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STABILITY ANALYSIS FOR A DISSIPATIVE FEEDBACK CONTROL LAW

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1. Introduction

Piezo devices such as piezoceramic patches knwon as collocated rate sensor and actuators are commonly used in control of flexible structure (see, e.g., [1]) and noise reduction. Recently, Ito and Kang ([4]) developed a nonlinear feedback control synthesis for regulating fluid flow using these devices. The control law is designed for driving a given fluid flow to a prescribed equilibrium state and enhancing energy dissipation effects. The controlled system becomes stable while control is activated. In this paper, the two-dimensional Navier-Stokes (N-S) equations with periodic boundary conditions are employed. A sufficient condition on control distribution vectors for changing flow state is obtained. Also, energy dissipation effects (exponential stability property) of the controlled dynamics is analyzed. These results extend those of [4]. Specifically, the condition on control distribution vectors are weakened so that they can be chosen with more flexibilities. For a convergent numerical scheme for solving the N-S equations and computational experiments, see [3,4,5].

A control problem and a weak variational form for the N-S equations are explained briefly in Section 2. In Section 3, the well-posedness and the exponential stability properties of the controlled system are given.

Throughout this paper notations are very standard. We will use the notation $|\cdot|$ without any subindex for vector or operator norm. In all such cases the appropriate index for $|\cdot|$ will be understood from the

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context. For a given Banach space X, X^* and $\langle\cdot,\cdot\rangle_{X^*,X}$ denote the strong dual space of X and the dual product, respectively. If X is a Hilbert space, $\langle\cdot,\cdot\rangle$ is the scalar inner product.

2. Controlled dynamics

Consider a control problem for the two-dimensional Navier-Stokes equations with periodic boundary conditions. For simplicity, the period in space is chosen to be 1. Let $\Omega = (0,1)\times(0,1)$, and let $e_1 = (1,0)$ and $e_2 = (0,1)$ be the canonical basis elements of $\mathbf{R}^2$. The governing equations are given by

$$
\begin{aligned}
\frac{\partial u}{\partial t} - \nu\Delta u + (u\cdot\nabla)u + \nabla p &= -\gamma(t)\sum_{i=1}^{m} b_i \int_\Omega b_i\cdot(u-u_e)\,dx,\\
\nabla\cdot u &= 0,\\
u(t,x+e_i) &= u(t,x), \qquad x\in\mathbf{R}^2, \quad t>0,\\
u(0,x) &= u_0(x),
\end{aligned}
\tag{2.1}
$$

where $x=(x_1,x_2)\in\mathbf{R}^2$, $u=u(t,x)=(u_1(t,x),u_2(t,x))$ is the velocity vector, $\nu>0$ is the nondimensional viscosity, $p=p(t,x)$ is the pressure, $b_i=(b_{i1}(x),b_{i2}(x))$ with $\nabla\cdot b_i=0$, $1\le i\le m$, are control input vectors, $u_e=u_e(x)$ is a desired equilibrium velocity field, and $\gamma(t)$ is the control law given by

$$
\gamma(t) = \frac{-\nu\,|\nabla(u-u_e)|^2 + \sqrt{g_1(u,u_e) + |\mathcal{B}^*(u-u_e)|_U^2 g_2(u,u_e)}}{|\mathcal{B}^*(u-u_e)|_U^2},
\tag{2.2}
$$

where $|\nabla(u-u_e)|^2 = \sum_{i=1}^2\int_\Omega|\nabla(u_i-u_{ei})|^2dx$, $u=(u_1,u_2)$, $u_e=(u_{e1},u_{e2})$, $U=\mathbf{R}^m$ is the control space,

$$
\begin{aligned}
g_1(u,u_e) &= \nu^2|\nabla(u-u_e)|^4,\\
g_2(u,u_e) &= |\nabla(u-u_e)|^2 + \alpha^2|\mathcal{B}^*(u-u_e)|_U^2,
\end{aligned}
\tag{2.3}
$$

α with $1\le\alpha\le\frac{1}{2\nu}$ is the weight on control action, $|\mathcal{B}^*(u-u_e)|_U^2 = \sum_{i=1}^m\left(\int_\Omega b_i(x)\cdot(u(t,x)-u_e(x))dx\right)^2$, and $\mathcal{B}^*$ is the adjoint operator of $\mathcal{B}$ which is to be defined later by equation (2.13). This $\gamma(t)$ drives the given initial state $u(0,\cdot) = u_0(\cdot)$ to the desired equilibrium state $u_e(\cdot)$ and enhances the energy dissipation effects of the system (2.1) (see [4]). It is a "*suboptimal*" control synthesis obtained from the following nonlinear optimal control problem:

Find the optimal control $\gamma(t) \in L^2(0,\infty;\mathbf{R}^+)$ *that minimizes the cost functional*

$$J(\gamma) = \frac{1}{2}\int_0^\infty \left(|\nabla(u-u_e)|^2 + (\alpha^2+\gamma^2)|\mathcal{B}^*(u-u_e)|^2\right) dt$$

subject to the control system (2.1).

Here, by the *suboptimal* control law we mean that it contains an essential part of a linear optimal control law, and one of its special forms satisfies the Hamilton-Jacobi-Bellman equation arising in a nonlinear programming problem which corresponds to the above optimal control problem.

For the well-posedness and the stability analyses of the controlled system (2.1), consider the following function spaces (see [7]).

$$\begin{aligned} H_p^m(\Omega) &= \{\, u \in H_{loc}^m(\mathbf{R}^2) \,:\, u(x+e_i) = u(x),\ i=1,2\,\}, \\ V &= \{\, u \in H_p^1(\Omega)\times H_p^1(\Omega) \,:\, \nabla\cdot u = 0\,\}, \\ H &= \{\, u \in H_p^0(\Omega)\times H_p^0(\Omega) \,:\, \nabla\cdot u = 0\,\}, \end{aligned} \tag{2.4}$$

where the subscript p stands for "periodic." The Stokes operator $\mathcal{A}$ is defined by

$$\langle \mathcal{A}u, v\rangle = \sum_{i=1}^{2}\int_\Omega \nabla u_i\cdot\nabla v_i\,dx \tag{2.5}$$

for $u, v \in V$ and it is given by

$$\mathcal{A}u = -\Delta u, \qquad \text{for all } u \in \mathcal{D}(\mathcal{A}) = (H_p^2(\Omega)\times H_p^2(\Omega))\cap H \tag{2.6}$$

due to the periodic boundary conditions. Define a bilinear form a on $V \times V$ by

$$a(u,v) = \nu\langle\mathcal{A}u, v\rangle + \gamma(t)\sum_{i=1}^{m}\Big(\int_\Omega b_i\cdot u\,dx\Big)\Big(\int_\Omega b_i\cdot v\,dx\Big), \tag{2.7}$$

where $\gamma(t)$ is the control law defined by equation (2.2). For any u, v, $w \in V$, define a trilinear form

$$b(u; v, w) = \sum_{i,j=1}^{2} \int_\Omega u_i(D_i v_j) w_j \, dx, \tag{2.8}$$

where $D_i = \frac{\partial}{\partial x_i}$, $i = 1, 2$, and a bilinear continuous operator B from $V \times V$ into V^* by

$$\langle B(u, v), w\rangle_{V^*, V} = b(u; v, w), \tag{2.9}$$

where V^* is the dual space of V. It is easy to see that, by integration by parts,

$$b(u; v, w) + b(u; w, v) = 0 \tag{2.10}$$

for all $u, v, w \in V$ due to the periodic boundary conditions.

With the trilinear form b, the variational form of the control system (2.1) becomes

$$\begin{aligned} &\left\langle \frac{d}{dt} u, v \right\rangle + \nu\langle \mathcal{A}u, v\rangle + b(u; u, v) \\ &+ \gamma(t) \sum_{i=1}^{m} \left(\int_\Omega b_i \cdot (u - u_e) dx \right) \left(\int_\Omega b_i \cdot v dx \right) = 0, \quad v \in V, \\ &u(0) = u_0, \end{aligned} \tag{2.11}$$

where $u(t) = u(t, \cdot)$. Here, the equilibrium state u_e satisfies $-\nu \Delta u_e + (u_e \cdot \nabla) u_e = 0$ in V^*, i.e., $\langle \nu \mathcal{A} u_e, v\rangle + b(u_e; u_e, v) = 0$ for all $v \in V$. Thus, from equation (2.11),

$$\begin{aligned} &\left\langle \frac{d}{dt}(u - u_e), v \right\rangle + a(u - u_e, v) + b(u - u_e; u - u_e, v) \\ &+ b(u - u_e; u_e, v) + b(u_e; u - u_e, v) = 0 \quad \text{for all } v \in V. \end{aligned} \tag{2.12}$$

It is easy to observe that the pressure term ∇p in equation (2.1) is dropped in the variational forms (2.11) and (2.12) due to the divergence free condition.

We now define a nonlinear operator $\mathcal{F} \in \mathcal{L}(V, V^*)$ and a control input operator $\mathcal{B} \in \mathcal{L}(U, H)$ by

$$\mathcal{F}(u) = \mathrm{P}B(u, u) \qquad \text{and} \qquad \mathcal{B}f = \sum_{i=1}^{m} b_i(x) f_i, \tag{2.13}$$

for all $u \in V$ and $f = (f_1, f_2, \cdots, f_m) \in U$, where P is the projection operator from $H_p^0(\Omega) \times H_p^0(\Omega)$ onto the state space H, B is the bilinear operator defined by equation (2.9), and $b_i \in H$, $1 \leq i \leq m$. Then the adjoint operator $\mathcal{B}^* \in \mathcal{L}(H, U)$ of $\mathcal{B}$ is given by

$$\mathcal{B}^* u = \left(\int_\Omega b_1 \cdot u dx, \int_\Omega b_2 \cdot u dx, \cdots, \int_\Omega b_m \cdot u dx \right) \quad \text{for all } u \in H.$$

Thus, the variational form (2.11) is equivalent, in V^*, to

$$\begin{aligned} \frac{d}{dt} u(t) + \nu \mathcal{A} u(t) + \mathcal{F}(u(t)) &= -\gamma(t) \mathcal{B}\mathcal{B}^*(u(t) - u_e), \\ u(0) &= u_0. \end{aligned} \tag{2.14}$$

The term $-\gamma(t)\mathcal{B}\mathcal{B}^*(u(t) - u_e)$ is one of passive feedback forms arising in piezo device control mechanisms. These piezo devices have a special sensing and actuating structure. That is, control actions are given at the same locations where sensors are located.

3. Well-posedness and stability

The following well-posedness property of the controlled system (2.1) is an application of arguments in [2,7,8].

THEOREM 3.1. *The control system (2.14) (equivalently, the system (2.1)) has a unique global weak solution* $u(\cdot) \in L^2(0, \infty; V) \cap C_{loc}(0, \infty; H) \cap H^1(0, \infty; V^*)$ *provided that* $u(0) = u_0 \in H$.

Proof. The bilinear form a defined equation (2.7) is continuous and V-coersive, i.e.,

$$\begin{aligned} |a(u, v)| &\leq M_1 |u|_V \, |v|_V \\ \text{and} \quad |a(u, u)| &\geq \nu \, |u|_V^2 - \nu \, |u|_H^2 \end{aligned}$$

for some constant $M_1 > 0$ and for all $u, v \in V$. From equations (2.8) and (2.10), the trilinear form b satisfies that

$$b(u; v, w) + b(u; w, v) = 0$$
$$\text{and} \quad |b(u; w, u)| \leq M_2\, |u|_H\, |u|_V\, |w|_V$$

for some constant $M_2 > 0$ and for all $u, v, w \in V$. Also,

$$|b(u - u_e; u - u_e, \cdot) + b(u - u_e; u_e, \cdot) + b(u_e; u - u_e, \cdot)|_{V^*} \leq (M_3|u - u_e|_H + M_4|u_e|_V)|u - u_e|_V$$

for some constants $M_3 > 0$ and $M_4 > 0$. Recall that $\gamma(t) \in L^2(0, \infty; \mathbf{R}^+)$ and $b_i \in H$, $1 \leq i \leq m$. Hence, by the standard arguments such as Lemma 8.4 in [2] or [7, p. 282], the system (2.12) (equivalently, the system (2.14)) has a unique solution $u - u_e \in L^2(0, T; V) \cap C(0, T; H) \cap H^1(0, T; V^*)$ for arbitrary $T > 0$. The continuity property $u - u_e \in C(0, T; H)$ follows from [6]. □

The exponential stability property of the controlled system (2.1) (equivalently, the system (2.14)) follows from the next theorem.

THEOREM 3.2. *Assume that the nondimensional viscosity $\nu > 0$ is sufficiently small, say, $0 < \nu << 1$, and that the control vectors $b_i \in H$, $1 \leq i \leq m$, and $\alpha > 0$ in equations (2.1)-(2.2) are chosen so that*

$$\nu\, |\nabla(u - u_e)|^2 + \alpha\, |\mathcal{B}^*(u - u_e)|_U^2 \geq \beta |u - u_e|_H^2 \tag{3.1}$$

for some $\beta > 0$ and for all $u \in V$. Then

$$|u(t, \cdot) - u_e(\cdot)|_H \leq e^{-\beta t} |u(0, \cdot) - u_e(\cdot)|_H. \tag{3.2}$$

Proof. By substituting $v = u - u_e$ in equation (2.12), we have the following estimate.

$$\left\langle \frac{d}{dt}(u - u_e), u - u_e \right\rangle + a(u - u_e, u - u_e) + b(u - u_e; u - u_e, u - u_e) + b(u - u_e; u_e, u - u_e) + b(u_e; u - u_e, u - u_e) = 0.$$

From equations (2.8) and (2.10), we have $b(u - u_e; u_e, u - u_e) = b(u - u_e; u - u_e, u - u_e) = b(u_e; u - u_e, u - u_e) = 0$. Hence, by the definitions

(2.7) and (2.13) of the bilinear form a and the control input operator $\mathcal{B}$, the above equation becomes

$$(3.3)\qquad \frac{1}{2}\frac{d}{dt}|u-u_e|^2+\nu|\nabla(u-u_e)|^2+\gamma(t)|\mathcal{B}^*(u-u_e)|^2=0.$$

Since for any $\omega>0$,

$$(3.4)\qquad \frac{1}{2}\frac{d}{dt}(e^{\omega t}|u-u_e|^2)=\frac{1}{2}e^{\omega t}\frac{d}{dt}|u-u_e|^2+\frac{\omega}{2}e^{\omega t}|u-u_e|^2,$$

by integrating equation (3.4) from 0 to t and from equation (3.3),

$$\begin{aligned}&\frac{1}{2}e^{\omega t}|u(t)-u_e|^2-\frac{1}{2}|u(0)-u_e|^2\\&=\int_0^t\left(\frac{1}{2}\frac{d}{ds}|u-u_e|^2+\frac{\omega}{2}|u-u_e|^2\right)e^{\omega s}ds\\&=\int_0^t\left(-\nu|\nabla(u-u_e)|^2-\gamma(s)|\mathcal{B}^*(u-u_e)|^2+\frac{\omega}{2}|u-u_e|^2\right)e^{\omega s}ds.\end{aligned}$$

By the definition (2.2) of $\gamma(t)$, the above equation becomes

$$(3.5)\qquad \begin{aligned}&\frac{1}{2}e^{\omega t}|u(t)-u_e|^2-\frac{1}{2}|u(0)-u_e|^2\\&=-\int_0^t\left(\sqrt{g_1(u,u_e)+|\mathcal{B}^*(u-u_e)|^2g_2(u,u_e)}-\frac{\omega}{2}|u-u_e|^2\right)e^{\omega s}ds,\end{aligned}$$

where $g_1(u,u_e)$ and $g_2(u,u_e)$ are given by equation (2.3). If the condition (3.1) holds,

$$\begin{aligned}&\sqrt{g_1(u,u_e)+|\mathcal{B}^*(u-u_e)|^2g_2(u,u_e)}\\&\quad\geq\sqrt{(\beta|u-u_e|^2)^2+(1-2\nu\alpha)|\nabla(u-u_e)|^2|\mathcal{B}^*(u-u_e)|^2}.\end{aligned}$$

Since $1\leq\alpha\leq\frac{1}{2\nu}$ and $0<\nu<<1$, $1-2\nu\alpha\geq 0$. Hence, from equation (3.5),

$$\frac{1}{2}e^{\omega t}|u(t)-u_e|^2+\int_0^t\left(\beta-\frac{\omega}{2}\right)|u-u_e|^2e^{\omega s}ds\leq\frac{1}{2}|u(0)-u_e|^2,$$

and, by setting $\omega=2\beta$, we have $e^{2\beta t}|u(t)-u_e|^2\leq|u(0)-u_e|^2$. □

REMARK 3.3. In Theorem 3.2, it is easy to see, from the Poincaré inequality (see, e.g., [8, p. 117]), that such $\beta > 0$ exists. The condition on $b_i \in H$ in equation (3.1) is a constraint for the control distribution vectors. Also, note that the velocity field $u(t, \cdot)$ approaches the desired equilibrium state $u_e(\cdot)$ in H as $t \to \infty$, and the approaching rate is exponential.

REMARK 3.4. It is easy to observe, from the proof of Theorem 3.2, that the exponential stability property (3.2) holds when the nondimensional viscosity ν satisfies $0 < \nu \leq \frac{1}{2}$.

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