NOTE ON NONPATH-CONNECTED ORTHOMODULAR LATTICES

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Abstract. Some nonpath-connected orthomodular lattices are given: Every infinite direct product of orthomodular lattices containing infinitely many non-Boolean factors is a nonpath-connected orthomodular lattice. The orthomodular lattice of all closed subspaces of an infinite dimensional Hilbert space is a nonpath-connected orthomodular lattice.

1. Preliminaries

An orthomodular lattice (abbreviated by OML) is an ortholattice $L$ which satisfies the orthomodular law: if $x \leq y$, then $y = x \lor (x' \land y)$ [5]. A Boolean algebra $B$ is an ortholattice satisfying the distributive law: $x \lor (y \land z) = (x \lor y) \land (x \lor z)$ $\forall x, y, z \in B$.

A subalgebra of an OML $L$ is a nonempty subset $M$ of $L$ which is closed under the operations $\lor$, $\land$ and $'$. We write $M \leq L$ if $M$ is a subalgebra of $L$. If $M \leq L$ and $a, b \in M$ with $a \leq b$, then the relative interval sublattice $M[a, b] = \{x \in M \mid a \leq x \leq b\}$ is an OML with the relative orthocomplementation $\downarrow$ on $M[a, b]$ given by $c^\downarrow = (a \lor c') \land b = a \lor (c' \land b)$ $\forall c \in M[a, b]$. In particular, $L[a, b]$ will be denoted by $[a, b]$ if there is no ambiguity.

The commutator of $a$ and $b$ of an OML $L$ is denoted by $a \ast b$, and is defined by $a \ast b = (a \lor b) \land (a \lor b') \land (a' \lor b) \land (a' \lor b')$. The set of all commutators of $L$ is denoted by $ComL$ and $L$ is said to be commutator-finite if $|ComL|$ is finite. For elements $a, b$ of an OML, we say $a$ commutes with $b$, in symbols $a \mathcal{C} b$, if $a \ast b = 0$. If $M$ is a subset of an OML $L$, the set $C(M) = \{x \in L \mid x \mathcal{C} m \ \forall m \in M\}$ is called the commutant of $M$ in $L$ and the set $Cen(M) = C(M) \cap M$ is called the center of $M$. The

1991 AMS Classification: 06C15.
Key words: Orthomodular Lattice, Path-connected, Nonpath-connected.
set $C(L)$ is called the center of $L$ and then $C(L) = \bigcap \{C(a) | a \in L\}$. An OML $L$ is called irreducible if $C(L) = \{0, 1\}$, and $L$ is called reducible if it is not irreducible.

A block of an OML $L$ is a maximal Boolean subalgebra of $L$. The set of all blocks of $L$ is denoted by $\mathcal{A}_L$. Note that $\bigcup \mathcal{A}_L = L$ and $\bigcap \mathcal{A}_L = C(L)$. An OML $L$ is said to be block-finite if $|\mathcal{A}_L|$ is finite.

For any $e$ in an OML $L$, the subalgebra $S_e = [0, e'] \cup [e, 1]$ is called the (principal) section generated by $e$. Note that for $A, B \in \mathcal{A}_L$, if $e \in (A \cap B)$ and $A \cap B = S_e \cap (A \cup B)$, then $A \cap B = S_e \cap A = S_e \cap B$.

**Definition 1.1.** For blocks $A, B$ of an OML $L$ define $A \overset{w}{\sim} B$ if and only if $A \cap B = S_e \cap (A \cup B)$ for some $e \in A \cap B$; $A \sim B$ if and only if $A \neq B$ and $A \cup B \leq L$; $A \approx B$ if and only if $A \sim B$ and $A \cap B \neq C(L)$.

A path in $L$ is a finite sequence $B_0, B_1, ..., B_n$ ($n \geq 0$) in $\mathcal{A}_L$ satisfying $B_i \sim B_{i+1}$ whenever $0 \leq i < n$. The path is said to join the blocks $B_0$ and $B_n$. The number $n$ is said to be the length of the path. A path is said to be proper if and only if $n = 1$ or $B_i \approx B_{i+1}$ holds whenever $0 \leq i < n$. A path is called to be strictly proper if and only if $B_i \approx B_{i+1}$ holds whenever $0 \leq i < n$ [1].

Note that $A \approx B$ implies $A \sim B$, and $A \sim B$ implies $A \overset{w}{\sim} B$. Some authors, for example Greechie, use the phrase “$A$ and $B$ meet in the section $S_e$” to describe $A \overset{w}{\sim} B$ [3].

**Definition 1.2.** Let $L$ be an OML, and $A, B \in \mathcal{A}_L$. We will say that $A$ and $B$ are path-connected in $L$, strictly path-connected in $L$ if $A$ and $B$ are joined by a proper path, a strictly proper path, respectively. We will say $A$ and $B$ are nonpath-connected if there is no proper path joining $A$ and $B$, and $L$ is called nonpath-connected if there exist two blocks which are nonpath-connected. An OML $L$ is called path-connected in $L$, strictly path-connected in $L$ if any two blocks in $L$ are joined by a proper path, a strictly proper path, respectively. An OML $L$ is called relatively path-connected iff each $[0, x]$ is path-connected for all $x \in L$.

The following two lemmas are well known.

**Lemma 1.3.** If $L$ is an OML with two blocks $A, B$ and $a \in A \setminus B$ and $b \in B \setminus A$, then $A \cap B = S_{a \ast b}$. If $A \cap B = S_e$, then $c = a \ast b$ [1].
Let $A, B$ be two blocks of an OML $L$. If $A \sim B$ holds, then there exists a unique element $e \in A \cap B$ satisfying $A \cap B = (A \cup B) \cap S_e$ by lemma 2.2. Therefore we say that $A$ and $B$ are linked at $e$ (strongly linked at $e$) if $A \sim B$ ($A \approx B$) and use the notation $A \sim_e B$ ($A \approx_e B$).

**Lemma 1.4.** [Bruns] If $L_1, L_2$ are OMLs, $L = L_1 \times L_2$, $A, B \in A_{L_1}$ and $C, D \in A_{L_2}$, then $A \times C \sim B \times D$ holds in $L$ if and only if either $A = B$ and $C \sim D$ or $A \sim B$ and $C = D$. If $A$ and $B$ are linked at $a$ then $A \times C$ and $B \times C$ are linked at $(a, 0)$. If $C$ and $D$ are linked at $c$ then $A \times C$ and $A \times D$ are linked at $(0, c)$ [1].

An OML $L$ is called the horizontal sum of a family $(L_i)_{i \in I}$ (denoted by $\sigma(L_i)_{i \in I}$) of at least two subalgebras, if $\bigcup L_i = L$, and $L_i \cap L_j = \{0, 1\}$ whenever $i \neq j$, and one of the following equivalent conditions is satisfied:

1. if $x \in L_i \setminus L_j$ and $y \in L_j \setminus L_i$, then $x \lor y = 1$;
2. every block of $L$ belongs to some $L_i$;
3. if $S_i$ is a subalgebra of $L_i$, then $\bigcup S_i$ is a subalgebra of $L$ [2].

Note that the horizontal sum of a family $(L_i)_{i \in I}$ of path-connected OML $L_i (i \in I)$ is a path-connected OML.

Bruns introduced a construction which is more general than the horizontal sum, and proved the following lemma 1.5 [1].

An OML $L$ is said to be the weak horizontal sum of a family $(L_i)_{i \in I}$ of subalgebras if and only if there exists an isomorphism $f$ of $L$ onto a product of $L_0 \times L'$ of a Boolean algebra $L_0$ and an OML $L'$ such that the subalgebra $L_i$ of $L$ correspond via $f$ to subalgebras of the form $L_0 \times L'_i$ and $L'$ is the the horizontal sum of the family $(L'_i)_{i \in I}$.

**Lemma 1.5.** Every OML $L$ with only two blocks is isomorphic with an OML of the form $B \times (A \circ C)$ where $A$, $B$, $C$ are Boolean algebras and $A \circ C$ is the horizontal sum of $A$ and $C$. In other words, every OML with only two blocks is the weak horizontal sum of its blocks [1].

2. Non path-connected orthomodular lattices

We know that every finite direct product of path-connected OMLs is path-connected [8], and we have the following class of nonpath-connected OMLs. We can find some examples and properties of path-connected OMLs in [1, 2, 7, 8].
PROPOSITION 2.1. Every infinite direct product of OMLs containing infinitely many non-Boolean factors is a nonpath-connected OML.

PROOF. Let \( L \cong \prod_{\alpha \in \mathcal{I}} L_{\alpha} \) where \(|\mathcal{I}| \geq \omega \) and each \( L_{\alpha} \) is OML. Let \( J = \{ j \in \mathcal{I} \mid L_j \text{ is a Boolean algebra} \} \). Then \( B = \prod_{j \in J} L_j \) is a Boolean factor of \( L \). Thus \( L \cong B \times \prod_{i \in I \setminus J} L_i \) such that \(|I \setminus J| \geq \omega \) and each \( L_i \ (i \in I \setminus J) \) is non Boolean path-connected OML. Therefore it is sufficient to show that \( \prod_{i \in I \setminus J} L_i \) is not path-connected by lemma 1.4. Since each \( L_i \ (i \in I \setminus J) \) is a path-connected OML containing at least two distinct blocks, there exist distinct \( A_i, B_i \in A_{L_i}, \forall i \in I \setminus J \) such that \( A_i \cup B_i \leq L_i \). Let \( A = \prod_{i \in (I \setminus J)} A_i \) and \( B = \prod_{i \in (I \setminus J)} B_i \). Then \( A \) and \( B \) are not path-connected since there is no path of finite length from \( A \) to \( B \) by lemma 1.4.

For the remainder of this paper, let \( H \) be an infinite dimensional Hilbert space over the real or complex numbers. A linear manifold is a nonempty subset \( M \) of \( H \) such that if \( x \) and \( y \) are in \( M \), then \( ax + by \in M \) for every pair of complex numbers \( a \) and \( b \). A closed subspace is a closed linear manifold. The closed subspace spanned by an arbitrary subset \( M \) of \( H \) is defined to be the intersection of all closed subspaces containing \( M \). The vector sum of two closed subspaces \( M \) and \( N \) in symbols \( M + N \), is defined to be the set of all vectors of the form \( x + y \) with \( x \in M \) and \( y \in N \). If \( M \) and \( N \) are closed subspaces, we use the symbol \( M \vee N \) for the closed subspace spanned by \( M \) and \( N \). It follows by this definition that \( M \vee N \) is the smallest closed subspace containing both \( M \) and \( N \).

We need the following lemmas to prove that the OML \( C(H) \) of all closed subspaces of an infinite dimensional Hilbert space \( H \) is a nonpath-connected OML.

LEMMA 2.2. Let \( A, B \) be distinct atomic blocks of \( C(H) \) with \( A \cup B \leq L \). Then there exists \( \alpha \in \text{Com} \ C(H) \) such that \( A \cup B = (A \cup B)[0, \alpha'] \oplus (A \cup B)[0, \alpha] \) where \((A \cup B)[0, \alpha] \cong MO2 \) and \((A \cup B)[0, \alpha'] \) is a Boolean algebra. In particular, \( h(\alpha) = 2 \) in \( C(H) \).

PROOF. We know that \( A \cup B = (A \cup B)[0, \alpha'] \oplus (A \cup B)[0, \alpha] \) for some \( \alpha \in \text{Com} \ C(H) \) by lemma 1.3 where \((A \cup B)[0, \alpha] \) is a horizontal sum of \( A[0, \alpha] \) and \( B[0, \alpha] \), and \((A \cup B)[0, \alpha'] \) is a Boolean algebra by lemma 1.5. Therefore it is sufficient to show that \( h(\alpha) = 2 \). Let \( a, b \) be
two distinct atoms of $C(H)$ such that $a \in A \setminus B$ and $b \in B \setminus A$. Then $\alpha = a \lor b$ since $(A \cup B)[0, \alpha]$ is a horizontal sum of $A[0, \alpha]$ and $B[0, \alpha]$. Thus $h(\alpha) = h(a \lor b) = 2$ since $a, b$ are atoms. We are done. ■

**Lemma 2.3.** If $F$ is a finite dimensional linear manifold in a Hilbert space $H$, and if $S$ is a closed subspace in $H$, then the vector sum $F + S$ is necessarily closed (and hence is therefore equal to the span $F \lor S$) [p9, 6].

As a consequence of lemma 2.3 every finite dimensional linear manifold is closed, since $S = \{0\}$ is a closed subspace of $H$.

**Lemma 2.4.** If $F$ is a finite dimensional closed subspace of a Hilbert space $H$ and $S$ is a closed subspace of $H$ such that $S^\perp$ is an infinite dimensional closed subspace of $H$. Then the closed subspace $F \lor S$ in $C(H)$ spanned by $F$ and $S$ is a proper closed subspace of $H$.

**Proof.** The closed subspace $F \lor S$ of $H$ spanned by $F$ and $S$ is equal to $F + S$ by lemma 2.3. Thus the quotient space $(F \lor S)/S = (F + S)/S$ is a proper closed subspace in $S^\perp$ since $F$ is finite dimensional and $S^\perp$ is infinite dimensional. Hence $S \lor F$ is a proper closed subspace of $H$. ■

**Lemma 2.5.** If $B$ is a block of $A(C(H))$, then there exists a unique element $x \in C(H)$ such that $B = B[0, x] \oplus B[0, x']$ where $B[0, x]$ is atomic and $B[0, x']$ totally nonatomic. Moreover, $B$ is atomic iff $x = 1$; and $B$ is totally nonatomic iff $x = 0$.

**Proof.** Let $\{a_i\}_{i \in I}$ be the set of all atoms in a block $B$ of the OML $C(H)$. Then $\bigvee_{i \in I} a_i$ exists since $C(H)$ is complete [p65, 5]. Let $x = \bigvee_{i \in I} a_i$. Then $x \in B$ since $B$ is subcomplete, $B[0, x]$ is atomic and $B[0, x']$ is totally nonatomic. ■

**Lemma 2.6.** Let $A$ be an atomic block of $C(H)$, and $B$ be a nonatomic block of $C(H)$. Then $A \cup B \not\subseteq C(H)$.

**Proof.** Suppose $A \cup B \subseteq C(H)$. Then $A \cup B = (A \cup B)[0, \alpha] \oplus (A \cup B)[0, \alpha']$ for some $\alpha \in Com C(H)$ by lemma 1.3 where $(A \cup B)[0, \alpha]$ is a horizontal sum of $A[0, \alpha]$ and $B[0, \alpha]$, and $(A \cup B)[0, \alpha']$ is a Boolean algebra by lemma (1.5). Moreover, $(A \cup B)[0, \alpha']$ is atomic since $A[0, \alpha']$
is atomic and \((A \cup B)[0, \alpha'] = A[0, \alpha']\). By lemma 2.5, there exists \(x \in B\) such that \(B[0, x]\) is totally nonatomic. Since \(A[0, \alpha']\) is atomic, \(B[0, x \wedge \alpha']\) is totally nonatomic and \(x \wedge \alpha' \in B[0, \alpha'] = A[0, \alpha']\) it follows that \(x \wedge \alpha' = 0\). Thus \(x = (x \wedge \alpha) \lor (x \wedge \alpha') = x \wedge \alpha\) so that \(x \leq \alpha\).

We may assume that \(0 < x < \alpha\). Moreover \(h(x) = \infty\) since \([0, x]\) is nonatomic. Let \(a\) be an atom of \(A[0, \alpha]\). Then \(a \lor x = a \lor (x' \wedge \alpha) = \alpha\), since \((A \cup B)[0, \alpha]\) is a horizontal sum of \(A[0, \alpha]\) and \(B[0, \alpha]\) and \(x' \wedge \alpha\) is the relative orthocomplement of \(X\) in \((A \cup B)[0, x]\). This contradicts \(a \lor (x' \wedge \alpha) < \alpha\) by applying lemma 2.4 to the Hilbert space \(\alpha\) since \(h(a) = 1\), \(h(\alpha) = \infty\), \(x\) is the orthocomplement of \(x' \wedge \alpha\) in \(B[0, \alpha]\), and \(h(x) = \infty\). We are done.

**Lemma 2.7.** Let \(A\) and \(B\) be atomic path-connected blocks of a \(C(H)\) with a path \(A = C_0 \sim C_1 \sim C_2 \sim \ldots \sim C_{(n-1)} \sim C_n = B\). Then \(A \cap B \supseteq S_x \cap (A \cup B)\) for some \(x \in (A \cap B)\) with \(h(x) \leq 2n\).

**Proof.** We will prove the conclusion by induction on the length \(k\) of the path joining atomic blocks \(A\) and \(B\) of a \(C(H)\). If \(k = 1\), then \(A = C_0 \sim C_1 = B\). Thus \(A \cap B = S_x \cap (A \cup B)\) for some \(x \in (A \cap B)\) where \(h(x) = 2\) by lemma 2.2. Assume that the conclusion of the lemma is true for each path joining two blocks of \(C(H)\) with the length less than or equal to \(n - 1\). Let \(A = C_0 \sim C_1 \sim \ldots \sim C_{n-1} \sim C_n = B\) be a path from \(A\) to \(B\) of length \(n\). We may assume that \(C_{n-1} \neq A\) otherwise, \(A = C_{n-1} \sim B\) we are done by the case \(k = 1\). By induction hypothesis, \((A \cap C_{n-1}) \supseteq S_x \cap (A \cup C_{n-1})\) for some \(x \in (A \cap C_{n-1})\) with \(h(x) \leq 2(n - 1)\), and \((C_{n-1} \cap B) \supseteq S_y \cap (C_{n-1} \cup B)\) for some \(y \in (C_{n-1} \cap B)\) with \(h(y) \leq 2\). Thus \((A \cap B) \supseteq (A \cap C_{n-1}) \cap (C_{n-1} \cap B) \supseteq (S_x \cap (A \cup C_{n-1})) \cap (S_y \cap (C_{n-1} \cup B))\), and \((S_x \cap (A \cup C_{n-1})) \cap (S_y \cap (C_{n-1} \cup B)) \supseteq S_{(x \lor y)} \cap (A \cup B)\) since \(S_x \cap (A \cup C_{n-1}) = S_x \cap A = S_x \cap C_{n-1},\)

\(S_y \cap (C_{n-1} \cup B) = S_y \cap C_{n-1} = S_y \cap B\) and \(S_x \cap S_y \supseteq S_{(x \lor y)}\). Moreover, \(x \lor y \in (A \cap B)\) and \(h(x \lor y) \leq h(x) + h(y) \leq 2(n - 1) + 2 = 2n\). We are done.

**Lemma 2.8.** [Greechie] Let \(L\) be an OML, let \(\{e_\alpha \mid \alpha \in I\}\) be a maximal orthogonal family of nonzero elements of \(L\), let \(\{B_\alpha \mid \alpha \in I\}\) be a collection of atomic blocks of \(L\) such that \(e_\alpha \in B_\alpha\) for all \(\alpha \in I\), let \(M = \bigcup \{B_\alpha[0, e_\alpha] \mid \alpha \in I\}\), and let \(B = C(M)\). Then \(B\) is an atomic block of \(L\) [3].
Now, we are ready to prove one of our main theorems.

**Theorem 2.9.** The OML $\mathcal{C}(H)$ of all closed subspaces of an infinite dimensional Hilbert space $H$ is a nonpath-connected OML.

**Proof.** First, assume that $H$ is a separable Hilbert space. Let $(e_1, e_2, e_3, \ldots)$ be an orthonormal basis of the separable Hilbert space $H$. Let $f_{2i-1} = \frac{e_{2i-1} + e_{2i}}{\sqrt{2}}$ and $f_{2i} = \frac{e_{2i-1} - e_{2i}}{\sqrt{2}} \ \forall (1 \leq i < \infty)$. Then $(f_1, f_2, f_3, \ldots)$ is an orthonormal basis of $H$. Let $[e_i] = \{ \lambda e_i \mid \lambda \in \mathbb{C} \}$ and $[f_i] = \{ \lambda f_i \mid \lambda \in \mathbb{C} \} \ \forall 1 \leq i$ where $\mathbb{C}$ is the complex numbers, let $A = \mathbb{C}([e_i] \mid 1 \leq i)$, and let $B = \mathbb{C}([f_i] \mid 1 \leq i)$. Then $A$ and $B$ are atomic blocks in $\mathcal{C}(H)$ by lemma 2.8 since $\{ [e_i] \mid i \in I \}$ and $\{ [f_i] \mid i \in I \}$ are maximal orthogonal families of atoms of $\mathcal{C}(H)$. We claim that $A$ and $B$ are nonpath-connected in $\mathcal{C}(H)$. Suppose $A$ and $B$ are path-connected with a path $A = C_0 \sim C_1 \sim \ldots \sim C_{n-1} \sim C_n = B$. $A \neq B$ and $A \not\sim B$, since $A \cup B \not\subseteq \mathcal{C}(H)$ by lemma 2.2. Thus $n \geq 2$. If the path joining $A$ and $B$ contains only atomic blocks, then by lemma 2.7 $A \cap B \supseteq S_x \cap (A \cup B)$ for some $x \in A \cap B$ with $h(x) \leq 2n$, contradicting there is no $x \in A \cap B$ such that $A \cap B \supseteq S_x \cap (A \cup B)$ with $h(x) \leq 2n$ by our choice of $A$ and $B$. Thus we may assume that one of $C_1, C_2, \ldots, C_{n-1}$ is nonatomic. Let $C_i$ be the nonatomic block with the smallest index in the path joining $A$ and $B$, and hence $C_{i-1}$ is atomic. Thus $C_{i-1} \sim C_i$, and hence $C_{i-1} \cup C_i \subseteq \mathcal{C}(H)$ contradicting $C_{i-1} \cup C_i \not\subseteq \mathcal{C}(H)$ by lemma 2.6. Therefore $A$ and $B$ are nonpath-connected.

Finally, if $H$ is nonseparable infinite dimensional Hilbert space, then there exists $x \in \mathcal{C}(H)$ such that $x$ is a separable infinite dimensional Hilbert subspace of $H$. Let $(g_1, g_2, g_3, \ldots)$ be an orthonormal basis of $x$. Let $h_{2i-1} = \frac{g_{2i-1} + g_{2i}}{\sqrt{2}}$ and $h_{2i} = \frac{g_{2i-1} - g_{2i}}{\sqrt{2}} \ \forall (1 \leq i < \infty)$. Then $(h_1, h_2, h_3, \ldots)$ is an orthonormal basis of $x$. Let $[g_i] = \{ \lambda g_i \mid \lambda \in \mathbb{C} \}$ and $[h_i] = \{ \lambda h_i \mid \lambda \in \mathbb{C} \} \ \forall 1 \leq i$ where $\mathbb{C}$ is the set of all complex numbers, let $D = \mathbb{C}([g_i] \mid 1 \leq i)$, and let $E = \mathbb{C}([h_i] \mid 1 \leq i))$. Then $D$ and $E$ are an atomic blocks in $x$ by the above argument. Let $F$ be an atomic block of $x'$. Then $D \oplus F$ and $E \oplus F$ are distinct atomic blocks of $\mathcal{C}(H)$. Now, the desired conclusion follows by applying lemmas 2.2, 2.7 and 2.6 to the blocks $D \oplus F$ and $E \oplus F$. $\blacksquare$
Corollary 2.10. $H$ is a finite dimensional Hilbert space if and only if $C(H)$ is path-connected.

Proof. If $H$ is finite dimensional, then $C(H)$ is chain-finite. Thus $C(H)$ is path-connected since every chain-finite OML is path-connected[7]. Conversely, if $H$ is infinite dimensional, then $C(H)$ is nonpath-connected by theorem 2.9.

References


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