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THE DOUBLY STOCHASTIC MATRICES OF A MULTIVARIATE MAJORIZATION

GI-SANG CHEON* AND YOU-HO LEE

1. Introduction

Throughout this paper, let $M_{mn}(R)$ be the set of all $m \times n$ real matrices, and let R^n the set of all real row vectors with n components.

For integers k, n with $1 \leq k \leq n$, let $V_{k,n}$ denote the set of all $1 \times n$ (0,1)-matrices whose entries have sum k. For $\mathbf{a}, \mathbf{b} \in \mathbf{R}^n$, $\mathbf{a}$ is said to be *majorized* by $\mathbf{b}$, written as $\mathbf{a} \prec \mathbf{b}$, if

$$\max\{\mathbf{a}\mathbf{v}^{\mathbf{T}} | \mathbf{v} \in V_{k,n}\} \leq \max\{\mathbf{b}\mathbf{v}^{\mathbf{T}} | \mathbf{v} \in V_{k,n}\} \tag{1.1}$$

for all $k = 1, \cdots, n$ and the equality holds in (1.1) when $k = n$.

The definition of the majorization $\mathbf{a} \prec \mathbf{b}$ is motivated as a way of making precise the idea that the components of $\mathbf{a}$ are "less spread out" than the components of $\mathbf{b}$.

As usual, let Ω_n denote the set of all $n \times n$ doubly stochastic matrices, which is known to be a convex polytope of dimension $(n-1)^2$ with $n!$ vertices in the n^2-dimensional Euclidean space. It is well known that $\mathbf{a} \prec \mathbf{b}$ for $\mathbf{a}$, $\mathbf{b} \in \mathbf{R}^n$ if and only if there exists $X \in \Omega_n$ with $\mathbf{a} = \mathbf{b}X$.

Let $\mathbf{a}, \mathbf{b} \in \mathbf{R}^n$ with $\mathbf{a} \prec \mathbf{b}$, and let

$$\Omega_n(\mathbf{a} \prec \mathbf{b}) = \{X \in \Omega_n | \mathbf{a} = \mathbf{b}X\}. \tag{1.2}$$

Then $\Omega_n(\mathbf{a} \prec \mathbf{b})$ forms a convex polytope and hence a subpolytope of Ω_n. We call it the *polytope of the majorization* $\mathbf{a} \prec \mathbf{b}$.

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For $A,\ B \in M_{mn}(R)$, A is said to be *multivariate majorized* by B, written $A \prec B$, if there exists $X \in \Omega_n$ with $A = BX$. For $A,\ B \in M_{mn}(R)$ with $A \prec B$, let

$$\Omega_{mn}(A \prec B) = \{X \in \Omega_n | A = BX\}. \tag{1.3}$$

Then it follows that $\Omega_{mn}(A \prec B)$ is a convex polytope. We call it the *multivariate majorization polytope.*

In this paper, we obtain necessary and sufficient conditions for Ω_{mn} $(A \prec B)$ to be a set of a single matrix, and we determine when Ω_{mn} $(A \prec B)$ contains a positive matrix. We also determine the support matrix of the multivariate majorization $A \prec B$.

2. Multivariate majorization polytope $\Omega_{mn}(A \prec B)$

For $A,\ B \in M_{mn}(R)$, throughout this paper let

$$A = (\mathbf{a}_1^R, \cdots, \mathbf{a}_m^R)^T \quad \text{and} \quad B = (\mathbf{b}_1^R, \cdots, \mathbf{b}_m^R)^T \tag{2.1}$$

where $\mathbf{a}_i^R$ and $\mathbf{b}_i^R$ are row vectors of length n of A and B respectively.

Notice that $A \prec B$ if and only if there exists a doubly stochastic matrix X such that

$$\mathbf{a}_i^R = \mathbf{b}_i^R X, \quad i = 1, \cdots, m. \tag{2.2}$$

This implies that

$$\mathbf{a}_i^R \prec \mathbf{b}_i^R, \quad i = 1, \cdots, m. \tag{2.3}$$

But (2.3) does not imply (2.2) because there is no guarantee that one can take $X_1 = \cdots = X_m$ in Ω_n such that $\mathbf{a}_i^R = \mathbf{b}_i^R X_i$ for each $i = 1, \cdots, m$.

A vector $\mathbf{x} = (x_1, \cdots, x_n) \in R^n$ is called a *scalar vector* if $x_1 = \cdots = x_n$, and let J_s denote the $s \times s$ matrix all of whose entries are $1/s$.

THEOREM 2.1. *For A, $B \in M_{mn}(R)$ in (2.1), let $\mathbf{a}_i^R \prec \mathbf{b}_i^R$, $i = 1, \cdots, m$. Then $A \prec B$ if $\mathbf{a}_i^R$ and $\mathbf{b}_i^R$ are conformally partitioned as $(\mathbf{a}_i^{(p_1)}, \cdots, \mathbf{a}_i^{(p_s)})$ and $(\mathbf{b}_i^{(p_1)}, \cdots, \mathbf{b}_i^{(p_s)})$ respectively, where $\mathbf{a}_i^{(p_j)} \prec \mathbf{b}_i^{(p_j)}$ and $\mathbf{a}_i^{(p_j)}$ is a scalar vector with length p_j for each $j = 1, \cdots, s$.*

Proof. From the assumption on $\mathbf{a}_i^R$ and $\mathbf{b}_i^R$ we have

$$\mathbf{a}_i^R = \mathbf{b}_i^R P, \quad i = 1, \cdots, m$$

where $P = J_{p_1} \oplus \cdots \oplus J_{p_s}$, $p_1 + \cdots + p_s = n$. Thus $A \prec B$ which completes the proof.

LEMMA 2.2. *Let A, $B \in M_{mn}(R)$ with $A \prec B$. Then*

$$\Omega_{mn}(A \prec B) = \cap_{i=1}^{m} \Omega_n(\mathbf{a}_i^R \prec \mathbf{b}_i^R). \tag{2.4}$$

Proof. Let $X \in \cap_{i=1}^{m} \Omega_n(\mathbf{a}_i^R \prec \mathbf{b}_i^R)$. Then we get $\mathbf{a}_i^R = \mathbf{b}_i^R X$ for each $i = 1, \cdots, m$. Thus $A = BX$ and it follows that $X \in \Omega_{mn}(A \prec B)$.

The converse is easily proved by the definition of $A \prec B$. Hence the Lemma holds.

COROLLARY 2.3. *The set $\Omega_{mn}(A \prec B)$ forms a convex polytope.*

A square matrix X is called *partly decomposable* if there exist permutation matrices P and Q such that

$$PXQ = \begin{bmatrix} A & O \\ C & B \end{bmatrix},$$

where A and B are square matrices of order ≥ 1. If X is not partly decomposable, it is called a *fully indecomposable.*

For a matrix $X = [x_{ij}]$, $X[\alpha|\beta]$ will denote the $h \times k$ matrix whose (p, q) entry is x_{pq}, $p \in \alpha$ and $q \in \beta$ where $\alpha = \{1, \cdots, h\}$ and $\beta = \{1, \cdots, k\}$.

THEOREM 2.4. *Let A, $B \in M_{mn}(R)$ with $A \prec B$. Then $\Omega_{mn}(A \prec B)$ has consists of a single doubly stochastic matrix if and only if one of the following holds ;*

(i) *dim $\Omega_n(\mathbf{a}_i{}^R \prec \mathbf{b}_i{}^R) = 0$ for some $i = 1, \cdots, m$;*

(ii) *rank* $(B) = n - 1$ *or* n.

Proof. Note that $\Omega_{mn}(A \prec B)$ consists of all nonnegative solutions of the following linear matrix equation in the $(n-1)^2$ variables x_{ij} of $X = [x_{ij}] \in \Omega_n$:

$$A = BX. \tag{2.5}$$

First suppose that neither (i) nor (ii). Then it is easily shown that the matrix equation (2.5) has many solutions.

Next suppose that (i) holds. Then the polytope $\Omega_n(\mathbf{a}_i^R \prec \mathbf{b}_i^R)$ has an unique doubly stochastic matrix X with $\mathbf{a}_i^R = \mathbf{b}_i^R X$. Hence it follows from Lemma 2.2 that there is an unique doubly stochastic matrix in $\Omega_{mn}(A \prec B)$.

Now suppose that (ii) holds. That is, first let rank $(B) = n - 1$. Then $m \geq n - 1$ and without loss of generality, we may assume that

$$\det B[\alpha|\alpha] \neq 0 \quad \text{for } \alpha = \{1, \cdots, n-1\}.$$

Thus from (2.5) we have, for each $j = 1, \cdots, n$ and for $A = [a_{ij}]$, $B = [b_{ij}] \in M_{mn}(R)$,

$$(*)\begin{cases} b_{11}x_{1j} + \cdots + b_{1\ n-1}x_{n-1\ j} = a_{1j} - b_{1n}x_{nj} \\ \quad \cdots \\ b_{n-1\ 1}x_{1j} + \cdots + b_{n-1\ n-1}x_{n-1\ j} = a_{n-1\ j} - b_{n-1\ n}x_{nj} \end{cases}$$

and since $x_{nj} = 1 - (x_{1j} + \cdots + x_{n-1\ j})$ for each $j = 1, \cdots, n$, the linear system $(*)$ has the unique solution depend on $x_{1j}, \cdots, x_{n-1\ j}$. It follows that X is uniquely determined. Next if rank $(B) = n$ then it is clear. Hence the proof is completed.

From now on, we will assume that dim $\Omega_n(\mathbf{a}_i{}^R \prec \mathbf{b}_i{}^R) \neq 0$ for all $i = 1, \cdots, m$ and rank $(B) \leq n - 2$.

3. The support matrix of the multivariate majorization

It will be almost impossible to determine the all nonnegative solutions X in (2.5). But we will be able to determine the zero-one pattern of X.

For $\mathbf{a} = (a_1, \cdots, a_n)$ and $\mathbf{b} = (b_1, \cdots, b_n)$ in R^n with $\mathbf{a} \prec \mathbf{b}$, without loss of generality, we may assume that $\mathbf{a}$ and $\mathbf{b}$ are *monotone* in the sense that $a_1 \geq \cdots \geq a_n$ and $b_1 \geq \cdots \geq b_n$.

For integers k, n with $1 \leq k \leq n$, we say that $\mathbf{a} \prec \mathbf{b}$ has a *coincidence* at k if $a_1 + \cdots + a_k = b_1 + \cdots + b_k$, and is *$k$-decomposable* if it has a coincidence at k ($< n$) and $b_k > b_{k+1}$.

Notice that (Levow [4]) if $\mathbf{a} \prec \mathbf{b}$ is k-decomposable then $\Omega_n(\mathbf{a} \prec \mathbf{b})$ consists of all the matrices of the form

$$X = \begin{bmatrix} X_1 & O \\ O & X_2 \end{bmatrix},$$

where X_1 is a $k \times k$ matrix.

Thus from Lemma 2.2 it is sufficient to look at only those $\mathbf{a}_i^R \prec \mathbf{b}_i^R$ which are not k-decomposable for all $i = 1, \cdots, m$.

Suppose that $\mathbf{a} \prec \mathbf{b}$ has a coincidence at k ($< n$), but is not k-decomposable. Then there exist integers j_i, k_i, k_i' and l_i, ($i = 1, \cdots, p$), such that

$$\begin{cases} \text{(i)} & \text{the only coincidences of } \mathbf{a} \prec \mathbf{b} \text{ occur at } k_i,\ k_i + 1, \cdots, k_i'; \\ \text{(ii)} & k_{i-1}' < l_{i-1} < j_i \leq k_i \leq k_i'; \\ \text{(iii)} & x_{j_i - 1} > x_{j_i} = \cdots = x_{k_i} = \cdots = x_{k_i'} = \cdots = x_{l_i} > x_{l_i+1}. \end{cases} \tag{3.1}$$

Note that $k_p' = l_p = n$, and we shall also use $k_0' = l_0 = 0$ and $j_0 = 1$.

For the matrix $X = [x_{ij}] \in \Omega_n$, we define the *support matrix* of X to be the $n \times n$ (0,1)-matrix with the $\text{supp}(X) = [d_{ij}]$ satisfying

$$d_{ij} = \begin{cases} 1 & \text{if } x_{ij} > 0 \ \ (i, j = 1, \cdots, n), \\ 0 & \text{otherwise.} \end{cases}$$

And we define the *support matrix of the majorization* $\mathbf{a} \prec \mathbf{b}$ to be the $n \times n$ (0,1)-matrix D satisfying the following two properties :

(i) $X \in \Omega_n(\mathbf{a} \prec \mathbf{b})$ implies $X \leq D$;
(ii) there is a matrix $Y \in \Omega_n(\mathbf{a} \prec \mathbf{b})$ such that $\text{supp}(Y) = D$.

Similarly, the *support matrix of the multivariate majorization* $A \prec B$ is defined.

THEOREM 3.1. (R.A. Brualdi [1]) *Let* $\mathbf{a}$, $\mathbf{b} \in \mathbf{R}^n$ *and suppose* $\mathbf{a} \prec \mathbf{b}$ *is not* k*-decomposable for each* $k = 1, \cdots, n-1$. *Then the support matrix of* $\mathbf{a} \prec \mathbf{b}$ *is the* $n \times n$ $(0,1)$*-matrix* $U = [u_{rs}]^T$ *such that for* $1 \leq r, s \leq n$, $u_{rs} = 1$ *if and only if for some* $i = 1, \cdots, p$,

$$k'_{i-1} + 1 \leq r \leq k_i,\ j_{i-1} \leq s \leq l_i \ \text{or}\ k_i + 1 \leq r \leq k'_i,\ j_i \leq s \leq l_i.$$

From now on, we suppose that $A = (\mathbf{a}_1^R, \cdots, \mathbf{a}_m^R)^T$ and $B = (\mathbf{b}_1^R, \cdots, \mathbf{b}_m^R)^T$ are *monotone* in the sense that $\mathbf{a}_i^R$ and $\mathbf{b}_i^R$ are monotone for all $i = 1, \cdots, m$.

For $1 \times n$ matrix $\mathbf{e}_n = [1, \cdots, 1]$ and A, $B \in M_{mn}(R)$ with $A \prec B$, let

$$\widehat{A} = \begin{bmatrix} \mathbf{e}_n \\ A \end{bmatrix} \quad \text{and} \quad \widehat{B} = \begin{bmatrix} \mathbf{e}_n \\ B \end{bmatrix}.$$

Then $\widehat{A} \prec \widehat{B}$ and $\Omega_{mn}(A \prec B) = \Omega_{m+1\ n}(\widehat{A} \prec \widehat{B})$. That is, the solution set of (2.5) is identical with the solution set of

$$\widehat{A} = \widehat{B}\, X. \tag{3.2}$$

Suppose that the system of equations, (3.2), is transformed by row operations into the system of equations,

$$\widehat{A}' = \widehat{B}'\, X \tag{3.3}$$

where $\widehat{A}'$ and $\widehat{B}'$ are obtained by reducing the augmented matrix $[\widehat{B}|\widehat{A}]$ for (3.2) to its row-echelon form. Thus we have the following.

LEMMA 3.2. *Let* A, $B \in M_{mn}(R)$ *with* $A \prec B$ *and let* $\widehat{A}'$, $\widehat{B}'$ *be the same as (3.3). Then* $\widehat{A}' \prec \widehat{B}'$ *and*

$$\Omega_{mn}(A \prec B) = \Omega_{m+1\ n}(\widehat{A}' \prec \widehat{B}').$$

For rank $B = k \leq n-2$, we may assume that

$$\widehat{A}' = \begin{bmatrix} 1 & \cdots & 1 \\ a'_{11} & \cdots & a'_{1n} \\ \vdots & & \vdots \\ a'_{k1} & \cdots & a'_{kn} \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \end{bmatrix} \quad \text{and} \quad \widehat{B}' = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 0 & \mathbf{b}'_1 & & \\ \vdots & \ddots & \ddots & \\ 0 & \cdots & 0 & \mathbf{b}'_k \\ 0 & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & 0 \end{bmatrix}. \tag{3.4}$$

THEOREM 3.3. *Let A, $B \in M_{mn}(R)$ with $A \prec B$ such that $\mathbf{a}_i^R \prec \mathbf{b}_i^R$ has no coincidences for all $i = 1, \cdots, m$, and let a'_{ij} and $\mathbf{b}'_i$ be as in (3.4). Then there exists a positive matrix in $\Omega_{mn}(A \prec B)$ if and only if one of the followings holds:*

(i) *$\mathbf{a}_i^R$ is a scalar vector for all $i = 1, \cdots, m$;*

(ii) *for all $i = 1, \cdots, k$ and $j = 1, \cdots, n$, $a'_{ij} \neq 0$ and there is no a'_{ij} such that $\mathbf{b}'_i = a'_{ij}\mathbf{e}$, where $\mathbf{e}$ is a vector of 1's of suitable sizes.*

Proof. First suppose that neither (i) nor (ii) holds. Then $\mathbf{a}_s^R$ and $\mathbf{b}_s^R$ are not scalar vectors for some $s \in \{1, \cdots, m\}$, and hence $\mathbf{b}'_s$ is not scalar vector since $\mathbf{a}_s^R \prec \mathbf{b}_s^R$ has no coincidences. It is easily seen that for a scalar vector $\mathbf{a}_i^R = [a_{ij}]_{1\times n}$, if $a'_{ik} = 0$ or $\mathbf{b}'_i = a'_{ik}\mathbf{e}$ for some $k \in \{1, \cdots, n\}$ then $\mathbf{b}_i^R$ is also scalar vector. Thus $\mathbf{a}_i^R \prec \mathbf{b}_i^R$ has coincidences.

Hence we may assume that $a'_{st} = 0$ for some $s \in \{1, \cdots, k\}$ and $t \in \{1, \cdots, n\}$. Then from (3.3) we get

$$[\mathrm{O}\ \mathbf{b}'_s][x_{1t} \cdots x_{nt}]^T = 0.$$

Since all the entries of $\mathbf{b}'_s$ have equal sign and are not zero, there exists a zero entry in the tth column of X.

Now if $\mathbf{b}'_s = a'_{sh}\mathbf{e}$ for some $h \in \{1, \cdots, n\}$ then from (3.3) we get

$$[\mathrm{O}\ t \cdots t][x_{1h} \cdots x_{nh}]^T = t, \quad (a'_{sh} = t).$$

Thus there exists a zero entry in the hth column of X.

Next suppose that (i) holds. Then the positive matrix with all entries equal to 1/n belongs to $\Omega_{mn}(A \prec B)$.

Now suppose that (ii) holds. Then we can choose a positive matrix in the solution set of (3.3). Hence the proof is completed.

LEMMA 3.4. *Let D be the support matrix of $A \prec B$ and let D_i be the support matrix of $\mathbf{a}_i^R \prec \mathbf{b}_i^R$, $i = 1, \cdots, m$. Then*

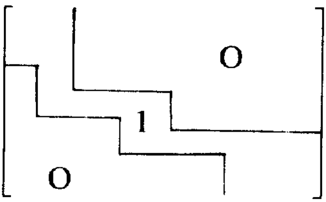
$$D \leq D_1 * \cdots * D_m$$

where $*$ stands for the Hadamard (entrywise) product.

Proof. Let $X \in \Omega_{mn}(A \prec B)$. Then from Lemma 2.2 we get $X \in \Omega_n(\mathbf{a}_i^R \prec \mathbf{b}_i^R)$ for each $i = 1, \cdots, m$. Thus we have $X \leq D_i$ for each $i = 1, \cdots, m$. It follows that $D \leq D_1 * \cdots * D_m$.

Since D_i's, $i = 1, \cdots, m$, are double staircase matrices, we may assume that

$$D_1 * \cdots * D_m =$$



, (3.5)

and since D_i's $i = 1, \cdots, m$ are fully indecomposable, the matrix (3.5) is fully indecomposable.

EXAMPLE 1. (1) For $A,\ B \in M_{24}(R)$, let

$$A = \begin{bmatrix} 3 & 3 & 2 & 2 \\ 3 & 2 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 2 & 2 & 2 \\ 3 & 3 & 0 & 0 \end{bmatrix}.$$

Then $A \prec B$ and since, from Theorem 3.1,

$$D_1 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

we have

$$D_1 * D_2 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}.$$

In fact, $\Omega_{24}(A \prec B)$ consists of all doubly stochastic matrix of the form

$$X = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/6 & 1/3 & 0 \\ 0 & & * & \\ 0 & & & \end{bmatrix} \qquad (3.6)$$

where $*$ can be chosen freely as long as X remains doubly stochastic. Thus the support matrix of $A \prec B$ is $D_1 * D_2$.

(2) For $E,\ F \in M_{24}(R)$, let

$$E = \begin{bmatrix} 3 & 3 & 2 & 2 \\ 7 & 6 & 4 & 3 \end{bmatrix}, \quad F = \begin{bmatrix} 4 & 2 & 2 & 2 \\ 8 & 6 & 4 & 2 \end{bmatrix}.$$

Then $E \prec F$ and since, from Theorem 3.1,

$$D_1 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

we have

$$D_1 * D_2 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

But, if $X = [x_{ij}] \in \Omega_{24}(E \prec F)$ then $x_{31} = x_{41} = 0$. Thus the support matrix of $E \prec F$ is not $D_1 * D_2$.

Let $X = [x_{ij}]$ be a matrix of positive variables, and let

$$Y = [y_{ij}] = (D_1 * \cdots * D_m) * X. \tag{3.7}$$

Then $\operatorname{supp}(Y) = D_1 * \cdots * D_m$.

Let $\widehat{B}'$ in (3.4) be an $(m+1)\times n$ matrix $[\mathbf{e}_n\ \widehat{\mathbf{b}}'_1 \cdots \widehat{\mathbf{b}}'_k\ O_{m-k}]^T$ and let Y in (3.7) be an $n \times n$ matrix $[\mathbf{y}_1^C \cdots \mathbf{y}_n^C]$ where $\mathbf{y}_j^C$ is the jth column vector of Y. Then we may assume that

$$\widehat{\mathbf{b}}'_i = [0 \cdots 0\ b'_{iu_i} \cdots b'_{in}], \quad 2 \le u_i \le n \tag{3.8}$$

and

$$\mathbf{y}_j^C = [0 \cdots 0\ y_{s_j j} \cdots y_{l_j j}\ 0 \cdots 0]^T, \quad 1 \le s_j < l_j \le n. \tag{3.9}$$

Let

$$\mathcal{S} = \{(i,j) \mid \widehat{\mathbf{b}}'_i \cdot \mathbf{y}_j^C \neq 0\} \tag{3.10}$$

where $\cdot$ denotes the dot product.

THEOREM 3.6. *Let D be the support matrix of $A \prec B$ and let D_i be the support matrix of $\mathbf{a}_i^R \prec \mathbf{b}_i^R$, $i = 1, \cdots, m$. Then*

$$D = D_1 * \cdots * D_m$$

if and only if both of the following conditions holds:

(i) *$a'_{ij} \neq 0$ for all (i,j) in $\mathcal{S}$;*

(ii) *for all $(i,j) \in \mathcal{S}$ such that $u_i > s_j$ in (3.8) and (3.9), not all overlapping nonzero elements of $\widehat{\mathbf{b}}'_i$ and $(\mathbf{y}_j^C)^T$ are equal to a'_{ij}.*

Proof. First suppose that (i) or (ii) does not hold. Since all the entries in $\mathbf{b}'_i$, $i = 1, \cdots, k$, have equal sign, note that $\widehat{\mathbf{b}}'_i \cdot \mathbf{y}_j^C \neq 0$ means that there exist nonzero entries $b'_{iu_k}, \cdots, b'_{il_j}$, $\max\{u_i, s_j\} \leq u_k \leq l_j$, such that

$$a'_{ij} = \widehat{\mathbf{b}}'_i \cdot \mathbf{y}_j^C = b'_{iu_k} y_{u_k j} + \cdots + b'_{il_j} y_{l_j j} \tag{3.11}$$

Let $a'_{ij} = 0$ for some (i,j) in $\mathcal{S}$. Then from $\widehat{A}' = \widehat{B}'Y$ and (3.11), we have

$$y_{u_k j} = \cdots = y_{l_j j} = 0$$

which implies that $\text{supp}(Y) \neq D$.

Now let there exist s'_j, $s_j < s'_j \leq l_j$, such that $b'_{is'_j} = \cdots = b'_{il_j} = a'_{ij}$. From (3.11) we get $y_{s'_j j} + \cdots + y_{l_j j} = 1$. Thus $y_{s_j j} = 0$, which implies that supp $(Y) \neq D$.

Next suppose that (i) and (ii) hold. Then $D = D_1 * \cdots * D_m$ and $X \leq D$ for any $X \in \Omega_{m+1\,n}(\widehat{A}' \prec \widehat{B}')$. Thus we can choose Y such that $\text{supp}(Y) = D$ in the solution set of linear matrix equation

$$\widehat{A}' = \widehat{B}' X, \quad X \in \Omega_n.$$

From Lemma 3.4 and Lemma 3.2 the proof is completed.

In (1) of Example 1, we get

$$\widehat{A}' = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & -2 & -2 \\ 0 & -1 & -2 & -3 \end{bmatrix}, \quad \widehat{B}' = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & -2 & -2 \\ 0 & 0 & -3 & -3 \end{bmatrix},$$

and from (3.10) we get

$$\mathcal{S} = \{(2,1), ((2,2), (2,3), (2,4), (3,2), (3,3), (3,4)\}.$$

Since (i) and (ii) in Theorem 3.6 hold for all $(i,j) \in \mathcal{S}$, the support matrix of $A \prec B$ is $D_1 * D_2$.

REMARK. The majorization polytope $\Omega_n(\mathbf{a} \prec \mathbf{b})$ is known to contain doubly stochastic matrices of very special type [5,p.40] such as, products of a finite number (or at most $n-1$) T-*transforms*; *orthostochastic matrices*; and *uniformly tapered matrices*.

But, it is easily shown that the doubly stochastic matrices of the form as X in (3.6) are not above special matrices.

Thus we conclude that all multivariate majorization polytopes Ω_{mn} $(A \prec B)$ do not contain products of T-transforms, orthostochastic, and uniformly tapered matrices.

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Gi-Sang Cheon
Department of Mathematics
Dae Jin University
Pocheon 487-800, Korea

You-Ho Lee
Department of Mathematics
Sung Kyun Kwan University
Suwon 440-746, Korea