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SPECTRAL ρ—DILATIONS AND POLYNOMIALLY BOUNDED OPERATORS

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Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. We recall that an operator T in $\mathcal{L}(\mathcal{H})$ is said to be ***polynomially bounded*** if there exists a positive number M such that for every polynomial p,

$$||p(T)|| \leq M \sup_{|z|\leq 1} |p(z)|.$$

We denote the set of all polynomially bounded operators in $\mathcal{L}(\mathcal{H})$ by $PB(\mathcal{H})$. One knows from the von Neumann inequality that all contractions are polynomially bounded. Whether every polynomially bounded operator in $\mathcal{L}(\mathcal{H})$ is similar to a contraction is one of the most interesting and difficult open problems in operator theory(cf. [4], [5], [6]). In particular, W. S. Li [6] studied properties of polynomially bounded operators and obtained some conditions for polynomially bounded operator concerning similarity to a contraction. Holbrook [5] introduced spectral dilations of operator representations of subalgebras of $C(X)$, where $C(X)$ is the Banach algebra of all complex continuous functions on a compact Hausdorff space X with sup norm, and obtained an important example of an operator representation corresponding to polynomially bounded operators. In particular, Parrott [9] gives a simple example of an operator representation which has no spectral dilation.

In this paper, we will discuss spectral ρ–dilations as generalizations of spectral dilations of operator representations, and show that the above problem is equivalent to the problem of the existence of a spectral ρ–dilation for certain generalized operator representations. Moreover, we will obtain a relationship between the well-known unitary

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ρ-dilations of a polynomially bounded operator and similarity to a contraction.

Let X be a compact Hausdorff space, and let $C(X)$ denote the Banach algebra of complex continuous functions on X with sup norm $||\cdot||_\infty$. Suppose that $\mathcal{F}$ is a subalgebra of $C(X)$ containing the constant functions.

DEFINITION 1. An *operator representation* of $\mathcal{F}$ is a mapping Φ from $\mathcal{F}$ into the algebra $\mathcal{L}(\mathcal{H})$ on some Hilbert space $\mathcal{H}$, with the requirement that Φ satisfy the following:
(1) $\Phi : \mathcal{F} \to \mathcal{L}(\mathcal{H})$ is an algebra homomorphism with $\Phi(1) = I$, and
(2) $||\Phi|| \leq 1$, that is, $||\Phi(f)|| \leq ||f||_\infty$ for all $f \in \mathcal{F}$.

A *spectral ρ-dilation* of such an operator representation Φ, $1 \leq \rho$, is an operator representation $\Psi : C(X) \to \mathcal{L}(\mathcal{K})$, where $\mathcal{K}$ is a Hilbert space containing $\mathcal{H}$ as a subspace, such that
(3) $\Phi(f) = \rho P_{\mathcal{H}} \Psi(f)|\mathcal{H}$ for every nonconstant function f in $\mathcal{F}$, where $P_{\mathcal{H}}$ denotes the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$.

Note that (3) is not valid for $f = 1$ unless $\rho = 1$.

DEFINITION 2. A *generalized operator representation* of $\mathcal{F}$ is a mapping $\Phi : \mathcal{F} \to \mathcal{L}(\mathcal{H})$ such that (1) holds and in place of (2), we have $||\Phi|| < \infty$, that is, there is some constant M such that $||\Phi(f)|| \leq M||f||_\infty$ for all $f \in \mathcal{F}$.

EXAMPLE 3. Let X be the unit circle $\mathbf{T}$ in the complex plane $\mathbf{C}$ and let $\mathcal{P}$ be the algebra of complex polynomials restricted to $\mathbf{T}$, and require that $\Phi(p) = p(T)$, where T is a contraction operator in $\mathcal{L}(\mathcal{H})$. Then Φ is an operator representation of $\mathcal{P}$.

EXAMPLE 4. If $A \in PB(\mathcal{H})$, then the Φ is a generalized operator representation of $\mathcal{P}$. Furthermore, if A has a unitary ρ-dilation $U \in \mathcal{L}(\mathcal{K})$ for $\rho \geq 1$, i.e., there exist a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ and a unitary operator $U \in \mathcal{L}(\mathcal{K})$ such that

$$A^n x = \rho P_{\mathcal{H}} U^n x, x \in \mathcal{H}, n \in \mathbf{N},$$

then $\Psi(h) = h(U), h \in C(X)$, is a spectral ρ-dilation of Φ.

REMARK. If we take $X = \mathbf{T}, \mathcal{F} = \mathcal{P}$ and, for each polynomial $p \in \mathcal{P}, \Phi(p) = p(T)$, where T is some operator in $\mathcal{L}(\mathcal{H})$, then Φ is a generalized operator representation of $\mathcal{P}$ if and only if $T \in PB(\mathcal{H})$.

By a spectral ρ–dilation of a generalized operator representation of $\Phi : \mathcal{F} \to \mathcal{L}(\mathcal{H})$ we shall mean a generalized operator representation $\Psi : C(X) \to \mathcal{L}(\mathcal{K})$ satisfying (3).

One of important steps in the proof of main theorem in this section can be seem most clearly in the following lemma.

LEMMA 5. *Suppose that $A \in \mathcal{L}(\mathcal{H})$. Then A is similar to a contraction on $\mathcal{H}$ if and only if there exist a Hilbert space $\mathcal{K}$, a contraction $C \in \mathcal{L}(\mathcal{K})$, bounded operators $S : \mathcal{K} \to \mathcal{H}, T : \mathcal{H} \to \mathcal{K}$, and a constant $1 \leq \rho < 2$ such that*

(4) $\quad A^n = \rho S C^n T$ *for all* $n \in \mathbf{N}$,

and

(5) $\quad ST = I_{\mathcal{H}}$.

Proof. If A is similar to a contraction $C \in \mathcal{L}(\mathcal{H})$, so that $A = SCS^{-1}$ for some invertible $S \in \mathcal{L}(\mathcal{H})$, then we can put $\mathcal{K} = \mathcal{H}$ and $T = S^{-1}$. Thus (4) and (5) are proved.

On the other hand, suppose that (4) and (5) hold. It is well–known that A is similar to a contraction on $\mathcal{H}$ if, and only if, A is a contraction with respect to some equivalent inner product norm $|\cdot|$ on $\mathcal{H}$. We shall construct such a norm.

We denote the given norms on $\mathcal{H}$ and $\mathcal{K}$ by $||\cdot||$, and we define $|h|$, for $h \in \mathcal{H}$ by the relation

$$|h| = \inf \left\{ || \sum_{n=0}^{\infty} C^n T h_n || : h_n \in \mathcal{H}, \sum_{n=0}^{\infty} A^n h_n = h, ||h_0|| \leq ||h|| \right\},$$

where $h_n = 0$ with a finite number of exceptions. It is easy to see that $|\cdot|$ is a semi–norm on $\mathcal{H}$. Since

$$h = h + A0 + A^2 0 + A^3 0 + \cdots,$$

we have $|h| \leq ||T|| \cdot ||h||$. If $h = \sum_{n=0}^{\infty} A^n h_n$ with $||h_0|| \leq ||h||$, then

$$\begin{aligned} ||h|| = ||\sum_{n=0}^{\infty} A^n h_n|| = ||h_0 + \sum_{n=1}^{\infty} \rho S C^n T h_n|| \\ \leq ||\sum_{n=0}^{\infty} \rho S C^n T h_n|| + |1 - \rho| ||h_0|| \\ \leq ||\sum_{n=0}^{\infty} \rho S C^n T h_n|| + (\rho - 1) ||h||. \end{aligned}$$

Thus $||h|| \leq \frac{\rho}{2-\rho} ||S|| \cdot |h|$, and hence $|\cdot|$ is a norm on $\mathcal{H}$ that is equivalent to $||\cdot||$.

Now we claim that $|Ah| \leq |h|$, i.e., A is a contraction with respect to $|\cdot|$. Suppose that $|h| < \alpha$. Then

$$\alpha > ||\sum_{n=0}^{\infty} C^n T h_n||$$

for some h_n such that $h = \sum_{n=0}^{\infty} A^n h_n$ and $||h_0|| \leq ||h||$. Since

$$Ah = \sum_{n=0}^{\infty} A^{n+1} h_n,$$

we have

$$\begin{aligned} |Ah| \leq ||\sum_{n=0}^{\infty} C^{n+1} T h_n|| \\ \leq ||C|| \cdot ||\sum_{n=0}^{\infty} C^n T h_n|| \\ \leq ||\sum_{n=0}^{\infty} C^n T h_n|| < \alpha. \end{aligned}$$

So, $|Ah| \leq |h|$.

Finally, we show that $|\cdot|$ is generated by an inner product. It is sufficient to prove that

(6) $$|h+g|^2 + |h-g|^2 = 2(|h|^2 + |g|^2), \quad h, g \in \mathcal{H}.$$

If $2(|h|^2+|g|^2)<\alpha$, there exist h_n, g_n in $\mathcal{H}$ such that $\sum_{n=0}^{\infty} A^n h_n = h, \sum_{n=0}^{\infty} A^n g_n = g$ with

$$||h_0 \pm g_0|| \leq ||h \pm g||,$$

and

$$\alpha > 2\Big(||\sum_{n=0}^{\infty} C^n T h_n||^2 + ||\sum_{n=0}^{\infty} C^n T g_n||^2\Big).$$

By the parallelogram law in $\mathcal{H}$,

$$\alpha > \Big\|\sum_{n=0}^{\infty} C^n T(h_n+g_n)\Big\|^2 + \Big\|\sum_{n=0}^{\infty} C^n T(h_n-g_n)\Big\|^2.$$

Since

$$\sum_{n=0}^{\infty} A^n(h_n \pm g_n) = h \pm g,$$

we have

$$\alpha > |h+g|^2 + |h-g|^2$$

and

$$|h+g|^2 + |h-g|^2 \leq 2(|h|^2+|g|^2), \quad h, g \in \mathcal{H}.$$

Since the reverse inequality follows immediately upon replacing h by $h+g$ and g by $h-g$, we obtain (6).

REMARK. If we have

$$A^n = \rho P_{\mathcal{H}} C^n|\mathcal{H}, n \in \mathbf{N}, 1 \leq \rho < 2$$

and C is some contraction, then A must be similar to a contraction.

THEOREM 6. *The following are equivalent:*

(1) *Every generalized operator representation of $\mathcal{P}$ has a spectral ρ-dilation for some $1 \leq \rho < 2$.*
(2) *Every polynomially bounded operator is similar to a contraction.*

Proof. If $A \in PB(\mathcal{H})$, we can obtain a generalized operator representation Φ of $\mathcal{P}$ by setting $\Phi(p) = p(A)$. Let $\Psi : C(\mathbf{T}) \to \mathcal{L}(\mathcal{K})$ is a spectral ρ-dilation of Φ. Let $\Psi(z) = V$. For $n = 0, \pm 1, \pm 2, \cdots$,

$$||V^n|| = ||\Psi(z^n)|| \leq ||\Psi||.$$

By Theorem of Sz-Nagy (cf.[7]), V is similar to a unitary operator $U \in \mathcal{L}(\mathcal{K})$. Let W be invertible such that $V = WUW^{-1}$. For $h \in \mathcal{H}$ and $n \in \mathbf{N}$,

$$\begin{aligned} A^n h &= \Phi(z^n)h = \rho P_{\mathcal{H}} \Psi(z^n) h \\ &= \rho P_{\mathcal{H}} V^n h = \rho P_{\mathcal{H}} W U^n W^{-1} h. \end{aligned}$$

Let $S = P_{\mathcal{H}} W$ and $T = W^{-1} | \mathcal{H}$. Then $A^n = \rho S U^n T$ and $ST = I_{\mathcal{H}}$. By Lemma 5, A is similar to a contraction on $\mathcal{H}$. The converse was proved by Holbrook (cf. [5, Theorem 1]).

We recall that an operator T in $\mathcal{L}(\mathcal{H})$ is said to be *weakly centered* if TT^* commutes with T^*T. Weakly centered operators have been studied in [1], [2], and [3].

The intresting theorem for the dilation theory of polynomially bounded operators was proved by Petrović (cf. [10]).

THEOREM 7 [10, THEOREM 1]. *For any $T \in PB(\mathcal{H})$, there exist a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ and $\tilde{T} \in PB(\mathcal{K})$ such that*

(1) *$\mathcal{H}$ is a semi-invariant subspace for $\tilde{T}$*
(2) $\sigma(\tilde{T}) = \mathbf{T}$
(3) *$\tilde{T}$ is weakly centered.*

The following corollary shows a sufficient condition for a polynomially bounded operator to be similar to a contraction using its dilation.

COROLLARY 8. *If $0 \notin W(\tilde{T})$, the numerical range of $\tilde{T}$, where $\tilde{T}$ is as in Theorem 7, then T is similar to a contraction.*

Proof. If $0 \notin W(\tilde{T})$, then $\tilde{T}$ is normal (cf.[1],[2]). By the Functional Calculus for $\tilde{T}$ on $C(\mathbf{T})$, T is similar to a contraction.

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