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ON A SPECTRAL EQUIVALENCE

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ABSTRACT. In this note we prove that Apostol's conjecture is true for the operators having totally disconnected spectra and also give a result related on compalence.

1. Introduction

Throughout this note suppose X is a complex Banach space and write $\mathcal{L}(X)$ for the set of all bounded linear operators on X. Also we write $\sigma(T)$ and $\sigma_e(T)$ for the *spectrum* and the *essential spectrum*, respectively, of T, and ∂K and iso K for the topological boundary and the set of all isolated points, respectively, of $K \subseteq \mathbb{C}$. We recall ([**1, 7, 8, 9, 10**]) that if Ω is an open set in $\mathbb{C}$, and for each λ in Ω if $T(\lambda) \in \mathcal{L}(X)$ and $S(\lambda) \in \mathcal{L}(X)$, then the operator functions $T(\cdot)$ and $S(\cdot)$ are called *(globally) equivalent on* Ω if there exist operator functions $E : \Omega \to \mathcal{L}(X)$ and $F : \Omega \to \mathcal{L}(Y)$, which are analytic on Ω, such that

$$T(\lambda) = F(\lambda)S(\lambda)E(\lambda), \quad \lambda \in \Omega,$$

and, in addition, $E(\lambda)$ and $F(\lambda)$ are invertible for each $\lambda \in \Omega$. In this note we shall be concerned mainly with the case when $T(\lambda) = \lambda - T_1$ and $S(\lambda) = \lambda - T_2$ with $T_1, T_2 \in \mathcal{L}(X)$. Given an operator function $T : \Omega \to \mathcal{L}(X)$ and a Banach space Z, we call the operator function

$$\begin{bmatrix} T(\cdot) & 0 \\ 0 & I_Z \end{bmatrix} : \Omega \longrightarrow \mathcal{L}(X \oplus Z)$$

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the *Z-extension* of $T(\cdot)$.

The following lemma shows that for an operator function of the form $\lambda - T$ the procedure of linearization by extension and equivalence does not simplify further the operator T and leads to operators that are similar to T.

LEMMA 1. ([13, Theorem 2]) *Let $T_1, T_2 \in \mathcal{L}(X)$ and suppose for some Banach space Z the extensions $(\lambda - T_1) \oplus I_Z$ and $(\lambda - T_2) \oplus I_Z$ are equivalent on some open set Ω containing $\sigma(T_1) \cup \sigma(T_2)$. Then T_1 and T_2 are similar. In fact, if the equivalence is given by*

$$\begin{bmatrix} \lambda - T_1 & 0 \\ 0 & I_Z \end{bmatrix} = F(\lambda) \begin{bmatrix} \lambda - T_2 & 0 \\ 0 & I_Z \end{bmatrix} E(\lambda), \quad \lambda \in \Omega,$$

then $ST_1 = T_2 S$, where $S \in \mathcal{L}(X)$ is an invertible operator defined by

$$S = \frac{1}{2\pi i} \int_\Gamma (\lambda - T_2)^{-1} \rho F(\lambda)^{-1} \tau d\lambda,$$

where Γ is the boundary of some bounded Cauchy domain Δ such that $(\sigma(T_1) \cup \sigma(T_2)) \subset \Delta \subset \overline{\Delta} \subset \Omega$, the map $\rho : X \oplus Z \to X$ is the projection of $X \oplus Z$ onto X and the map $\tau : X \to X \oplus Z$ is the natural embedding of X into $X \oplus Z$.

For linear functions $\lambda - T_1$ and $\lambda - T_2$, global equivalence on $\mathbb{C}$ means just that T_1 and T_2 are similar. The converse is also true because

$$ST_1 = T_2 S \text{ for an invertible } S \in \mathcal{L}(X) \implies \lambda - T_1 = S^{-1}(\lambda - T_2)S.$$

Thus we have

(1.1) T_1 and T_2 are similar if and only if $\lambda - T_1$ and $\lambda - T_2$ are equivalent on $\mathbb{C}$.

We also recall that given λ_0 in Ω, we say that $T(\cdot)$ and $S(\cdot)$ are *(locally) equivalent at* λ_0 if there exists an open neighborhood $\mathcal{U}$ of λ_0 in Ω such that

$$T(\lambda) = F(\lambda) S(\lambda) E(\lambda), \quad \lambda \in \mathcal{U},$$

where $E(\lambda)$ and $F(\lambda)$ are invertible operators which depend analytically on λ in $\mathcal{U}$. In other words, the operator functions $T(\cdot)$ and $S(\cdot)$ are equivalent at λ_0 if they are globally equivalent on an open neighborhood of λ_0.

If two operator functions $T(\cdot)$ and $S(\cdot)$ are globally equivalent on an open set Ω, then, obviously, $T(\cdot)$ and $S(\cdot)$ are locally equivalent at each point of Ω. For certain special classes of operator functions the converse statement is also true, however, in general, local equivalence at each point of Ω does not imply global eqivalence on Ω (see [**9, 14, 15**]). But the question whether or not local equivalence implies global equivalence in case when $T(\lambda) = \lambda - T_1$ and $S(\lambda) = \lambda - T_2$ with $T_1, T_2 \in \mathcal{L}(X)$ is an unsolved problem (see [**8**]). Apostol ([**1**]) conjectured that the answer is affirmative. In view of (1.1), the conjecture can be rephrased as follows:

Apostol's Conjecture. If $\lambda - T_1$ and $\lambda - T_2$ are equivalent at every point of $\mathbb{C}$ then T_1 and T_2 are similar.

The above conjecture was proved for certain special classes of operators: for example, for normal operators ([**1**]), the unilateral shifts of finite multiplicity ([**1**]), and the compact operators ([**7**]).

In this note we prove that Apostol's conjecture is true for the operators having totally disconnected spectra and also give a result related on compalence.

2. Equivalence and similarity

Our main result is the following:

THEOREM 2. *Let $T_1, T_2 \in \mathcal{L}(X)$ and suppose T_1 and T_2 have totally disconnected spectra. If $\lambda - T_1$ and $\lambda - T_2$ are equivalent at every point of $\mathbb{C}$ then T_1 and T_2 are similar.*

PROOF. Suppose for some open neighborhood $\mathcal{U}(\mu)$ of each $\mu \in \mathbb{C}$,

$$\lambda - T_1 = F(\lambda)(\lambda - T_2)E(\lambda), \quad \lambda \in \mathcal{U}(\mu), \tag{2.1}$$

where $E(\lambda)$ and $F(\lambda)$ are invertible and analytic on $\mathcal{U}(\mu)$. Let $\sigma(T_1)$ be totally disconnected. Since $\sigma(T_1)$ is compact it follows that if $\mu \in \sigma(T_1)$

and $\mathcal{U}(\mu)$ is an open neighborhood of μ which satisfies (2.1), then there is a subset $\mathcal{V}$ of $\sigma(T_1)$ that is both open and closed and such that $\mu \in \mathcal{V} \subseteq \mathcal{U}(\mu)$. Thus both $\mathcal{V}$ and $\sigma(T_1)\setminus\mathcal{V}$ are closed subsets of $\sigma(T_1)$, that is, $\mathcal{V}$ is an isolated part of $\sigma(T_1)$. Further, by (2.1), $\mathcal{V}$ is also an isolated part of $\sigma(T_2)$. Thus we can find the Riesz projections P_1 and P_2 of, respectively, T_1 and T_2, corresponding to $\mathcal{V}$. Write T_1 and T_2 as 2×2 operator matrices relative to the decomposition $X = \operatorname{Im} P_i \oplus \operatorname{Ker} P_i$ $(i = 1, 2)$:

$$T_1 = \begin{bmatrix} T_{11} & 0 \\ 0 & T_{12} \end{bmatrix} \quad \text{and} \quad T_2 = \begin{bmatrix} T_{21} & 0 \\ 0 & T_{22} \end{bmatrix}.$$

Then $\sigma(T_{11}) = \sigma(T_{21}) = \mathcal{V}$ and $\lambda - T_{12}$ and $\lambda - T_{22}$ are invertible for all $\lambda \in \mathcal{V}$. Define

$$E_0(\lambda) = \begin{bmatrix} I & 0 \\ 0 & \lambda - T_{22} \end{bmatrix} E(\lambda), \quad \lambda \in \mathcal{V}$$

and

$$F_0(\lambda) = \begin{bmatrix} I & 0 \\ 0 & (\lambda - T_{12})^{-1} \end{bmatrix} F(\lambda), \quad \lambda \in \mathcal{V}.$$

Then $E_0(\lambda)$ and $F_0(\lambda)$ are invertible and analytic on $\mathcal{V}$. Furthermore,

$$\begin{bmatrix} \lambda - T_{11} & 0 \\ 0 & I \end{bmatrix} = F_0(\lambda) \begin{bmatrix} \lambda - T_{21} & 0 \\ 0 & I \end{bmatrix} E_0(\lambda), \quad \lambda \in \mathcal{V}.$$

Since $\sigma(T_{11}) = \sigma(T_{21}) = \mathcal{V}$, it follows from Lemma 1 that T_{11} and T_{21} are similar. This process with $\sigma(T_1) \cap \mathcal{V}^c$ must stop after a finite number of steps since $\sigma(T_1)$ is compact. Thus we can construct piecewise the needed similarity.

Note that in Theorem 2 it is not necessary to assume that both T_1 and T_2 have totally disconnected spectra. In fact it suffices to assume that one of the operators has a totally disconnected spectrum, and then the similarity implies that the other also has a totally disconnected spectrum.

COROLLARY 3. *Let $T_1, T_2 \in \mathcal{L}(X)$ and suppose $\sigma_\epsilon(T_1)$ is totally disconnected. If $\lambda - T_1$ and $\lambda - T_2$ are equivalent at every point of $\mathbb{C}$ then T_1 and T_2 are similar.*

PROOF. Since by the punctured neighborhood theorem ([**7, 11, 12**]),

$$\partial\,\sigma(T) \setminus \sigma_e(T) \subseteq \operatorname{iso}\sigma(T),$$

it follows that if $\sigma(T)$ has a connected part then $\sigma_e(T)$ contains a connected set, say the boundary of the component. Thus if $\sigma_e(T_1)$ is totally disconnected then so is $\sigma(T_1)$. Therefore by Theorem 2 and the preceeding remark, T_1 and T_2 are similar.

3. Equivalence and compalence

In the section suppose H is a complex separable Hilbert space. We recall ([**17**]) that if $T_1, T_2 \in \mathcal{L}(H)$ then T_1 and T_2 are said to be *compalent* if there exists an unitary operator $W \in \mathcal{L}(H)$ and a compact operator $K \in \mathcal{L}(H)$ such that $WT_1W^* + K = T_2$ and that an operator $T \in \mathcal{L}(H)$ is *essentially normal* if $T^*T - TT^*$ is a compact operator. The *spectral picture* (cf. [**3, 5, 17**]) of an operator $T \in \mathcal{L}(H)$, denoted by $\mathcal{SP}(T)$, is the structure consisting of $\sigma_e(T)$, the collection of holes and pseudoholes in $\sigma_e(T)$, and the indices associated with these holes and pseudoholes. Then the celebrated Brown-Douglas-Fillmore Theorem ([**2**]) says that if T_1 and T_2 are essentially normal then

$$\text{(3.1)}\qquad T_1 \text{ and } T_2 \text{ are compalent} \quad \text{if and only if} \quad \mathcal{SP}(T_1) = \mathcal{SP}(T_2).$$

We are ready for:

THEOREM 4. *Suppose $A, B \in \mathcal{L}(H)$ such that $\sigma_e(A)$ is an arc and B is a Riesz operator. Let $T_1 = A \otimes B$ and T_2 be subnormal on $H \otimes H$. If $\lambda - T_1$ and $\lambda - T_2$ are equivalent at every point of $\mathbb{C}$ then T_1 and T_2 are compalent.*

PROOF. Since B is a Riesz operator, we have ([**4**])

$$\sigma_e(B) \subseteq \{0\} \quad \text{and} \quad \sigma(B) \text{ consists of isolated points.}$$

Thus if $\sigma_e(A)$ is an arc then

$$\sigma_e(T_1) = \sigma_e(A)\cdot\sigma(B) \bigcup \sigma(A)\cdot\sigma_e(B),$$

which has planar Lebesgue measure zero. Observe that the local equivalence of $\lambda - T_1$and $\lambda - T_2$ gives that

$$\sigma(T_1) = \sigma(T_2), \quad \sigma_e(T_1) = \sigma_e(T_2), \quad \text{and}$$
$$\text{index}(\lambda - T_1) = \text{index}(\lambda - T_2) \ \text{ for all } \lambda \notin \sigma_e(T_1),$$

which implies that T_1 and T_2 have the same spectral picture. If $\pi : \mathcal{L}(H) \to \mathcal{C}(H)$ ($\mathcal{C}(H)$ is the Calkin algebra) is the Calkin homomorphism then $\pi(T_1)$ and $\pi(T_2)$ are also subnormal. Thus by an argument of Stampfli ([**16**]), $\pi(T_1)$ and $\pi(T_2)$ are normal and hence T_1 and T_2 are essentially normal. Therefore, by (3.1), T_1 and T_2 are compalent.

COROLLARY 5. *Let $T_1, T_2 \in \mathcal{L}(H)$ be pure quasinormal and dim(ran $T_1)^\perp < \infty$. If $\lambda - T_1$ and $\lambda - T_2$ are equivalent at every point of $\mathbb{C}$ then T_1 and T_2 are compalent.*

PROOF. If T is pure quasinormal, then by [**6**, Theorem III.3.2], T is unitarily equivalent to $U_\alpha \otimes |T|^o$, where U_α denotes the unilateral shift of multiplicity α and $|T|^o = (T^*T)^{\frac{1}{2}}|_{(\text{ran } T)^\perp}$. Further, since α is the same as the dimension of $(\text{ran } T)^\perp$, the result follows from Theorem 4.

In view of Theorem 4 and Corollary 5, we have an interesting problem:

PROBLEM. Under what condition, does compalence imply similarity?

We were unable to solve this problem. We however conjecture that compalence under local equivalence implies similarity.

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