{7} - 4]$ over a semiring $S = (\mathbb{Z}[\sqrt{7}])_+$. Then it is trivially that $r(A) = 1$. Since $(6 - 2\sqrt{7}) + (2\sqrt{7} - 4) = 2$, $2$ is spanned by the last two columns of $A$. Then we have $(6 - 2\sqrt{7}) = 2(3 - \sqrt{7})$ and $2\sqrt{7} - 4 = 2(\sqrt{7} - 2)$ with $3 - \sqrt{7}, \sqrt{7} - 2 \in S$, which means that $\{2, 3\}$ is a basis of the column space of $A$. So $c(A) = 2$. But, any column of $A$ cannot be spanned by the other two columns. That is, $sc(A) = 3$. ■

Let $\Gamma$ be a nonempty subset of $S^k$ and let $g \in S^k$. We'll say that $g$ is a common factor of $\Gamma$ if $\Gamma \subseteq \{\sigma g \mid \sigma \in S\}$.

**Lemma 2.2.** ([1]) Let $\Gamma$ be any nonempty subset of $(U_+)^k$. Each pair of nonzero vectors in $\Gamma$ has a common nonzero scalar multiple in $(U_+)^k$ if and only if $\Gamma$ has a common factor in $(U_+)^k$. ■

**Example 2.3.** If $k > 1$, let

\[
A(k) = \begin{pmatrix}
1 & 1 & k - 1 \\
1 & k & 0 \\
1 & 0 & k
\end{pmatrix}.
\]
If \( 0 < k < 1 \), let \( p = \lfloor \frac{1}{k} \rfloor \), \( q = p - 1 \) and

\[
A(k) = \begin{pmatrix}
1 & 1 - kq & kp - 1 \\
1 & k & 0 \\
1 & 0 & k
\end{pmatrix}.
\]

If \( k \) is a nonzero nonunit in \( S \), then \( c(A(k)) = 3 \) by definition of column rank. Multiplying the first column of \( A(k) \) by \( k \) reduces its column rank to 2. From this matrix \( A(k) \) we can obtain an \( m \times n \) matrix of column rank \( r \) such that the matrix obtained by multiplying the \( j \)th column of it by \( k \) has column rank \( r - 1 \) as follows; let \( P \) be the matrix obtained from \( I_n \) by interchanging its first and \( j \)th column, and let \( B \) be any \( (m - 3) \times (n - 3) \) matrix over \( S \) of column rank \( r - 3 \). Then \( X = (A \oplus B)P \) is the required matrix of column rank \( r \).

3. Column rank 1 matrix

If \( X \) is a matrix over a semiring \( S \) and \( X = xa^t \), then the vectors \( x, a \) are called left and right factors of \( X \) respectively. In particular, \( a \) is called a basic right factor of \( X \) if \( a^t \) has column rank 1.

**Theorem 3.1.** For \( A \in M_{m,n}(S), c(A) = 1 \) if and only if \( A \) can be factored as \( xa^t \) for some \( a \in S^n, x \in S^m \), where \( x \neq 0 \) and \( a^t \) is a basic right factor.

**Proof.** Suppose that \( c(A) = 1 \) and denote the columns of \( A \) by \( a_1, \ldots, a_n \). Let \( \{x\} \) be a basis of the column space of \( A \) over \( S \), so that \( x = \sum_{j=1}^n \gamma_j a_j \) for some constants \( \gamma_1, \ldots, \gamma_n \) in \( S \). In particular, \( x \in S^m \) and \( x \neq 0 \). Now for each \( j \) between 1 and \( n \), we have \( a_j = \alpha_j x \) for some \( \alpha_j \in S \), since \( x \) spans the column space of \( A \). Letting \( a^t = [\alpha_1, \ldots, \alpha_n] \), we have \( a \in S^n \) and \( A = xa^t \). Further, \( x = \sum_{j=1}^n \gamma_j a_j = \sum_{j=1}^n \gamma_j \alpha_j x \), and hence \( 1 = \sum_{j=1}^n \gamma_j \alpha_j \) since \( x \) is not zero. Thus 1 is in the column space of \( a^t \), and it follows that \( c(a^t) = 1 \). Consequently, \( a \) is a basic right factor of \( A \), as desired.

The converse is clear.

Identifying \( S^{mn} \) with \( M_{m,n}(S) \), we transfer the definitions to \( M_{m,n}(S) \). If \( V \neq \{0\} \) is a vector space in \( M_{m,n}(S) \) whose members have column rank at most 1, then \( V \) is a column rank 1 space. If \( V \) is a
vector space all of whose members have the same basic right factor $b$, then $V$ is called a basic right factor space. Notice that in that case $W = \{a \in \mathbb{S}^m | ab' \in V\}$ is a vector space in $\mathbb{S}^m$. Conversely, if $W$ is a vector space in $\mathbb{S}^m$ and $c(b') = 1$ then $\{ab' | a \in W\}$ is a basic right factor space in $\mathbb{M}_{m,n}(\mathbb{S})$. Evidently basic right factor spaces are column rank 1 spaces.

Define a relation $\lambda$ on the $m \times n$ column rank 1 matrices over $\mathbb{S}$ by $A \lambda B$ if $A$ and $B$ have a common basic right factor.

**Proposition 3.2.** (1) $\lambda$ is an equivalence relation on the $m \times n$ column rank 1 matrices over $\mathbb{U}_+$.  
(2) For any nonempty set $E$ of $m \times n$ column rank 1 matrices over $\mathbb{U}_+$, the members of $E$ have a common basic right factor if and only if $X \lambda Y$ for all $X,Y$ in $E$.

**Proof.** (1) Evidently $\lambda$ is reflexive and symmetric. Suppose $A,B,C$ are $m \times n$ column rank 1 matrices over $\mathbb{U}_+$ that satisfy $A \lambda B$ and $B \lambda C$. Then $A,B$ and $C$ can be factored as $A = xa', ya' = B = zb'$ and $C = wb'$ by Theorem 3.1, where $a'$ and $b'$ have column rank 1. Now $a,b$ have a common nonzero scalar multiple because the left factors of $B$ are nonzero. Therefore $a,b$ have a common factor $f$ by Lemma 2.2, and $f'$ has column rank 1. So $A$ and $C$ can be factored as $A = (\alpha x)f'$ and $C = (\beta w)f'$ for some $\alpha, \beta \in \mathbb{U}_+$. Consequently $A \lambda C$ and hence $\lambda$ is transitive.

(2) Suppose $X \lambda Y$ for all $X,Y$ in $E$. For each $X$ in $E$, select a basic right factor $g_X$ and put $\Gamma = \{g_X | X \in E\}$. By the proof of (1), if $A,B$ are in $E$, then $A$ and $B$ have a common basic right factor. Thus $g_A$ and $g_B$ have a common nonzero scalar multiple. Therefore $\Gamma$ has a common factor $f$ by Lemma 2.2, and $f'$ has column rank 1. Thus $f$ is a common basic right factor of all $X$ in $E$.

The converse is immediate. \[\blacksquare\]

Thus the $\lambda$-equivalence classes are the maximal basic right factor spaces in $\mathbb{M}_{m,n}(\mathbb{U}_+)$. These in turn are of the form $V(a) = \{xa' | x \in \mathbb{U}_+\}$. where $c(a') = 1$.

4. **Spanning column rank 1 spaces**

In this section, we obtain a structure theorem for the vector space
of matrices whose members have spanning column rank at most 1. For this purpose we need some definitions and lemmas.

If \( A \) is a matrix over a semiring \( S \) and \( A \) has the form \( fa' \), then \( a' \) is called a strong right factor of \( A \) if \( a' \) has spanning column rank 1. Hwang, Kim and Song [4] showed the following Lemma:

**Lemma 4.1.** ([4]) For \( A \in \mathbb{M}_{m,n}(S) \), \( sc(A) = 1 \) if and only if \( A \) can be factored as \( fa' \) for some \( a \in S^n \) and \( f \in S^m \), where \( f \neq 0 \) and \( a' \) is a strong right factor.

If \( V \neq \{0\} \) is a vector space in \( \mathbb{M}_{m,n}(S) \) whose members have spanning column rank at most 1, then \( V \) is called a spanning column rank 1 space. If \( V \) is a vector space all of whose members have the same strong right factor \( b \), then \( V \) is called a strong right factor space. As the case of basic right factor space, \( W = \{a \in S^m \mid ab' \in V\} \) is a vector space in \( S^m \). Conversely, if \( W \) is a vector space in \( S^m \) and \( sc(b') = 1 \) then \( \{ab' \mid a \in W\} \) is a strong right factor space in \( \mathbb{M}_{m,n}(S) \). Evidently strong right factor spaces are spanning column rank 1 spaces.

Beasley and Pullman [1] obtained a Lemma for the common factor of two matrices as follows:

**Lemma 4.2.** ([1]) Suppose \( A \) and \( B \) are \( m \times n \) matrices of semiring rank 1 over \( U_+ \) and \( \min(m,n) \geq 2 \). Then \( r(A+B) = 1 \) if and only if \( A \) and \( B \) have a common factor.

For the common strong right factor of two matrices, we obtain the following Lemma:

**Lemma 4.3.** Suppose \( A, B \in \mathbb{M}_{m,n}(U_+) \) with \( sc(A) = sc(B) = 1 \) and \( \min(m,n) \geq 2 \). Then \( A \) and \( B \) have a common strong right factor if and only if \( sc(\alpha A + \beta B) = 1 \) for any \( \alpha, \beta \in U_+ \), not both zero.

**Proof.** By Lemma 4.1, we can write \( A = fa' \), and \( B = gb' \) for some \( f, g \in (U_+)^m \) and \( a, b \in (U_+)^n \) with \( sc(a') = sc(b') = 1 \). Assume that \( A \) and \( B \) have a common strong right factor \( r \). Then, for any \( \alpha, \beta \in U_+ \), \( \alpha A + \beta B = (\alpha \sigma f + \beta \tau g)r' \) for some \( \sigma, \tau \in U_+ \). Since \( sc(r') = sc(\sigma r') = sc(a') = 1 \), \( sc(\alpha A + \beta B) = 1 \) for any \( \alpha, \beta \), not both zero.

Conversely, assume that \( sc(\alpha A + \beta B) = 1 \) for any \( \alpha, \beta \in U_+ \), not both zero. Then we have \( r(\alpha A + \beta B) = 1 \) by (2.1). In particular, \( A \) and \( B \) have a common factor by Lemma 4.2.
Case 1) $A$ and $B$ have a common right factor $r$. Then we can write $A + B = (\sigma f + \tau g)r^t$ for some $\sigma, \tau \in \mathbf{U}$. Since $sc(r^t) = sc(\sigma r^t) = sc(a^t) = 1$, $A$ and $B$ have a common strong right factor $r$.

Case 2) $A$ and $B$ have a common left factor $d$. Then we may write $A = d\alpha a^t$ and $B = d\beta b^t$, where $\alpha a = (a_1, \cdots, a_n)^t$, and $\beta b = (b_1, \cdots, b_n)^t$ are strong right factors of $A$ and $B$, respectively. Since there are infinitely many primes in $\mathbf{U}$ (for the existence of infinite primes, see Lemma 2.2 in [4]), we can choose a prime $\pi$ such that $\pi$ does not divide all nonzero $b_i, i = 1, \cdots, n$. Consider

$$\pi^p A + B = d[\pi^p a_1 + b_1, \pi^p a_2 + b_2, \cdots, \pi^p a_n + b_n]$$

which has spanning column rank 1 for any positive integer $p$. Since the columns of $\pi^p A + B$ are finite in number, there exists a column $j$ and a sequence of $p$'s with the properties that i) the $j$th columns of $\pi^p A + B$ spans the column space for each term $p$ in the sequence, and ii) the difference between two successive terms in the sequence is at most $n$. Therefore for infinitely many $p$.

(4.1)

$$\pi^p a_k + b_k = \mu_{pk}(\pi^p a_j + b_j)$$

for some $\mu_{hk} \in \mathbf{U}, k = 1, \cdots, n$. In (4.1), if $b_j = 0$, then $b_k$ must be divided by nonunit $\pi^p$. But it is impossible since $\pi$ does not divide $b_k$ for at least one nonzero $b_k$. Thus $b_j \neq 0$. If the column space of $\pi^q A + B$ is spanned by its $j$th column, then we get

(4.2)

$$\pi^q a_k + b_k = \mu_{qk}(\pi^q a_j + b_j)$$

for some $\mu_{qk} \in \mathbf{U}, k = 1, \cdots, n$. From (4.1) and (4.2), we get $| \mu_{qk} - \mu_{pk} | \in \mathbf{U}$ for $q > p$. Since there are only $n$ columns in $\pi^p A + B$ for each $p$, we can choose infinitely many pairs $p$ and $q$ such that they satisfy $p < q \leq p + n$ and the column spaces of $\pi^p A + B$ and $\pi^q A + B$ are spanned by their $j$th column respectively. For such pairs $p$ and $q$, consider

(4.3)

$$| \mu_{qk} - \mu_{pk} | = \frac{\pi^q a_k + b_k}{\pi^q a_j + b_j} - \frac{\pi^p a_k + b_k}{\pi^p a_j + b_j} = \frac{((\pi^q - \pi^p - 1)(a_kb_j - a_j b_k))\pi^p}{(\pi^q a_j + b_j)(\pi^p a_j + b_j)}$$
Assume that $\mu_{qk} \neq \mu_{pk}$ for all such pairs $p$ and $q$. Since $\pi$ is prime, $\pi$ is not divided by $\pi^p a_j + b_j$. If $\pi^p a_j + b_j$ has $\pi$ as its prime factor, then $\pi^p a_j + b_j = \beta \pi$ for some $\beta \in U_+$. Thus $\pi(\beta - \pi^{p-1} a_j) = b_j$ and hence $b_j$ is divided by $\pi$, which is a contradiction. Then $\pi^p$ does not have any factor of $(\pi^p a_j + b_j)(\pi^q a_j + b_j)$. Since $|a_k b_j - a_j b_k|$ is fixed and $|\pi^{q-p} - 1|$ takes at most $n$ values for any pairs $p$ and $q$ with $1 \leq q - p \leq n$, the prime factors of $|(\pi^q - 1)(a_k b_j - a_j b_k)|$ are finite in number. Thus we can choose sufficiently large pair $p$ and $q$ with $1 \leq q - p \leq n$ such that $|(\pi^{q-p} - 1)(a_k b_j - a_j b_k)|$ does not contain some prime factors of $(\pi^p a_j + b_j)(\pi^q a_j + b_j)$. Then the denominator of (4.3) contains some nonunit prime factors such that the numerator of (4.3) does not contain. Since $U_+$ contains no element of the form $y/x$, where $y$ has a prime factor which $x$ does not, the fractional expression of (4.3) is not an element of $U_+$. Thus we have a contradiction such that $|\pi_{qk} - \pi_{pk}| \notin U_+$ for some pair $p$ and $q$ with $p < q \leq p + n$. Hence $\mu_{qk} = \mu_{pk}$ for some $p$ and $q$. Subtracting (4.1) from (4.2), we have $a_k = \mu_{pk} a_j$ for all $k = 1, \ldots, n$. And we get $b_k = \mu_{pk} b_j$ for all $k = 1, \ldots, n$ from (4.1). That is, $a = a_j r$ and $b = b_j r$ where $r = [\mu_{p1}, \ldots, \mu_{pn}]$ with $\mu_{pj} = 1$.

By cases 1) and 2), $A$ and $B$ have a common strong right factor $r$. ■

Define a relation $\rho$ on the $m \times n$ spanning column rank 1 matrices over a semiring $S$ by $: A \rho B$ if $A, B$ have a common strong right factor. Then we have some properties on the relation $\rho$ that are similar to those on the relation $\lambda$ in section 3.

**Proposition 4.4.**

1) $\rho$ is an equivalence relation on the $m \times n$ spanning column rank 1 matrices over $U_+$.

2) For any nonempty set $F$ of $m \times n$ spanning column rank 1 matrices over $U_+$, the members of $F$ have a common strong right factor if and only if $X \rho Y$ for all $X, Y$ in $F$.

*Proof. Similar to the proof of Proposition 3.2. ■*

Thus the $\rho$-equivalence classes are the maximal strong right factor spaces in $M_{m,n}(U_+)$. These in turn are of the form $V(a) = \{xa^t \mid x \in (U_+)^m\}$, where $a^t$ has spanning column rank 1.
THEOREM 4.5. Suppose that $V$ is a subspace of $\mathbb{M}_{m,n}(U_+)$ with $\min(m,n) \geq 2$. Then $V$ is a spanning column rank 1 space if and only if $V$ is a strong right factor space.

Proof. Suppose $V$ is a spanning column rank 1 space. For every $A$ and $B$ in $V$, $sc(\alpha A + \beta B) = 1$ for any $\alpha, \beta \in U_+$, not both zero. Then $A$ and $B$ have a common strong right factor by Lemma 4.3. Therefore $V$ is a strong right factor space by Proposition 4.4.

The converse is immediate. ■

Thus we have a structure theorem for spanning column rank 1 space in $\mathbb{M}_{m,n}(U_+)$. 

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