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FREDHOLM ALTERNATIVES FOR AF ALGEBRAS

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ABSTRACT. We prove Fredholm alternatives for AF algebras: If an element have the finite null projection and the cofinite range projection then its image in the relative Calkin algebra is invertible.

1. Introduction

Let $\mathcal{H}$ be an infinite-dimensional separable Hilbert space,$\mathcal{L}(\mathcal{H})$ the set of all bounded linear operators on $\mathcal{H}$,and $\mathcal{K}(\mathcal{H})$ the norm-closed two-sided ideal of all compact operators. The Fredholm alternative says that if $T \in \mathcal{L}(\mathcal{H})$ is compact then the range space of $1-T$ is a closed subspace of $\mathcal{H}$ and the dimension of the null space of $1-T$ and the codimension of the range space of $1-T$ are finite and they are identical.

This theorem has been generalized in various directions. Among them is the following. It is well-known that an infinite semi-finite factor $\mathcal{M}$ possesses an ideal $\mathcal{K}(\mathcal{M})$ with properties analogous to the compact ideal $\mathcal{K}(\mathcal{H})$ of $\mathcal{L}(\mathcal{H})$, namely the norm-closed two-sided ideal generated by the set of all finite projections of $\mathcal{M}$. Generalized Fredholm alternatives due to Breuer[2,3] for an infinite semi-finite factor $\mathcal{M}$ says that if $A \in \mathcal{M}$ is compact then the relative dimensions of the null spaces of $1-A$ and $1-A^*$ are finite and they are identical. We mention that in general the range space of $1-A$ for $A \in \mathcal{K}(\mathcal{M})$ is not necessarily a closed subspace. Note that if the range space of $1-A$ is a closed subspace then the dimension of the null space of $1-A^*$ and the codimension of the range space of $1-A$ are identical.

In this note we study Fredholm alternatives for certain AF algebras. If an AF algebra contains the compact ideal generated by the set of all

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finite projections, then our result says that if a is a compact element then the null projection $1-a$ is nearly finite and the range projection of $1-a$ is nearly cofinite(see Definition 3.1 for near finiteness of the null projection). Although any AF algebra contains many projections (in fact, AF algebra contains a dense *-subalgebra generated by its projections), the set of all projections is not a complete lattice. Hence the null projection and the range projection of $1-a$ for a compact element a of AF algebra are not in general in the given AF algebra.

2. Preliminaries

A C*-algebra $\mathcal{A}$ is called an *approximately finite dimensional C*-algebra* , or in short AF algebra, if it is the norm closure of an increasing sequence of finite dimensional C*-algebras. AF algebra was introduced by Bratteli [1]. There are several good texts for the general theory of AF algebras. See Kye[6] for example.

Certain AF algebra contains an ideal which resembles the compact ideal $\mathcal{K}(\mathcal{H})$ of $\mathcal{L}(\mathcal{H})$.

A projection p in an AF algebra $\mathcal{A}$ is called *minimal* in $\mathcal{A}$ if there is no proper nonzero subprojection of p in $\mathcal{A}$. A projection p in $\mathcal{A}$ is called *finite* in $\mathcal{A}$ if it can be written as a finite sum of mutually orthogonal projections in $\mathcal{A}$. If a projection is not finite in $\mathcal{A}$, then it is called *infinite* in $\mathcal{A}$.

We remark that one can easily get AF algebra with sufficiently many finite projections and also AF algebras without a finite projection.

DEFINITION 2.1. Let $\mathcal{A}$ be an AF algebra. The norm-closed two-sided ideal generated by the set of all finite projections in $\mathcal{A}$ is called the *compact ideal* of $\mathcal{A}$.

The compact ideal of $\mathcal{A}$ will be denoted by $\mathcal{K}(\mathcal{A})$. For more details see [4].

LEMMA 2.2. *Let $\mathcal{A}$ be an AF algebra with $\mathcal{A}=\overline{\cup_{n=1}^{\infty}\mathcal{A}_n}$, where $\mathcal{A}_n$ are finite-dimensional C*-algebras. Let p be a projection in $\mathcal{A}$. Then for any $\epsilon>0$ there exists a projection $p_n\in\mathcal{A}_n$ such that $\| p-p_n \|<\epsilon$.*

PROOF. See Glimm[5,Lemma 1.6] □

LEMMA 2.3. *Let $\mathcal{A}$ and $\mathcal{A}_n$ be C^*-algebras as in Lemma 2.2. Let p and q be projections such that $p \in \mathcal{A}$ and $q \in \mathcal{A}_n$ with $\| p - q \| < \epsilon$ for sufficiently small positive number ϵ. Then there exists a partial isometry $w \in \mathcal{A}$ such that $ww^* = p, w^*w = q$*

PROOF. See Glimm[5,lemma 1.7] □

The following is a folk-lore theorem. But we give a proof for completeness. We also mention that it holds for more general context.

PROPOSIOTION 2.4. *Let $\mathcal{A}$ be an AF algebra with non-trivial compact ideal $\mathcal{K}(\mathcal{A})$. Let $p \in \mathcal{K}(\mathcal{A})$ be a projection. Then p is a finite projection.*

PROOF. Let $\mathcal{A} = \overline{\cup_{n=1}^{\infty} \mathcal{A}_n}$, where $\{\mathcal{A}_n\}$ is an increasing sequence of finite dimensional C^*-algebras. Let $p \in \mathcal{K}(\mathcal{A})$ be a projection. Since $\mathcal{K}(\mathcal{A}) = \overline{\cup_{n=1}^{\infty}(\mathcal{K}(\mathcal{A}) \cap \mathcal{A}_n)}$, by Lemma 2.2,there exists a sequence of finite projections $p_n \in \mathcal{A}_n$ such that $\| p_n - p \| \to 0$ as n goes to ∞. Then by Lemma 2.3, for sufficiently large n there exist partial isometries w_n such that $p_n = w_n w_n^*$ and $p = w_n^* w_n$. Since p_n is finite, the equivalent projection p is also finite. □

3. Fredholm alternatives

Throughout the remainder of this note $\mathcal{A}$ will denote an AF algebra with non-trivial compact ideal $\mathcal{K}(\mathcal{A})$.

DEFINITION 3.1. We say that an element a in $\mathcal{A}$ has *nearly finite null projection* if for any projection p in $\mathcal{A}$ with $ap = 0$ we have p finite.And also we say that an element a has *nearly cofinite range projection* if for any projection p in $\mathcal{A}$ with $pa = a$ we have $1 - p$ finite.

THEOREM 3.2. *Let a be a compact element of $\mathcal{A}$. Then $1 - a$ has nearly finite null projection and nearly cofinite range projection.*

PROOF. Suppose that $(1 - a)p = 0$ for a projection p. Then $p = ap$. Since $a \in \mathcal{K}(\mathcal{A})$ and $\mathcal{K}(\mathcal{A})$ is a two-sided ideal, $p \in \mathcal{K}(\mathcal{A})$. But by Proposition 2.4, p is a finite projection. Hence $1 - a$ has nearly finite projection.

Next suppose that $p(1-a) = 1-a$ for a projection p. Then $1-p = (1-p)a$. Hence as above, $1-p$ is a finite projection. Thus $1-a$ has nearly cofinite range projection. This completes the proof. □

DEFINITION 3.3. The quotient C^*-algebra $\mathcal{A}/\mathcal{K}(\mathcal{A})$ is called the *relative Calkin algebra* of $\mathcal{A}$. The canonical homomorphism from $\mathcal{A}$ onto $\mathcal{A}/\mathcal{K}(\mathcal{A})$ is denoted by π.

THEOREM 3.4. *Let $\pi(a)$ be invertible in the relative Calkin algebra of $\mathcal{A}$. Then the element a has nearly finite null projection and nearly cofinite range projection.*

PROOF. Let $\pi(b)$ the inverse of $\pi(a)$ in the relative Calkin algebra. Then there exist compact elements x and y such that $ab = 1-x$ and $ba = 1-y$. Suppose that $ap = 0$ for a projection p. Then $bap = (1-y)p = 0$. Hence by Theorem 3.2, p is finite. Next suppose that $pa = a$ for a projection p. Then $pab = ab$ and hence $p(1-x) = 1-x$. Therefore by Theorem 3.2 again, $1-p$ is finite. Hence a has nearly finite null projection and nearly cofinite range projection, which completes the proof. □

THEOREM 3.5. *Let $a \in \mathcal{A}$ have the finite null projection and the cofinite range projection in $\mathcal{A}$. Then $\pi(a)$ is invertible in the relative Calkin algebra.*

PROOF. Let p and q be the null and range projection of a in $\mathcal{A}$, repectively. Since both p and $1-q$ are finite projections , there exists a finite central projection r such that p is a subprojection of r and $1-r$ is a subprojection of $1-q$. Then $(1-r)a(1-r)$ is invertible in $(1-r)\mathcal{A}(1-r)$. Since $\pi(a) = \pi((1-r)a(1-r))$ and since the relative Calkin algebra of $\mathcal{A}$ and the relative Calkin algebra of $(1-r)\mathcal{A}(1-r)$ are isomorphic, $\pi(a)$ is invertible in the relative Calkin algebra of $\mathcal{A}$. □

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