ON A SPECTRAL EQUIVALENCE

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ABSTRACT. In this note we prove that Apostol's conjecture is true for the operators having totally disconnected spectra and also give a result related on compalance.

1. Introduction

Throughout this note suppose $X$ is a complex Banach space and write $\mathcal{L}(X)$ for the set of all bounded linear operators on $X$. Also we write $\sigma(T)$ and $\sigma_e(T)$ for the spectrum and the essential spectrum, respectively, of $T$, and $\partial K$ and $\text{iso} K$ for the topological boundary and the set of all isolated points, respectively, of $K \subseteq \mathbb{C}$. We recall ([1, 7, 8, 9, 10]) that if $\Omega$ is an open set in $\mathbb{C}$, and for each $\lambda$ in $\Omega$ if $T(\lambda) \in \mathcal{L}(X)$ and $S(\lambda) \in \mathcal{L}(X)$, then the operator functions $T(\cdot)$ and $S(\cdot)$ are called (globally) equivalent on $\Omega$ if there exist operator functions $E : \Omega \to \mathcal{L}(X)$ and $F : \Omega \to \mathcal{L}(Y)$, which are analytic on $\Omega$, such that

$$T(\lambda) = F(\lambda)S(\lambda)E(\lambda), \quad \lambda \in \Omega,$$

and, in addition, $E(\lambda)$ and $F(\lambda)$ are invertible for each $\lambda \in \Omega$. In this note we shall be concerned mainly with the case when $T(\lambda) = \lambda - T_1$ and $S(\lambda) = \lambda - T_2$ with $T_1, T_2 \in \mathcal{L}(X)$. Given an operator function $T : \Omega \to \mathcal{L}(X)$ and a Banach space $Z$, we call the operator function

$$\begin{bmatrix} T(\cdot) & 0 \\ 0 & I_z \end{bmatrix} : \Omega \to \mathcal{L}(X \oplus Z)$$

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the $Z$-extension of $T(\cdot)$.

The following lemma shows that for an operator function of the form $\lambda - T$ the procedure of linearization by extension and equivalence does not simplify further the operator $T$ and leads to operators that are similar to $T$.

**Lemma 1.** ([13, Theorem 2]) Let $T_1, T_2 \in \mathcal{L}(X)$ and suppose for some Banach space $Z$ the extensions $(\lambda - T_1) \oplus I_Z$ and $(\lambda - T_2) \oplus I_Z$ are equivalent on some open set $\Omega$ containing $\sigma(T_1) \cup \sigma(T_2)$. Then $T_1$ and $T_2$ are similar. In fact, if the equivalence is given by

$$
\begin{bmatrix}
\lambda - T_1 & 0 \\
0 & I_Z
\end{bmatrix} = F(\lambda) 
\begin{bmatrix}
\lambda - T_2 & 0 \\
0 & I_Z
\end{bmatrix} E(\lambda), 
\lambda \in \Omega,
$$

then $ST_1 = T_2 S$, where $S \in \mathcal{L}(X)$ is an invertible operator defined by

$$
S = \frac{1}{2\pi i} \int_{\Gamma} (\lambda - T_2)^{-1} \rho F(\lambda)^{-1} \tau d\lambda,
$$

where $\Gamma$ is the boundary of some bounded Cauchy domain $\Delta$ such that $(\sigma(T_1) \cup \sigma(T_2)) \subset \Delta \subset \overline{\Delta} \subset \Omega$, the map $\rho : X \oplus Z \to X$ is the projection of $X \oplus Z$ onto $X$ and the map $\tau : X \to X \oplus Z$ is the natural embedding of $X$ into $X \oplus Z$.

For linear functions $\lambda - T_1$ and $\lambda - T_2$, global equivalence on $\mathbb{C}$ means just that $T_1$ and $T_2$ are similar. The converse is also true because

$$
ST_1 = T_2 S \text{ for an invertible } S \in \mathcal{L}(X) \implies \lambda - T_1 = S^{-1}(\lambda - T_2) S.
$$

Thus we have

(1.1) \quad $T_1$ and $T_2$ are similar if and only if $\lambda - T_1$ and $\lambda - T_2$ are equivalent on $\mathbb{C}$.

We also recall that given $\lambda_0$ in $\Omega$, we say that $T(\cdot)$ and $S(\cdot)$ are (locally) equivalent at $\lambda_0$ if there exists an open neighborhood $\mathcal{U}$ of $\lambda_0$ in $\Omega$ such that

$$
T(\lambda) = F(\lambda) S(\lambda) E(\lambda), \quad \lambda \in \mathcal{U},
$$
where $E(\lambda)$ and $F(\lambda)$ are invertible operators which depend analytically on $\lambda$ in $\mathcal{U}$. In other words, the operator functions $T(\cdot)$ and $S(\cdot)$ are equivalent at $\lambda_0$ if they are globally equivalent on an open neighborhood of $\lambda_0$.

If two operator functions $T(\cdot)$ and $S(\cdot)$ are globally equivalent on an open set $\Omega$, then, obviously, $T(\cdot)$ and $S(\cdot)$ are locally equivalent at each point of $\Omega$. For certain special classes of operator functions the converse statement is also true, however, in general, local equivalence at each point of $\Omega$ does not imply global equivalence on $\Omega$ (see [9, 14, 15]). But the question whether or not local equivalence implies global equivalence in case when $T(\lambda) = \lambda - T_1$ and $S(\lambda) = \lambda - T_2$ with $T_1, T_2 \in \mathcal{L}(X)$ is an unsolved problem (see [8]). Apostol ([1]) conjectured that the answer is affirmative. In view of (1.1), the conjecture can be rephrased as follows:

Apostol's Conjecture. If $\lambda - T_1$ and $\lambda - T_2$ are equivalent at every point of $\mathbb{C}$ then $T_1$ and $T_2$ are similar.

The above conjecture was proved for certain special classes of operators: for example, for normal operators ([1]), the unilateral shifts of finite multiplicity ([1]), and the compact operators ([7]).

In this note we prove that Apostol's conjecture is true for the operators having totally disconnected spectra and also give a result related on compalance.

2. Equivalence and similarity

Our main result is the following:

**Theorem 2.** Let $T_1, T_2 \in \mathcal{L}(X)$ and suppose $T_1$ and $T_2$ have totally disconnected spectra. If $\lambda - T_1$ and $\lambda - T_2$ are equivalent at every point of $\mathbb{C}$ then $T_1$ and $T_2$ are similar.

**Proof.** Suppose for some open neighborhood $\mathcal{U}(\mu)$ of each $\mu \in \mathbb{C}$,

\[(2.1) \quad \lambda - T_1 = F(\lambda)(\lambda - T_2)E(\lambda), \quad \lambda \in \mathcal{U}(\mu), \]

where $E(\lambda)$ and $F(\lambda)$ are invertible and analytic on $\mathcal{U}(\mu)$. Let $\sigma(T_1)$ be totally disconnected. Since $\sigma(T_1)$ is compact it follows that if $\mu \in \sigma(T_1)$
and \( \mathcal{U}(\mu) \) is an open neighborhood of \( \mu \) which satisfies (2.1), then there is a subset \( \mathcal{V} \) of \( \sigma(T_1) \) that is both open and closed and such that \( \mu \in \mathcal{V} \subseteq \mathcal{U}(\mu) \). Thus both \( \mathcal{V} \) and \( \sigma(T_1) \setminus \mathcal{V} \) are closed subsets of \( \sigma(T_1) \), that is, \( \mathcal{V} \) is an isolated part of \( \sigma(T_1) \). Further, by (2.1), \( \mathcal{V} \) is also an isolated part of \( \sigma(T_2) \). Thus we can find the Riesz projections \( P_1 \) and \( P_2 \) of, respectively, \( T_1 \) and \( T_2 \), corresponding to \( \mathcal{V} \). Write \( T_1 \) and \( T_2 \) as \( 2 \times 2 \) operator matrices relative to the decomposition \( X = \text{Im} P_i \oplus \text{Ker} P_i \) \( (i = 1, 2) \):

\[
T_1 = \begin{bmatrix} T_{11} & 0 \\ 0 & T_{12} \end{bmatrix} \quad \text{and} \quad T_2 = \begin{bmatrix} T_{21} & 0 \\ 0 & T_{22} \end{bmatrix}.
\]

Then \( \sigma(T_{11}) = \sigma(T_{21}) = \mathcal{V} \) and \( \lambda - T_{12} \) and \( \lambda - T_{22} \) are invertible for all \( \lambda \in \mathcal{V} \). Define

\[
E_0(\lambda) = \begin{bmatrix} I & 0 \\ 0 & \lambda - T_{22} \end{bmatrix} E(\lambda), \quad \lambda \in \mathcal{V}
\]

and

\[
F_0(\lambda) = \begin{bmatrix} I & 0 \\ 0 & (\lambda - T_{12})^{-1} \end{bmatrix} F(\lambda), \quad \lambda \in \mathcal{V}.
\]

Then \( E_0(\lambda) \) and \( F_0(\lambda) \) are invertible and analytic on \( \mathcal{V} \). Furthermore,

\[
\begin{bmatrix} \lambda - T_{11} & 0 \\ 0 & I \end{bmatrix} = F_0(\lambda) \begin{bmatrix} \lambda - T_{21} & 0 \\ 0 & I \end{bmatrix} E_0(\lambda), \quad \lambda \in \mathcal{V}.
\]

Since \( \sigma(T_{11}) = \sigma(T_{21}) = \mathcal{V} \), it follows from Lemma 1 that \( T_{11} \) and \( T_{21} \) are similar. This process with \( \sigma(T_i) \cap \mathcal{V}^c \) must stop after a finite number of steps since \( \sigma(T_1) \) is compact. Thus we can construct piecewise the needed similarity.

Note that in Theorem 2 it is not necessary to assume that both \( T_1 \) and \( T_2 \) have totally disconnected spectra. In fact it suffices to assume that one of the operators has a totally disconnected spectrum, and then the similarity implies that the other also has a totally disconnected spectrum.

**Corollary 3.** Let \( T_1, T_2 \in \mathcal{L}(X) \) and suppose \( \sigma_e(T_1) \) is totally disconnected. If \( \lambda - T_1 \) and \( \lambda - T_2 \) are equivalent at every point of \( \mathbb{C} \) then \( T_1 \) and \( T_2 \) are similar.
PROOF. Since by the punctured neighborhood theorem ([7, 11, 12]),
\[ \partial \sigma(T) \setminus \sigma_e(T) \subseteq \text{iso } \sigma(T), \]
it follows that if \( \sigma(T) \) has a connected part then \( \sigma_e(T) \) contains a
connected set, say the boundary of the component. Thus if \( \sigma_e(T_1) \) is totally
disconnected then so is \( \sigma(T_1) \). Therefore by Theorem 2 and the preceding
remark, \( T_1 \) and \( T_2 \) are similar.

3. Equivalence and compalence

In the section suppose \( H \) is a complex separable Hilbert space. We
recall ([17]) that if \( T_1, T_2 \in \mathcal{L}(H) \) then \( T_1 \) and \( T_2 \) are said to be compalent
if there exists an unitary operator \( W \in \mathcal{L}(H) \) and a compact operator
\( K \in \mathcal{L}(H) \) such that \( WT_1W^* + K = T_2 \) and that an operator \( T \in \mathcal{L}(H) \)
is essentially normal if \( T^*T - TT^* \) is a compact operator. The spectral
picture (cf. [3, 5, 17]) of an operator \( T \in \mathcal{L}(H) \), denoted by \( \mathcal{SP}(T) \), is
the structure consisting of \( \sigma_e(T) \), the collection of holes and pseudoholes
in \( \sigma_e(T) \), and the indices associated with these holes and pseudoholes.
Then the celebrated Brown-Douglas-Fillmore Theorem ([2]) says that if
\( T_1 \) and \( T_2 \) are essentially normal then

\[(3.1) \quad T_1 \text{ and } T_2 \text{ are compalent } \text{ if and only if } \mathcal{SP}(T_1) = \mathcal{SP}(T_2). \]

We are ready for:

THEOREM 4. Suppose \( A, B \in \mathcal{L}(H) \) such that \( \sigma_e(A) \) is an arc and \( B \)
is a Riesz operator. Let \( T_1 = A \otimes B \) and \( T_2 \) be subnormal on \( H \otimes H \). If
\( \lambda - T_1 \) and \( \lambda - T_2 \) are equivalent at every point of \( \mathbb{C} \) then \( T_1 \) and \( T_2 \) are
compalent.

PROOF. Since \( B \) is a Riesz operator, we have ([4])
\[ \sigma_e(B) \subseteq \{0\} \quad \text{and} \quad \sigma(B) \text{ consists of isolated points}. \]
Thus if \( \sigma_e(A) \) is an arc then
\[ \sigma_e(T_1) = \sigma_e(A) \cdot \sigma(B) \bigcup \sigma(A) \cdot \sigma_e(B), \]
which has planar Lebesgue measure zero. Observe that the local equivalence of $\lambda - T_1$ and $\lambda - T_2$ gives that

$$
\sigma(T_1) = \sigma(T_2), \quad \sigma_\epsilon(T_1) = \sigma_\epsilon(T_2), \quad \text{and}
$$

$$
\text{index} (\lambda - T_1) = \text{index} (\lambda - T_2) \quad \text{for all } \lambda \notin \sigma_\epsilon(T_1),
$$

which implies that $T_1$ and $T_2$ have the same spectral picture. If $\pi : \mathcal{L}(H) \to \mathcal{C}(H)$ ($\mathcal{C}(H)$ is the Calkin algebra) is the Calkin homomorphism then $\pi(T_1)$ and $\pi(T_2)$ are also subnormal. Thus by an argument of Stampfli ([16]), $\pi(T_1)$ and $\pi(T_2)$ are normal and hence $T_1$ and $T_2$ are essentially normal. Therefore, by (3.1), $T_1$ and $T_2$ are compalent.

**Corollary 5.** Let $T_1, T_2 \in \mathcal{L}(H)$ be pure quasinormal and $\dim(\text{ran } T_1) < \infty$. If $\lambda - T_1$ and $\lambda - T_2$ are equivalent at every point of $\mathbb{C}$ then $T_1$ and $T_2$ are compalent.

**Proof.** If $T$ is pure quasinormal, then by [6, Theorem III.3.2], $T$ is unitarily equivalent to $U_\alpha \otimes |T|$, where $U_\alpha$ denotes the unilateral shift of multiplicity $\alpha$ and $|T| = (T^*T)^{1/2} |(\text{ran } T)^\perp$. Further, since $\alpha$ is the same as the dimension of $(\text{ran } T)^\perp$, the result follows from Theorem 4.

In view of Theorem 4 and Corollary 5, we have an interesting problem:

**Problem.** Under what condition, does compalence imply similarity?

We were unable to solve this problem. We however conjecture that compalence under local equivalence implies similarity.

**References**


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