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ESTIMATION OF THE SMALLER AND LARGER OF TWO PARETO SCALE PARAMETERS

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1. Introduction

Many authors have utilized a Pareto distribution because of its wide applicability in socio-economic, physical and biological phenomena. The problem considered in this paper is estimation of the minimum and maximum of two unknown Pareto scale parameters. Similar problems were considered by Blumenthal and Cohen(1968a, b, and c), who were estimating the large translation parameter of two symmetric(mainly normal) distribution. Kushary and Cohen(1989) considered a similar but somewhat different problem when it was known which populations correspond to each scale parameter. These authors found improved estimators of scale parameters, in a general setup, under the condition that the first sample corresponded to the smaller scale parameter and the second to the larger. Recently, Carpenter and Pal(1992) have considered the estimation of the smaller and larger of two exponentially unknown location parameters. Elfessi and Pal(1992) have considered the estimation of the smaller and larger of two uniform scale parameters.

Let $X_{i1}, \cdots, X_{in}$, i = 1,2, be a pair of independent random samples from populations which are Pareto distributed with unknown scale parameters λ_i, i = 1,2, and a common known shape parameter α as follows ; for i = 1,2,

$$f(x_i; \alpha, \lambda_i) = \alpha \lambda_i^{\alpha} x_i^{-(\alpha+1)}, \ 0 < \lambda_i < x_i, \ 0 < \alpha. \tag{1.1}$$

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Define θ_1 = minimum(λ_1, λ_2) and θ_2 = maximum(λ_1, λ_2). A sufficient statistic for λ_i is $Y_i = \max\{X_{i1}, \cdots, X_{in}\}$, i = 1,2. Our goal is to estimate θ_1 and θ_2 based on Y_1 and Y_2.

In the next section, we shall introduce several estimators of θ_1 and θ_2, and compare them in terms of standardized bias and risk under square error loss function. The following definitions are used to clarify the various criteria used in this paper.

DEFINITION 1.1. Let $\hat{\theta}_i$ be an estimator of θ_i, i = 1,2.

(a) The standardized bias(s-bias) of $\hat{\theta}_i$ is defined as

$$B(\hat{\theta}_i) = E\left(\frac{\hat{\theta}_i}{\theta_i} - 1\right), \tag{1.2}$$

(b) The risk of $\hat{\theta}_i$ under the square error loss is defined as

$$R(\hat{\theta}_i) = E\left(\frac{\hat{\theta}_i}{\theta_i} - 1\right)^2. \tag{1.3}$$

The reason why B($\hat{\theta}_i$) and R($\hat{\theta}_i$) are considered instead of the usual bias and mean square error is the fact that they are invariant under scale transformations.

2. Estimation of the smaller and larger of two scales

Define $Z_1 = \min\{Y_1, Y_2\}$ and $Z_2 = \max\{Y_1, Y_2\}$. ML estimators of θ_1 and θ_2 are

$$\hat{\theta}_1^{(1)} = Z_1 \text{ and } \hat{\theta}_2^{(1)} = Z_2. \tag{2.1}$$

Note that the reparametrization from (λ_1, λ_2) to (θ_1, θ_2) is not one-to-one. Zehna(1966) defined the induced likelihood function $L_1(\theta_1, \theta_2 | y_1, y_2)$ as

$$L_1(\theta_1, \theta_2 | y_1, y_2) = \sup_U L(\lambda_1, \lambda_2 | y_1, y_2). \tag{2.2}$$

where $L(\lambda_1, \lambda_2 | y_1, y_2) = f_{Y_1}(y_1) \cdot f_{Y_2}(y_2)$,

$U = \{(\lambda_1, \lambda_2) | \theta_1 = \min(\lambda_1, \lambda_2)\ ,\ \theta_2 = \max(\lambda_1, \lambda_2)\}$.

It is easy to see that $(\hat{\theta}_1^{(1)}, \hat{\theta}_2^{(1)})$ maximizes the induced likelihood function $L_1(\theta_1, \theta_2 | y_1, y_2)$, and hence $(\hat{\theta}_1^{(1)}, \hat{\theta}_2^{(1)})$ is said to be an extended maximum likelihood estimator (MLE) of (θ_1, θ_2). Another way to define an MLE is to use the joint density of (Z_1, Z_2) as (see Dudewicz(1968)) the likelihood function in hope that (Z_1, Z_2) carries all of the information relevant for estimation of (θ_1, θ_2). The following Lemma gives the likelihood function $L_2(\theta_1, \theta_2 | z_1, z_2)$ of (θ_1, θ_2) based on (Z_1, Z_2).

LEMMA 2.1. *The joint density of* (Z_1, Z_2) *is*

$$g(z_1, z_2 | \theta_1, \theta_2) = \begin{cases} \frac{(n\alpha)^2(\theta_1\theta_2)^{n\alpha}}{(z_1 z_2)^{n\alpha+1}}, & \theta_1 \le z_1 \le \theta_2 \le z_2 \\ \frac{2(n\alpha)^2(\theta_1\theta_2)^{n\alpha}}{(z_1 z_2)^{n\alpha+1}}, & \theta_1 \le \theta_2 \le z_1 \le z_2. \end{cases} \tag{2.3}$$

Proof. Note that the joint d.f. of (Z_1, Z_2) is

$$\begin{aligned} G(z_1, z_2 | \theta_1, \theta_2) =& P\{Z_1 \le z_1, Z_2 \le z_2\} \\ =& P\{Z_2 \le z_2\} - P\{Z_1 > z_1, Z_2 \le z_2\} \\ =& A_1 - A_2, \end{aligned}$$

where, $A_1 = P\{Y_1 \le z_2, Y_2 \le z_2\} = [1 - (\frac{\lambda_1}{z_2})]^{n\alpha}[1 - (\frac{\lambda_2}{z_2})]^{n\alpha} = [1 - (\frac{\theta_1}{z_2})]^{n\alpha}[1 - (\frac{\theta_2}{z_2})]^{n\alpha}$, and

$$\begin{aligned} A_2 =& P\{z_1 \le Y_1, Y_2 \le z_2\} \\ =& \begin{cases} \int_{z_1}^{z_2} \frac{n\alpha\theta_1^{n\alpha}}{y_1^{n\alpha+1}} dy_1 \int_{\theta_2}^{z_2} \frac{n\alpha\theta_2^{n\alpha}}{y_2^{n\alpha+1}} dy_2, & \theta_1 \le z_1 \le \theta_2 \le z_2 \\ \int_{z_1}^{z_2} \frac{n\alpha\theta_1^{n\alpha}}{y_1^{n\alpha+1}} dy_1 \int_{z_1}^{z_2} \frac{n\alpha\theta_2^{n\alpha}}{y_2^{n\alpha+1}} dy_2, & \theta_1 \le \theta_2 \le z_1 \le z_2 \end{cases} \\ =& \begin{cases} \left(\frac{\theta_1^{n\alpha}}{z_1^{n\alpha}} - \frac{\theta_1^{n\alpha}}{z_2^{n\alpha}}\right)\left(1 - \frac{\theta_2^{n\alpha}}{z_2^{n\alpha}}\right), & \theta_1 \le z_1 \le \theta_2 \le z_2 \\ \theta_1^{n\alpha}\theta_2^{n\alpha}\left(\frac{1}{z_1^{n\alpha}} - \frac{1}{z_2^{n\alpha}}\right)^2, & \theta_1 \le \theta_2 \le z_1 \le z_2. \end{cases} \end{aligned}$$

Now differentiating G(z_1, z_2) with respect to z_i, i = 1,2, we can get the joint density function of Z_1 and Z_2.

It is interesting to note that on the region $I_1 = \{(\theta_1, \theta_2)|\theta_1 \le z_1 \le \theta_2 \le z_2\}$, the likelihood function $L_1(\theta_1, \theta_2|z_1, z_2)(= g(z_1, z_2|\theta_1, \theta_2))$ is maximized at (z_1, z_2) and max $L_1(\theta_1, \theta_2|z_1, z_2) = (n\alpha)^2(z_1 \cdot z_2)^{-1}$. On the other hand, on the region $I_2 = \{(\theta_1, \theta_2)|\theta_1 \le \theta_2 \le z_1 \le z_2\}$ is maximized at (z_1, z_1) and max $L_2(\theta_1, \theta_2|z_1, z_2) = 2(n\alpha)^2 z_1^{n\alpha-1} \cdot z_2^{-n\alpha-1}$. Hence an estimator maximizing $L_2(\theta_1, \theta_2|z_1, z_2)$ is

$$
(2.4) \qquad \left(\hat{\theta}_1^{(2)}, \hat{\theta}_2^{(2)}\right) = \begin{cases} (Z_1, Z_2), & \text{if } \left(\frac{Z_1}{Z_2}\right)^{n\alpha} < \frac{1}{2} \\ (Z_1, Z_1), & \text{if } \left(\frac{Z_1}{Z_2}\right)^{n\alpha} \ge \frac{1}{2}. \end{cases}
$$

Obviously, $\left(\hat{\theta}_1^{(2)}, \hat{\theta}_2^{(2)}\right) \ne \left(\hat{\theta}_1^{(1)}, \hat{\theta}_2^{(1)}\right)$ with probability $P\{(Z_1/Z_2)^{n\alpha} \ge 1/2\} > 0$. The reason these estimators can differ is straightforward, since the estimators are chosen to maximize two different likelihood functions and these likelihood functions are not necessarily proportional. For further discussion on this phenomenon, see Pal and Berry (1991). We shall derive the s-bias and risk of $\hat{\theta}_2^{(2)}$ later.

In the following, we suggest various estimators of θ_1 and θ_2 and derive their biases and MSE expressions. In each case, the expressions depend on the parameter value only through $\delta = \theta_1/\theta_2, 0 < \delta \le 1$. Before going into the details, we first show that both θ_1 and θ_2 can not be estimated unbiasedly as long as we look at Z_1 and Z_2 only.

From (2.3), it is easy to show that for i = 1,2, $n\alpha \ge 2$,

$$
(2.5) \qquad E[Z_i] = \frac{n\alpha}{n\alpha - 1}\theta_i - (-\delta)^{i-1}\delta^{n\alpha-1}\theta_i \frac{n\alpha}{(n\alpha-1)(2n\alpha-1)}.
$$

Therefore,

$$
(2.6) \qquad \begin{aligned} |B(\hat{\theta}_1^{(1)})| &= \left| \frac{1}{n\alpha - 1} - \delta^{n\alpha-1}\frac{n\alpha}{(n\alpha-1)(2n\alpha-1)} \right|, \\ |B(\hat{\theta}_2^{(1)})| &= \left| \frac{1}{n\alpha - 1} + \delta^{n\alpha}\frac{n\alpha}{(n\alpha-1)(2n\alpha-1)} \right|. \end{aligned}
$$

So, the usual estimators are biased. It is tempting to estimate $\theta_1\delta^{n\alpha-1}$ and $\theta_2\delta^{n\alpha}$ using suitable unbiased estimators and hence to obtain unbiased estimators of θ_i, i = 1,2. But the following result shows that it is impossible to obtain such unbiased estimators based on the Z_i's alone.

THEOREM 2.2. *If we assume $\theta_1 < \theta_2$ (i.e., $\lambda_1 \neq \lambda_2$), then there does not exist any unbiased estimators (function of Z_1 and Z_2 only) of θ_i, $i = 1,2$.*

Proof. From (2.5), note that θ_1 and θ_2 are unbiasedly estimable if and only if $\theta_1\delta^{n\alpha-1}$ and $\theta_2\delta^{n\alpha}$ are unbiasedly estimable. So it is enough to show that $\theta_1\delta^{n\alpha-1}$ and $\theta_2\delta^{n\alpha}$ are not unbiasedly estimable and we will prove it by contradition.

Suppose there is a $T_1 = T_1(Z_1, Z_2)$ such that $E[T_1] = \theta_1\delta^{n\alpha-1}$, $0 < \theta_1 < \theta_2$. Then

$$
\begin{aligned}
&(n\alpha)^2 \int_{\theta_2}^{\infty}\int_{\theta_1}^{\theta_2} t_1(z_1, z_2)(z_1 \cdot z_2)^{-(n\alpha+1)} dz_1 dz_2 \\
&\quad + 2(n\alpha)^2 \int_{\theta_2}^{\infty}\int_{\theta_2}^{z_2} t_1(z_1, z_2)(z_1 \cdot z_2)^{-(n\alpha+1)} dz_1 dz_2 \qquad (2.7)\\
&= \theta_2^{-2n\alpha}, \quad 0 < \theta_1 < \theta_2.
\end{aligned}
$$

The right hand side(RHS) of (2.7) is a function of θ_2 only, whereas the first term on the left hand side(LHS) depends on θ_2 as well as θ_1. Therefore, (2.7) is true for all θ_1 and θ_2 $(0 < \theta_1 < \theta_2 < \infty)$, if and only if

$$
\begin{aligned}
&\int_{\theta_1}^{\theta_2} t_1(z_1, z_2) z_1^{-(n\alpha+1)} dz_1 = 0 \ ; \ \text{i.e.,} \qquad (2.8)\\
&\int_{a}^{b} t_1(z_1, z_2) z_1^{-(n\alpha+1)} dz_1 = 0, \ \forall a, b \ ; \ 0 < a < b < \infty.
\end{aligned}
$$

But (2.8) implies that $\int_{\theta_2}^{z_2} t(z_1, z_2) z_1^{-(n\alpha+1)} dz_1 = 0$ for any $\theta_2 < z_2$, which implies the fact that LHS of (2.7) is zero, and it is yielding a contradiction since the RHS is nonzero. Similarly, one can prove that there does not exist any $T_2 = T_2(Z_1, Z_2)$ such that $E[T_2] = \theta_2\delta^{n\alpha}$.

To improve over the standard estimators $\hat{\theta}_1^{(1)}$ and $\hat{\theta}_2^{(1)}$, we consider the class of scale equivariant estimators of θ_i, i = 1,2, given as

$$\mathbf{C}_i = \{\hat{\theta}_{i(c)} = c \cdot Z_i \ ; \ 0 < c\}, \ i = 1, 2.$$

Note that,

$$(2.9)\qquad \begin{aligned} |B(\hat\theta_{1(c)})| &= \left| c\frac{n\alpha}{n\alpha-1} - 1 - c\delta^{n\alpha-1}\frac{n\alpha}{(n\alpha-1)(2n\alpha-1)} \right|, \\ |B(\hat\theta_{2(c)})| &= \left| 1 - c\frac{n\alpha}{n\alpha-1} - c\delta^{n\alpha}\frac{n\alpha}{(n\alpha-1)(2n\alpha-1)} \right|. \end{aligned}$$

Clearly, it is impossible to find a c_o such that $|B(\hat\theta_{i(c_o)})|$ is minimun in the class $\mathbf{C}_i$ for all $\delta, 0 < \delta \le 1$. Therefore , we adopt the minimax absolute s-bias approach i.e., choose $c_o(> 0)$ such that

$$(2.10)\qquad \max_{0<\delta\le 1} |B(\hat\theta_{i(c_o)})| = \min_{0<c}\max_{0<\delta\le 1} |B(\hat\theta_{i(c)})|.$$

The following Theorem gives the optimal estimators of θ_i in $\mathbf{C}_i$, i = 1,2, under the criterion (2.10).

THEOREM 2.3. *Let* $c_i = \frac{2(n\alpha-1)(2n\alpha-1)}{n\alpha(4n\alpha+2i-5)}$, *i = 1,2. The estimators* $\hat\theta_{i(c_i)} = c_i Z_i$ *is the minimax bias estimator(according to (2.10)) for* θ_i, *i = 1,2.*

Proof. First we consider estimator of θ_1. Let

$$B_1^{(1)} = c\frac{n\alpha}{n\alpha-1} - 1 - c\delta^{n\alpha-1}\frac{n\alpha}{(n\alpha-1)(2n\alpha-1)}.$$

Then, for any estimator $\hat\theta_{1(c)} = cZ_1, (c > 0)$,

$$|B(\hat\theta_{1(c)})| = \begin{cases} B_1^{(1)}, & \text{if } \ c \ge \frac{2n\alpha-1}{2n\alpha} \\ -B_1^{(1)}, & \text{if } \ c < \frac{n\alpha-1}{n\alpha}. \end{cases}$$

Case(i). $c \ge \frac{2n\alpha-1}{2n\alpha}$. In this case $\frac{d}{d\delta}|B(\hat\theta_{1(c)})| < 0$ and

$$\max_{0<\delta\le 1} |B(\hat\theta_{1(c)})| = c\frac{n\alpha}{n\alpha-1} - 1,$$

which is increasing in c. Therefore, $\min_{0<c}\max_{0<\delta\le 1} |B(\hat\theta_{1(c)})| = \frac{1}{2n\alpha-2}$.

Case(ii). $0 < c \le \frac{n\alpha-1}{n\alpha}$. In this case $\frac{d}{d\delta}|B(\hat{\theta}_{1(c)})| > 0$ and

$$\max_{0<\delta\le 1} |B(\hat{\theta}_{1(c)})| = 1 - c\frac{2n\alpha}{2n\alpha-1},$$

which is decreasing in c. Therefore, $\min_{0<c} \max_{0<\delta\le 1} |B(\hat{\theta}_{1(c)})| = \dfrac{1}{2n\alpha-1}$.

Case(iii). $\frac{n\alpha-1}{n\alpha} < c < \frac{2n\alpha-1}{2n\alpha}$.

Let $\delta_n = \left\{2n\alpha - 1 - \frac{(n\alpha-1)(2n\alpha-1)}{cn\alpha}\right\}^{\frac{1}{n\alpha-1}}$. Then

$$\max_{0<\delta\le 1} |B(\hat{\theta}_{1(c)})| = \max\left\{\max_{0<\delta\le\delta_n} B_1^{(1)}, \max_{\delta_n\le\delta\le 1} B_1^{(2)}\right\},$$

where $B_1^{(2)} = -B_1^{(1)}$.

So, $\max_{0<\delta\le 1} |B(\hat{\theta}_{1(c)})| = \max\left\{c\frac{n\alpha}{n\alpha-1} - 1, 1 - c\frac{2n\alpha}{2n\alpha-1}\right\}$,

which implies that $\min_{0<c} \max_{0<\delta\le 1} |B(\hat{\theta}_{1(c)})| = |B(\hat{\theta}_{1(c)})|_{c=c_1} = \dfrac{1}{4n\alpha-3}$,

where $c_1 = \frac{2(n\alpha-1)(2n\alpha-1)}{n\alpha(4n\alpha-3)}$.

Combining the preceding three cases, we can get the required result. The result for θ_2 is proved as the similar method.

The following Lemma provides the risks of $\hat{\theta}_{i(c)}$, i = 1,2, which are easy to derive.

LEMMA 2.4. *The risks of $\hat{\theta}_{i(c)}$, i = 1,2, are given by*

$$(a)\ R[\hat{\theta}_{1(c)}] = \frac{2cn\alpha}{(n\alpha-1)(2n\alpha-1)}\delta^{n\alpha-1} - \frac{c^2n\alpha}{(n\alpha-1)(n\alpha-2)}\delta^{n\alpha-2} + \left\{\frac{c^2n\alpha}{n\alpha-2} - \frac{2cn\alpha}{n\alpha-1} + 1\right\},$$

$$(b)\ R[\hat{\theta}_{2(c)}] = \left\{\frac{c^2n\alpha}{(n\alpha-1)(n\alpha-2)} - \frac{2cn\alpha}{(n\alpha-1)(2n\alpha-1)}\right\}\delta^{n\alpha} + \left\{\frac{c^2n\alpha}{n\alpha-2} - \frac{2cn\alpha}{n\alpha-1} + 1\right\}.$$

Again, there does not exist any $c_0 > 0$ such that $\hat{\theta}_{i(c_0)}$ has the uniformly smallest risk in the class $\mathbf{C}_i$, *$i = 1,2$. A minimax approach can be taken to an optimal value of c so that $\hat{\theta}_{i(c)}$ minimizes the maximum risk in the class* $\mathbf{C}_i$, $i = 1, 2$.

THEOREM 2.5. *Let $c^i = \frac{2(n\alpha-2)}{2n\alpha+i-3}$, $i = 1,2$. Then*

$$\max_{0<\delta\leq 1} R[\hat{\theta}_{i(c^i)}] = \min_{0<c}\max_{0<\delta\leq 1} R[\hat{\theta}_{i(c^i)}],\ i = 1, 2.$$

Proof. For estimating θ_1, note that $\frac{d}{d\delta}R[\hat{\theta}_{2(c)}] \geq 0$ if $\delta_n \leq \delta \leq 1$, and ≤ 0 if $0 < \delta \leq \delta_n$, where $\delta_n = \frac{c(2n\alpha-1)}{2(n\alpha-2)}$.

Case(i). $c \geq \frac{2(n\alpha-1)}{2n\alpha-1}$. Then $\delta_n \geq 1$ and $\min_{0<c}\max_{0<\delta\leq 1} R[\hat{\theta}_{1(c)}] = 1 - \dfrac{n\alpha(\alpha-2)}{(n\alpha-1)^2}$.

Case(ii). $c \geq \frac{2(n\alpha-1)}{2n\alpha-1}$. Then,

$$\begin{aligned}\max_{0<\delta\leq 1} R[\hat{\theta}_{1(c)}] =&\max\left\{\max_{0<\delta\leq\delta_n} R[\hat{\theta}_{1(c)}], \max_{\delta_n<\delta\leq 1} R[\hat{\theta}_{1(c)}]\right\}\\ =&\max\{M_1^{(1)}, M_1^{(2)}\},\end{aligned}$$

where, $M_1^{(1)} = \frac{c^2 n\alpha}{n\alpha-2} - \frac{2cn\alpha}{n\alpha-1} + 1$ and $M_1^{(2)} = \frac{c^2 n\alpha}{n\alpha-1} - \frac{4cn\alpha}{n\alpha-1} + 1$.
Also,

$$\frac{d}{dc}M_1^{(1)} \geq 0 \text{if} c \geq \frac{n\alpha-2}{n\alpha-1} \equiv c^o, \text{and} \frac{d}{dc}M_1^{(2)} \geq 0 \text{if} c \geq \frac{n\alpha-1}{2n\alpha-1} \equiv c^{oo}.$$

Moreover, $\min_c M_1^{(1)} = M_1^{(1)}|_{c=c^o}$ and $\min_c M_1^{(2)} = M_1^{(2)}|_{c=c^{oo}}$. Also, note that $M_1^{(1)} \geq M_1^{(2)} \leftrightarrow c \geq \frac{2(n\alpha-2)}{2n\alpha-1} \equiv c_{oo}$. So

$$\max_{0<\delta\leq 1} R[\hat{\theta}_{1(c)}] = \begin{cases} M_1^{(2)}, & c^{oo} \leq c \leq c_{o,} \\ M_1^{(1)}, & c > c_{oo}. \end{cases}$$

Since $M_1^{(2)}|_{c=c^{oo}} > M_1^{(1)}|_{c=c^o}, \min_{0<c}\max_{0<\delta\leq 1} R[\hat{\theta}_{1(c)}] = 1 - \dfrac{n\alpha(\alpha-2)}{(n\alpha-1)^2}$, and this is attained at $c = c^o$. Combining cases (i) and (ii), we get the required result.

To estimate θ_2, note that $\frac{d}{d\delta}R[\hat{\theta}_{2(c)}] \geq 0$ if $c \geq c^1$, and ≤ 0 if $c < c^1$. Hence,

$$\min_{c \geq c^1} \max_{0<\delta\leq 1} R[\hat{\theta}_{1(c)}] = \min_{c<c^1} \max_{0<\delta\leq 1} R[\hat{\theta}_{1(c)}] = 1 - \frac{4(n\alpha)^2(n\alpha-2)}{(n\alpha-1)(2n\alpha-1)^2}$$

and hence this is attained at $c = c^1$.

We now show that the minimax(absolute s-bias and risk) estimators obtained in the last two Theorems are uniformly better than the MLE's in terms of absolute s-bias and risk. From (2.10), Lemmas 2.1 and 2.4, Theorems 2.2 and 2.5, we can obtain the following result.

THEOREM 2.6.

(a) $$|B[\hat{\theta}_{i(c_i)}]| \leq |B[\hat{\theta}_i^{(1)}]|,$$

(b) $$R[\hat{\theta}_{i(c^i)}] \leq R[\hat{\theta}_i^{(1)}],\ 0 < \theta \leq 1,\ i = 1, 2.$$

Now we focus our attention on the problem of estimating θ_2 *only because the* $\hat{\theta}_2^{(2)}$ *(a modified MLE) developed in (2.4) is nonsmooth and quite different from the other estimators. The following lemma gives the first and second moments of* $\hat{\theta}_2^{(2)}$.

LEMMA 2.7. *Let* $\lambda = 2^{\frac{1}{n\alpha}}$. *Then*

(a) $$E[\hat{\theta}_2^{(2)}] = \begin{cases} A_1 + A_3 & \text{if } 1 \geq \delta \geq \lambda^{-1} \\ A_2 + A_3 & \text{if } 0 < \delta < \lambda^{-1}, \end{cases}$$

(b) $$E[\hat{\theta}_2^{(2)}]^2 = \begin{cases} B_1 + B_3 & \text{if } 1 \geq \delta \geq \lambda^{-1} \\ B_2 + B_3 & \text{if } 0 < \delta < \lambda^{-1}, \end{cases}$$

where, $A_1 = \frac{n\alpha}{n\alpha-1}\theta_2\delta^{-1} + \frac{n\alpha[\lambda n\alpha - n\alpha+1]}{2(n\alpha-1)(2n\alpha-1)}\theta_2\delta^{-(n\alpha-1)}$
$- \frac{n\alpha[\lambda n\alpha+3n\alpha+1]}{2(n\alpha-1)(2n\alpha-1)}\theta_2\delta^{n\alpha}$,

$A_2 = \frac{n\alpha}{n\alpha-1}\theta_2 + \frac{n\alpha}{2(n\alpha-1)(2n\alpha-1)}[n\alpha + \frac{4n\alpha}{\lambda} - \lambda n\alpha - 1]\theta_2\delta^{n\alpha}$,

$A_3 = \frac{n\alpha[\lambda n\alpha+n\alpha-1]}{(n\alpha-1)(2n\alpha-1)}\theta_2\delta^{n\alpha}$,

$B_1 = \frac{n\alpha}{n\alpha-2}\theta_1^2 + \frac{n\alpha[\lambda^2 n\alpha-n\alpha+2]}{4(n\alpha-1)(n\alpha-2)}\theta_2^2\delta^{-(n\alpha-2)} - \frac{n\alpha[\lambda^2 n\alpha+3n\alpha-2]}{4(n\alpha-1)(n\alpha-2)}\theta_2^2\delta^{n\alpha}$,

$B_2 = \frac{n\alpha}{n\alpha-2}\theta_2^2 + \frac{n\alpha}{4(n\alpha-1)(n\alpha-2)}[n\alpha + 6 + \frac{4n\alpha}{\lambda^2} - \lambda^2 n\alpha]\theta_2^2\delta^{n\alpha}$,

and $B_3 = \frac{n\alpha[\lambda^2 n\alpha+n\alpha-2]}{2(n\alpha-1)(n\alpha-2)}\theta_2\delta^{n\alpha}$.

Proof. From Lemma 2.1, it can be shown.

Next, we shall consider Bayes estimators of $\theta_i (i = 1, 2)$ under square error loss and a noninformative prior.

Consider the noninformative prior $\pi(\theta_1, \theta_2) \propto (\theta_1 \cdot \theta_2)$, $0 < \theta_1 \le \theta_2 < \infty$.

According to the Bayes theorem, the joint posterior distribution and the marginal posterior distribution of θ_1 and θ_2, respectively, are obtained by

$$g(\theta_1, \theta_2 | z_1, z_2) = \begin{cases} \frac{(n\alpha)^2 (\theta_1 \theta_2)^{n\alpha - 1}}{(z_1 z_2)^{n\alpha}}, & \theta_1 \le z_1 \le \theta_2 \le z_2 \\ \frac{2(n\alpha)^2 (\theta_1 \theta_2)^{n\alpha - 1}}{(z_1 z_2)^{n\alpha}}, & \theta_1 \le \theta_2 \le z_1 \le z_2, \end{cases}$$

$$\begin{aligned} g(\theta_1 | z_1, z_2) =& (n\alpha)\theta_1^{n\alpha - 1}(z_1^{-n\alpha} + z_2^{-n\alpha}) \\ & - 2n\alpha\theta_1^{2n\alpha - 1}(z_1 \cdot z_2)^{-n\alpha}, \qquad 0 < \theta_1 \le z_1, \end{aligned}$$

(2.11)

$$g(\theta_2 | z_1, z_2) = \begin{cases} 2n\alpha(z_1 \cdot z_2)^{-n\alpha}\theta_2^{2n\alpha - 1}, & 0 < \theta_2 \le z_1 \\ (n\alpha) z_2^{-n\alpha}\theta_2^{n\alpha - 1}, & z_1 < \theta_2 \le z_2. \end{cases}$$

Therefore, the Bayes estimators of θ_1 and θ_2 under square error loss and a noninformative prior are

$$\hat{\theta}_i^{BN} = \left[\frac{n\alpha}{n\alpha + 1} + (-1)^i \hat{\delta}^{n\alpha} \frac{n\alpha}{(n\alpha + 1)(2n\alpha + 1)} \right] Z_i, \quad i = 1, 2,$$

where $\hat{\delta} = Z_1 / Z_2$ is an estimator of δ.

From (2.11), we can obtain the s-bias and risk of $\hat{\theta}_i^{BN}$, i = 1,2.

LEMMA 2.8. *The s-bias of $\hat{\theta}_i^{BN}$, i = 1,2, are*

(a)

$$\begin{aligned} |B(\hat{\theta}_1^{BN})| = & \left| \frac{1}{(n\alpha - 1)(n\alpha + 1)} - \frac{(n\alpha)^2}{2(n\alpha - 1)(2n\alpha - 1)} \delta^{n\alpha - 1} \right. \\ & \left. + \frac{(n\alpha)^2}{2(n\alpha + 1)(2n\alpha + 1)} \delta^{n\alpha} \right|, \end{aligned}$$

(b)

$$\begin{aligned} |B(\hat{\theta}_2^{BN})| = & \left| \frac{1}{(n\alpha - 1)(n\alpha + 1)} + \frac{(n\alpha)^2}{2(n\alpha - 1)(2n\alpha - 1)} \delta^{n\alpha} \right. \\ & \left. - \frac{(n\alpha)^2}{2(n\alpha + 1)(2n\alpha + 1)} \delta^{n\alpha + 1} \right|, \end{aligned}$$

And the risk of $\hat{\theta}_i^{BN}, i = 1, 2,$ *are*

(c)

$$R[\hat{\theta}_1^{BN}] = \frac{(n\alpha)^2 + n\alpha + 2}{(n\alpha+1)^2(n\alpha-1)(n\alpha-2)} + \frac{(n\alpha)^2\delta^{n\alpha-1}}{(n\alpha-1)(2n\alpha-1)}$$
$$- \frac{(n\alpha)^2(n\alpha+2)\delta^{n\alpha}}{2(n\alpha+1)^2(2n\alpha+1)} - \frac{(n\alpha)^3\delta^{2n\alpha}}{3(n\alpha)^2(2n\alpha+1)^2(n\alpha+2)}$$
$$- \frac{(n\alpha)^3\{3(n\alpha)^3 + 15(n\alpha)^2 + 16n\alpha + 4\}\delta^{n\alpha-2}}{6(n\alpha+1)^2(n\alpha-2)(n\alpha+2)(n\alpha+1)(2n\alpha+1)}$$

(d)

$$R[\hat{\theta}_2^{BN}] = \frac{(n\alpha)^2 + n\alpha - 2}{(n\alpha+1)^2(n\alpha-1)(n\alpha-2)} + \frac{(n\alpha)^2(n\alpha-1)\delta^{n\alpha+1}}{(n\alpha+1)^2(4(n\alpha)^2-1)}$$
$$- \frac{(n\alpha)^2\{9(n\alpha)^4 - 14(n\alpha)^3 - 73(n\alpha)^2 - 12\}\delta^{n\alpha}}{3(n\alpha+1)^2(n\alpha-2)(n\alpha+2)(n\alpha-1)(2n\alpha+1)(2n\alpha-1)}$$
$$- \frac{(n\alpha)^3\delta^{2n\alpha+2}}{3(n\alpha+1)^2(2n\alpha+1)^2(n\alpha+2)}$$

From Theorem 2.3 and 2.5, Lemma 2.7 and 2.8, we can evaluate exact numerical values of biases and MSE's for the minimax bias, minimax risk, and Bayes estimators for the smaller and larger of two pareto scale parameters, and also obtain the following numerical small sample properties. The Bayes estimators for the smaller and larger of two pareto scale parameters are more useful than other estimators in a sense of bias. The minimax risk estimator for the smaller of two pareto scale parameters is more efficient than other estimators, while the Bayes estimator for the larger of two pareto scale parameters is more efficient than other estimators.

References

1. S. Blumental and A. Cohen, *Estimation of the Larger Translation Parameter*, Ann. Math. Statist. **39** (1968a), 502-516.
2. S. Blumental and A. Cohen, *Estimation of the Two Ordered Parameters*, Ann. Math. Statist. **39** (1968b), 517-530.
3. S. Blumental and A. Cohen, *Estimation of the Larger of Two Normal Means*, JASA **63** (1968c), 861-876.
4. A. Carpenter and N. Pal, *Estimation of the Smaller and Larger of Two Exponential Location Parameters*, Commun. Statist.-Theory and Meth. **21** (1992), 883-895.

5. E. J. Dudewicz, *Multiple Decision Procedures(Ranking and Selection) : Estimation*, Ph.D. Thesis, Cornell University (1968), USA.
6. A. Elffesi and N. Pal, *Estimation of the Smaller and Larger of Two Uniform Scale Parameters*, Commun. Statist.-Theory and Meth. **21** (1992), 2997-3015.
7. D. Kushary and A. Cohen, *Estimating Ordered Location and Scale Parameters*, Statistics and Decisions **7** (1989), 201-213.
8. N. Pal and J. C. Berry, *On Invariance and Maximum likelihood Estimation*, Technical Report, Department of Statistics (1992), University of Southwestern Lousiana.
9. P. W. Zehna, *Invariance of maximum Likelihood*, Ann. Math. Statist. **37** (1966), 744.

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