Comm. Korean Math. Soc. 10 (1995), No. 2, pp. 337–348

FEYNMAN'S OPERATIONAL CALCULUS APPLIED TO MULTIPLE INTEGRALS

BONG JIN KIM

ABSTRACT. In 1987, Johnson and Lapidus introduced the noncommutative operations $*$ and $\dotplus$ on Wiener functionals and gave a precise and rigorous interpretation of certain aspects of Feynman's operational calculus for noncommuting operators. They established the operational calculus for certain functionals which involve Lebesgue measure. In this paper we establish the operational calculus for the functionals applied to multiple integrals which involve some Borel measures.

1. Notations and preliminaries

In this section we present some necessary notations and lemmas which are need in our subsequent section. Insofar as possible, we adopt the definitions of [2 and 6].

A. Let $C[a,b]$ denote the space of all real-valued continuous functions on $[a,b]$ and let $C_0[a,b]$ be the Wiener space on $[a,b]$, that is, $C_0[a,b] = \{x \in C[a,b] : x(a) = 0\}$. m_w will denote Wiener measure on $C_0[a,b]$.

B. Let $M(a,b)$ denote the space of all complex Borel measures η on the open interval (a,b). And let $M^*(a,b)$ denote the subspace of $M(a,b)$ such that if μ is the continuous part of η in $M^*(a,b)$ then the Radon-Nikodym derivative $d|\mu|/dm_l$ exists and is essentially bounded, and if ν is the discrete part of η then ν is finitely supported.

C. For $2 < r \leq \infty$, let $L_{1r} := L_{1r}((a,b)^2 \times \mathbb{R}^2)$ be the space of Borel measurable $\mathbb{C}$-valued functions θ on $(a,b)^2 \times \mathbb{R}^2$ such that

$$(1.1) \qquad \|\theta\|_{1r} := \left\{ \int_a^b \int_a^b \|\theta(s,t,\cdot,\cdot)\|_1^r \, dsdt \right\}^{\frac{1}{r}} < \infty.$$

Received November 1, 1994.
AMS 1991 Subject Classifications: 28C20, 28A33, 47D45.
Key words: Wiener measure, function space integral, Feynman's operational calculus.

Note that $L_{1r} \subseteq L_{1s}$ if $1 \leq s \leq r \leq \infty$.

D. Let F be a real or complex functional defined on $C[a,b]$. Given $\lambda > 0, \psi \in L_1(\mathbb{R})$ and $\xi \in \mathbb{R}$, let

$$(1.2) \qquad (I_\lambda(F)\psi)(\xi) = \int_{C_0[a,b]} F(\lambda^{-\frac{1}{2}}x + \xi)\,\psi(\lambda^{-\frac{1}{2}}x(b) + \xi)\,dm_w(x)$$

If $I_\lambda(F)\psi$ is in $L_\infty(\mathbb{R})$ as a function of ξ and if the correspondence $\psi \longrightarrow I_\lambda(F)\psi$ gives an element of $\mathcal{L} := \mathcal{L}\big(L_1(\mathbb{R}), L_\infty(\mathbb{R})\big)$, we say that the operator-valued function space integral $I_\lambda(F)$ exists for λ. For each $\psi \in L_1(\mathbb{R})$ let a function $A(\lambda : \psi)$ exist as a weakly analytic vector-valued function of λ for $\lambda \in \Omega$, where Ω be a simply connected domain of the complex λ-plane whose intersection with the positive real axis is a single non-empty open interval (α, β), $A(\lambda : \psi) \in L_\infty(\mathbb{R})$ and let $A(\lambda : \psi) = I_\lambda(F)\psi$ for $\lambda \in (\alpha, \beta)$ and $\psi \in L_1(\mathbb{R})$. We define

$$I_\lambda^{an}(F)\psi = A(\lambda : \psi)$$

for $\lambda \in \Omega$ and $\psi \in L_1(\mathbb{R})$. $I_\lambda^{an}(F)$ is called the analytic operator valued function space integral.

We note that, if $I_\lambda^{an}(F)$ exists, it is uniquely defined and is a linear operator that takes $L_1(\mathbb{R})$ into $L_\infty(\mathbb{R})$.

Let β and η be in $M^*(a,b)$, say,

$$(1.3) \qquad \beta = \mu + \sum_{j=1}^{l} w_j \delta_{\tau_j}, \quad \eta = \nu + \sum_{k=1}^{n} \alpha_k \delta_{\gamma_k}$$

and let θ be a $\mathbb{C}$-valued function on $(a,b)^2 \times \mathbb{R}^2$ satisfying the following three conditions;

$$(1.4\,\text{a}) \qquad \theta \in L_{1r}, \quad r \in (2, \infty]$$

$$(1.4\,\text{b}) \qquad \theta(\tau_j, \gamma_k, v_j, u_k) = \phi_1(\tau_j, v_j)\,\phi_2(\gamma_k, u_k),$$

where $\phi_1(\tau_j, \cdot)$, $\phi_2(\gamma_k, \cdot) \in L_1(\mathbb{R}) \cap L_\infty(\mathbb{R})$ for each $j = 1, 2, \cdots, l$, $k = 1, 2, \cdots, n$, and

$$(1.4\,\text{c}) \qquad \theta(\tau_j, t, \cdot, \cdot) = \theta(s, \gamma_k, \cdot, \cdot) = 0,$$

where $t \neq \gamma_k$, $s \neq \tau_j$ for $j = 1, 2, \cdots, l$, $k = 1, 2, \cdots, n$. Let

$$(1.5) \qquad F(y) = \int_{(a,b)} \int_{(a,b)} \theta(s, t, y(s), y(t))\,d\beta(s)\,d\eta(t)$$

for any $y \in C[a,b]$ for which the integral exists.

Lemma 1.1. *Let F be given by (1.5) with the assumptions discussed above ((1.3)-(1.5)). Then for every $\lambda > 0$ and every $\xi \in \mathbb{R}$, $F(\lambda^{-\frac{1}{2}}x + \xi)$ is defined for $m_w \times m_l$–a.e. $(x, \xi) \in C_0[a,b] \times \mathbb{R}$.*

Proof. We can easily check that for every $\lambda > 0$ and $m_w \times m_l$-a.e. (x,ξ), $\theta(s,t,\lambda^{-\frac{1}{2}}x(s)+\xi, \lambda^{-\frac{1}{2}}x(t)+\xi)$ is defined, see [5, Lemma 0.1]. And

$$\begin{aligned}
&\int_{C_0[a,b]}\Big(\int_{(a,b)}\int_{(a,b)} |\theta(s,t,\lambda^{-\frac{1}{2}}x(s)+\xi, \lambda^{-\frac{1}{2}}x(t)+\xi)| \\
&\qquad d|\beta|(s)\,d|\eta|(t)\Big) dm_w(x) \\
&\overset{[I]}{=} \int_{C_0[a,b]}\Big(\int_a^b\int_a^b |\theta(s,t,\lambda^{-\frac{1}{2}}x(s)+\xi, \lambda^{-\frac{1}{2}}x(t)+\xi)|\,d|\mu|(s)\,d|\nu|(t) \\
&\quad + \sum_{j=1}^{l}\sum_{k=1}^{n} |w_j\alpha_k||\theta(\tau_j,\gamma_k,\lambda^{-\frac{1}{2}}x(\tau_j)+\xi, \lambda^{-\frac{1}{2}}x(\gamma_k)+\xi)|\Big)\,dm_w(x) \\
&\overset{[II]}{\leq} 2(\frac{\lambda}{2\pi})\Big(\|\frac{d|\mu|}{dm_l}\|_\infty \|\frac{d|\nu|}{dm_l}\|_\infty\Big)\Big[\int_a^b\int_a^b [(s-a)(t-s)]^{-\frac{r'}{2}}\,ds\,dt\Big]^{\frac{1}{r'}} \\
&\qquad \cdot \Big[\int_a^b\int_a^b \|\theta(s,t,\cdot,\cdot)\|_1^r\,dsdt\Big]^{\frac{1}{r}} + A \\
&= 2(\frac{\lambda}{2\pi})\Big(\|\frac{d|\mu|}{dm_l}\|_\infty \|\frac{d|\nu|}{dm_l}\|_\infty\Big)\|\theta\|_{1r}\Big[(2!)\frac{(b-a)^{1-r'}\{\Gamma(1-\frac{r'}{2})\}^2}{\Gamma(2-r')}\Big]^{\frac{1}{r'}} + A \\
&< \infty,
\end{aligned}$$

where $r \in (2,\infty]$, $\frac{1}{r} + \frac{1}{r'} = 1$, $a_{1,j} := \tau_j$, $\quad a_{2,k} := \gamma_k$ and

$$\begin{aligned}
A := \sum_{j=1}^{l}\sum_{k=1}^{n} |w_j\alpha_k| \int_{C_0[a,b]} &|\theta(a_{1,j}, a_{2,k}, \lambda^{-\frac{1}{2}}x(a_{1,j})+\xi, \\
&\lambda^{-\frac{1}{2}}x(a_{2,k})+\xi)|\,dm_w(x) < \infty.
\end{aligned}$$

Step $[I]$ follows from the condition (1.4 c). Step $[II]$ results from the Fubini's theorem and the Hölder's inequality. Hence $F(\lambda^{-\frac{1}{2}}x+\xi)$ is defined for $m_w \times m_l$- a.e. (x,ξ) in $C_0 \times \mathbb{R}$. $\square$

As an analogue of the Lemma 2.1 in [3], we have the following lemma.

LEMMA 1.2. *Consider* $A(2q_0 : a_1, \dots, a_q : r')$ *and a set* $\Delta^{a_1,\dots,a_q}_{2q_0:j_1,\dots,j_{q+1}}$, *where* $r > 2$, r' *satisfying* $\frac{1}{r} + \frac{1}{r'} = 1$ *and* q_0 *is nonnegative integer* [see,3], *then for* $0 \le \tilde{q} \le q$,

(1.11)

$$\begin{aligned}
&\tilde{A}(2q_0 : a'_1, \dots, a'_{q-\tilde{q}} : r') \\
&:= \Bigg\{ \sum_{j_1+\cdots+j_{q+1}=2q_0} \int_{\Delta^{a_1,\dots,a_q}_{2q_0:j_1,\dots,j_{q+1}}} [(r_1 - a)\cdots(r_{\sigma(1)+1} - r_{\sigma(1)})\cdots \\
&\qquad \cdot (r_{\sigma(\tilde{q})+1} - r_{\sigma(\tilde{q})})\cdots(b - r_{j_1+\cdots+j_{q+1}})]^{-\frac{r'}{2}}\, d \mathop{\times}_{i=1}^{2q_0} r_i \Bigg\}^{\frac{1}{r'}} \\
&\le \left(\frac{(2q_0+q)P_{\tilde{q}}}{{}_qP_{\tilde{q}}} \right)^{\frac{1}{r'}} A(2q_0 : a'_1, \dots, a'_{q-\tilde{q}} : r'),
\end{aligned}$$

where $\{a_1, \dots, a_q\} = \{a_{i,\tilde{q}} : i = 0, 1, \dots, \tilde{q}\} \cup \{a'_j : j = 1, \dots, q - \tilde{q}\}$ *such that* $a < a_1 < a_2 < \cdots < a_q < b$, $a_{i,\tilde{q}} < a_{j,\tilde{q}}$ *for* $i < j$ *and* $a'_i < a'_j$ *for* $i < j$. *And* σ *is a function from* $\{a_{i,\tilde{q}}, \dots, a_{\tilde{q},\tilde{q}}\}$ *to* $\left\{ \sum_{k=1}^{l} j_k : l = 1, \dots, q \right\}$ *defined by*

$$\sigma(i) := \sigma(a_{i,\tilde{q}}) = \sum_{k=1}^{t} j_k,$$

where $a_{i,\tilde{q}} = a_t$. *And* $j'_i = \sum_{k=c+1}^{d} j_k$ *where* $a'_{i-1} = a_c$, $a'_i = a_d$ *and* $j'_{q-\tilde{q}+1} = \sum_{k=1}^{q+1} j_q - \sum_{i=1}^{q-\tilde{q}} j'_i$, *for* $i = 1, \dots, q - \tilde{q}$.

PROOF. For $\tilde{q} = 0$, we have $A(2q_0 : a_1, \dots, a_q : r') = \tilde{A}(2q_0 : a'_1, \dots, a'_q : r')$. Furthermore, if $q = \tilde{q}$, that is, there are no a_i's in $\{a_1, \dots, a_q\}$, then we have

$$\begin{aligned}
&\tilde{A}(2q_0 : \cdot : r') \\
&:= \Bigg\{ \sum_{j_1+\cdots+j_{q+1}=2q_0} \int_{\Delta^{a_1,\dots,a_q}_{2q_0:j_1,\dots,j_{q+1}}} [(r_1 - a)\cdots(b - r_{j_1+\cdots+j_{q+1}})]^{-\frac{r'}{2}}\, d \mathop{\times}_{i=1}^{2q_0} r_i \Bigg\}^{\frac{1}{r'}} \\
&\le \Bigg\{ \sum_{j_1+\cdots+j_{q+1}=2q_0} \int_{\Delta_{2q_0}} [(r_1 - a)(r_2 - r_1)\cdots(b - r_{j_1+\cdots+j_{q+1}})]^{-\frac{r'}{2}}\, d \mathop{\times}_{i=1}^{2q_0} r_i \Bigg\}^{\frac{1}{r'}}
\end{aligned}$$

$$= \left(\frac{(2q_0+q)\cdots(2q_0+1)}{q!}\right)^{\frac{1}{r'}} A(2q_0 \,:\, \cdot\, : r').$$

Now for $0 < \tilde{q} < q$

$$\begin{aligned}
&\tilde{A}(2q_0 \,:\, a'_1, \ldots, a'_{q-\tilde{q}} \,:\, r') \\
:=&\Bigg\{ \sum_{j_1+\cdots+j_{q+1}=2q_0} \int_{\Delta^{a_1,\ldots,a_q}_{2q_0:j_1,\ldots,j_{q+1}}} [(r_1-a)\cdots(r_{\sigma(1)+1}-r_{\sigma(1)})\cdots \\
&\qquad \cdot (r_{\sigma(\tilde{q})+1}-r_{\sigma(\tilde{q})})\cdots(b-r_{j_1+\cdots+j_{q+1}})]^{-\frac{r'}{2}}\, d\mathop{\times}_{i=1}^{2q_0} r_i \Bigg\}^{\frac{1}{r'}} \\
\leq&\Bigg\{ \sum_{j_1+\cdots+j_{q+1}=2q_0} \int_{\Delta^{a'_1,\ldots,a'_{q-\tilde{q}}}_{2q_0:j'_1,\ldots,j'_{q-\tilde{q}+1}}} [(r_1-a)\cdots(r_{\sigma(1)+1}-r_{\sigma(1)})\cdots \\
&\qquad \cdot (r_{\sigma(\tilde{q})+1}-r_{\sigma(\tilde{q})})\cdots(b-r_{j_1+\cdots+j_{q+1}})]^{-\frac{r'}{2}}\, d\mathop{\times}_{i=1}^{2q_0} r_i \Bigg\}^{\frac{1}{r'}} \\
=&\left(\frac{{}_{(2q_0+q)}P_{\tilde{q}}}{{}_qP_{\tilde{q}}}\right)^{\frac{1}{r'}} A(2q_0 \,:\, a'_1, \ldots, a'_{q-\tilde{q}} \,:\, r').
\end{aligned}$$

Thus we have the lemma. □

2. The analytic operator valued function space integral $I^{an}_{\lambda}(F)$ and the operation $*$

We begin by introducing some maps and the noncommutative operation $*$ which is need in this section [6].

NOTATIONS. Throughout this section

(a) $C[a,b] := C^{a,b}$ and $C_0[a,b] := C_0^{a,b}$.

(b) $m_{a,b}$ will denote Wiener measure on $C_0^{a,b}$.

Frequently, we will have $a=0$. In this case we will write C^b, C_0^b and m_b rather than $C^{0,b}$, $C_0^{0,b}$ and $m_{0,b}$, respectively.

(c) For $a<b<c$ R_1, R_2 and T denote the restriction maps and the translation map on $C^{a,c}, C^{a,c}$ and $C^{a,b}$ into $C^{a,b}, C^{b,c}$ and C^{b-a}, respectively. And P_1, P_2 and P_3 also denote the bijective maps on $C_0^{a,c}$,

$C^{a,b}$ and $C^{a,c}$ into $C_0^{a,b} \times C_0^{b,c}$, $\mathbb{R} \times C_0^{a,b}$ and $\mathbb{R} \times C_0^{a,b} \times C_0^{b,c}$, respectively as those are defined in [6].

(d) Throughout the rest of this section t, t_1, and t_2 will denote positive real numbers.

DEFINITION 2.1. Let F and G be a functionals on C^{t_1} and C^{t_2}, respectively. We define the functional $F * G$ on $C^{t_1+t_2}$ by

$$(F * G)(x) := (F \circ R_1)(x)(G \circ T \circ R_2)(x) \tag{2.1}$$

for x in $C^{t_1+t_2}$.

In the definition of $*$ one can see that $*$ is noncommutative operation (even if $t_1 = t_2$).

Let $A_{a,b}$ be the set of all functionals F defined by the form

$$F(x) = \left[\int_{(a,b)} \int_{(a,b)} \theta(s, t, x(s), x(t))\, d\beta(s)\, d\eta(t)\right]^m \tag{2.2}$$

where x is in $C[a, b]$ and m is the nonnegative integer. Further θ, β and η satisfy the conditions of Lemma 1.1.

When $a = 0$, we will write A_b rather than $A_{0,b}$.

REMARK. Let θ, β and η be in Lemma 1.1 and let

$$\begin{aligned} &g_m(\beta, \eta : q_{1,1}, \ldots, q_{l,n} : v_1', \cdots, v_{q-\tilde{q}}') \\ &:= \prod_{j=1}^{l} \prod_{k=1}^{n} \theta(\tau_j, \gamma_k, v_j, v_k)^{q_{j,k}}, \end{aligned}$$

where $q_{1,1} + q_{1,2} + \cdots + q_{l,n} = m - q_0$ and $0 \le \tilde{q} \le q$.
Then $g_m(\beta, \eta : q_{1,1}, \ldots, q_{l,n} : \cdot, \overset{(q-\tilde{q})}{\cdots}, \cdot) \in L_1(\mathbb{R}^{q-\tilde{q}})$.

NOTATION. Throughout this paper we let

$$|||g_m||| := \sup_{q_{1,1}+q_{1,2}+\cdots+q_{l,n}=m-q_0} \|g_m(\beta, \eta : q_{1,1}, \ldots, q_{l,n} : \cdot \cdots \cdot)\|_1 .$$

THEOREM 2.2. *(β, η : finitely supported measures)*

Let θ, β and η be as in Lemma 1.1. Let $F \in A_{a,b}$. Then the operator $I_\lambda^{an}(F)$ exists for all $\lambda \in \mathbb{C}^+$(:$Re\lambda > 0$). Further for $\lambda \in \mathbb{C}^+$, $\psi \in L_1(\mathbb{R})$ and $\xi \in \mathbb{R}$,

(2.3)

$$
\begin{aligned}
&(I_\lambda^{an}(F)\psi)(\xi) \\
&= \sum_{q_0+q_{1,1}+\cdots+q_{l,n}=m} \frac{m!(w_1\alpha_1)^{q_{1,1}}\cdots(w_l\alpha_n)^{q_{l,n}}}{q_0!q_{1,1}!\cdots q_{l,n}!} \sum_{(m_1,\ldots,m_{q_0},k_1,\ldots,k_{q_0})\in P} \\
&\quad \sum_{j_1+\cdots+j_{q+1}=2q_0} \int_{\Delta^{a_1,\ldots,a_q}_{2q_0;j_1,\ldots,j_{q+1}}} Y \, d \mathop{\times}_{n=1}^{2q_0} \tilde{\mu}_{p,n}(r_n),
\end{aligned}
$$

where $\{r_1, \ldots, r_{2q_0}\}$ is the set of numbers $s_1, \ldots, s_{q_0}, t_1, \ldots, t_{q_0}$ in some rearrangement, P is the set of all permutations of $\{1, 2, \ldots, 2q_0\}$, $s_j := r_{m_j}$, $t_j := r_{k_j}$ and $\tau_j := a_{1,j}$, $\gamma_k := a_{2,k}$ and $\{a_1, \ldots, a_q\} = \{\tau_j, \gamma_k : j = 1, 2, \ldots, l, k = 1, 2, \ldots, n\}$ such that $a < a_1 < \cdots < a_q < b$ and $q_{j,k}$ is a nonnegative integer. And $\int f \, d\tilde{\mu}_{p,i}(r_i)$ means that $\int f \, d\mu(r_i)$ when $r_i = r_{m_j}$ for some r_{m_j} and $\int f \, d\tilde{\mu}_{p,i}(r_i)$ means that $\int f \, d\nu(r_i)$ when $r_i = r_{k_j}$ for some r_{k_j}.
And

(2.4)

$$
\begin{aligned}
Y &= \left(\frac{\lambda}{2\pi}\right)^{\frac{2q_0+q+1}{2}} \left[(r_1 - a)\cdots(a_1 - r_{j_1})(r_{j_1+1} - a_1)\cdots(b - r_{j_1+\cdots+j_{q+1}})\right]^{-\frac{1}{2}} \\
&\int_{-\infty}^{\infty} \overset{2q_0+q+1}{\cdots} \int_{-\infty}^{\infty} \prod_{j=1}^{q_0} \theta(r_{m_j}, r_{k_j}, v_{m_j}, v_{k_j}) \cdot \prod_{j=1}^{l} \prod_{k=1}^{n} \left[\theta(a_{1,j}, a_{2,k}, v'_{1,j}, v'_{2,k})\right]^{q_{j,k}} \\
&\psi(v'_{q+1}) \exp\left\{ -\frac{\lambda(v_1 - v_0)^2}{2(r_1 - a)} - \cdots - \frac{\lambda(v'_1 - v_{j_1})^2}{2(a_1 - r_{j_1})} - \frac{\lambda(v_{j_1+1} - v'_1)^2}{2(r_{j_1+1} - a_1)} - \right. \\
&\left. \cdots - \frac{\lambda(v'_{q+1} - v_{j_1+\cdots+j_{q+1}})^2}{2(b - r_{j_1+\cdots+j_{q+1}})} \right\} \mathop{\times}_{i=1}^{2q_0} dv_i \mathop{\times}_{j=1}^{q+1} dv'_j.
\end{aligned}
$$

In addition we have for $\lambda \in \mathbb{C}^+$,

$$
\|I_\lambda^{an}(F)\| \leq b_m(|\lambda|), \tag{2.5}
$$

where

$$b_m(|\lambda|) := m! \sum_{q_0+q_{1,1}+\cdots+q_{l,n}=m} \frac{\prod_{j=1}^{l}\prod_{k=1}^{n} |w_j\alpha_k|^{q_{j,k}}}{q_0!q_{1,1}!\cdots q_{l,n}!}((2q_0)!)$$
$$\left(\frac{(2q_0+q)!}{(2q_0)!q!}\right)^{\frac{r+1}{2r}} |||g_m||| \left(\frac{|\lambda|}{2\pi}\right)^{\frac{2q_0+q-\tilde{q}+1}{2}} \left(\frac{|\lambda|}{Re\lambda}\right)^{\frac{\tilde{q}}{2}}$$
$$\left(\|\frac{d|\mu|}{dm_l}\|_\infty \|\frac{d|\nu|}{dm_l}\|_\infty \|\theta\|_{1r}\right)^{q_0} \tilde{A}(2q_0 : a_1', \cdots, a_{q-\tilde{q}}' : r').$$

PROOF. Let $\psi \in L_1(\mathbb{R})$, $\xi \in \mathbb{R}$ and $\lambda > 0$ be given. Then by [D] in section 1, multinomial expansion and using the simplex trick,
(2.6)

$$\begin{aligned}
&(I_\lambda(F)\psi)(\xi)\\
&= \int_{C_0[a,b]} \sum_{q_0+q_{1,1}+\cdots+q_{l,n}=m} \frac{m!}{q_0!q_{1,1}!\cdots q_{l,n}!}\\
&\quad \left(\int_a^b \int_a^b \theta(s,t,\lambda^{-\frac{1}{2}}x(s)+\xi, \lambda^{-\frac{1}{2}}x(t)+\xi)\, d\mu(s)\, d\nu(t)\right)^{q_0}\\
&\quad \cdot \prod_{j=1}^{l}\prod_{k=1}^{n} \left(w_j\alpha_k\, \theta(\tau_j,\gamma_k,\lambda^{-\frac{1}{2}}x(\tau_j)+\xi, \lambda^{-\frac{1}{2}}x(\gamma_k)+\xi)\right)^{q_{j,k}}\\
&\qquad \cdot \psi(\lambda^{-\frac{1}{2}}x(b)+\xi)\, dm_w(x)\\
&= \sum_{q_0+q_{1,1}+\cdots+q_{l,n}=m} \frac{m!(w_1\alpha_1)^{q_{1,1}}\cdots(w_l\alpha_n)^{q_{l,n}}}{q_0!q_{1,1}!\cdots q_{l,n}!}\\
&\quad \cdot \sum_{(m_1,\ldots,m_{q_0},k_1,\ldots,k_{q_0})\in P}\ \sum_{j_1+\cdots+j_{q+1}=2q_0} \int_{\Delta^{a_1,\ldots,a_q}_{2q_0:j_1,\ldots,j_{q+1}}} Y\, d \overset{2q_0}{\underset{n=1}{\times}} \tilde{\mu}_{p,n}(r_n),
\end{aligned}$$

where
(2.7)

$$\begin{aligned}
Y := \int_{C_0[a,b]} &\prod_{j=1}^{q_0} [\theta(r_{m_j}, r_{k_j}, \lambda^{-\frac{1}{2}}x(r_{m_j})+\xi, \lambda^{-\frac{1}{2}}x(r_{k_j})+\xi)]\\
&\cdot \prod_{j=1}^{l}\prod_{k=1}^{n} \left[\theta(a_{1,j}, a_{2,k}, \lambda^{-\frac{1}{2}}x(a_{1,j})+\xi, \lambda^{-\frac{1}{2}}x(a_{2,k})+\xi)\right]^{q_{j,k}}\\
&\qquad \cdot \psi(\lambda^{-\frac{1}{2}}x(b)+\xi)\, dm_w(x).
\end{aligned}$$

The last equality in (2.6) comes from the Fubini's theorem which will be justified later in conjunction with the norm estimate (2.5). Using the basic Wiener integration formula [4] and a simple change of variables, we have (2.4). From (2.6), we have

(2.8)
$$
\begin{aligned}
&|(I_\lambda(F)\psi)(\xi)| \\
&\le \sum_{q_0+q_{1,1}+\cdots+q_{l,n}=m} \frac{m!|w_1\alpha_1|^{q_{1,1}}\cdots|w_l\alpha_n|^{q_{l,n}}}{q_0!q_{1,1}!\cdots q_{l,n}!} \sum_{(m_1,\ldots,m_{q_0},k_1,\ldots,k_{q_0})\in P} \\
&\cdot \sum_{j_1+\cdots+j_{q+1}=2q_0} \int_{\Delta^{a_1,\ldots,a_q}_{2q_0;j_1,\ldots,j_{q+1}}} |Y|\, d \mathop{\times}_{n=1}^{2q_0} |\tilde{\mu}_{p,n}|(r_n).
\end{aligned}
$$

Using the Chapman-Kolmogorov theorem [4], we have

$$
\begin{aligned}
|Y| \le &\left(\frac{\lambda}{2\pi}\right)^{\frac{2q_0+q-\tilde{q}+1}{2}} \\
&\big[(r_1-a)\cdots(r_{\sigma(1)+1}-r_{\sigma(1)})\cdots(r_{\sigma(\tilde{q})+1}-r_{\sigma(\tilde{q})})\cdots \\
&\cdot(b-r_{j_1+\cdots+j_{q+1}})\big]^{-\frac{1}{2}} |||g_m|||\,\|\psi\|_1 \prod_{j=1}^{q_0} \|\theta(r_{m_j}, r_{k_j}, \cdot, \cdot)\|_1,
\end{aligned} \tag{2.9}
$$

where σ is a function defined as in the Lemma 1.2. Thus
(2.10)
$$
\begin{aligned}
&\sum_{j_1+\cdots+j_{q+1}=2q_0} \int_{\Delta^{a_1,\ldots,a_q}_{2q_0;j_1,\ldots,j_{q+1}}} |Y|\, d \mathop{\times}_{n=1}^{2q_0} |\tilde{\mu}_{p,n}|(r_n) \\
&\overset{[I]}{\le} \left(\frac{\lambda}{2\pi}\right)^{\frac{2q_0+q-\tilde{q}+1}{2}} |||g_m|||\,\|\psi\|_1 \left(\|\frac{d|\mu|}{dm_l}\|_\infty \|\frac{d|\nu|}{dm_l}\|_\infty\right)^{q_0} \\
&\cdot\Bigg[\sum_{j_1+\cdots+j_{q+1}=2q_0} \Bigg\{\int_{\Delta^{a_1,\ldots,a_q}_{2q_0;j_1,\ldots,j_{q+1}}} [(r_1-a)\cdots(r_{\sigma(1)+1}-r_{\sigma(1)})\cdots \\
&\cdot(r_{\sigma(\tilde{q})+1}-r_{\sigma(\tilde{q})})\cdots\cdot(b-r_{j_1+\cdots+j_{q+1}})]^{-\frac{r'}{2}}\, d \mathop{\times}_{n=1}^{2q_0} r_n\Bigg\}^{\frac{2}{r'}}\Bigg]^{\frac{1}{2}} \\
&\cdot\Bigg[\sum_{j_1+\cdots+j_{q+1}=2q_0} \Bigg\{\int_{\Delta^{a_1,\ldots,a_q}_{2q_0;j_1,\ldots,j_{q+1}}} \prod_{j=1}^{q_0} \|\theta(r_{m_j}, r_{k_j}, \cdot, \cdot)\|_1^r\, d \mathop{\times}_{n=1}^{2q_0} r_n\Bigg\}^{\frac{2}{r}}\Bigg]^{\frac{1}{2}} \\
&\overset{[II]}{\le} \left(\frac{\lambda}{2\pi}\right)^{\frac{2q_0+q-\tilde{q}+1}{2}} |||g_m|||\,\|\psi\|_1 \left(\|\frac{d|\mu|}{dm_l}\|_\infty \|\frac{d|\nu|}{dm_l}\|_\infty\right)^{q_0}
\end{aligned}
$$

$$\cdot \left(\frac{(2q_0+q)!}{(2q_0)!q!} \right)^{\frac{r+1}{2r}} \|\theta\|_{1r}^{q_0} \hat{A}(2q_0 : a_1', \ldots, a_{q-\bar{q}}' : r').$$

By the Hölder's inequality and the Schwartz's inequality we obtain Step [I]. From Lemma 1.2 and the Hölder's inequality, we obtain Step [II]. Combining (2.7) – (2.10) we get the norm estimate (2.5) for $\lambda > 0$. This also justifies the use of Fubini's theorem in (2.6), and we see that $I_\lambda(F)\psi$ is a member of $L_\infty(\mathbb{R})$.

For Re$\lambda > 0$, let $K(\xi, \lambda)$ be defined as the right-hand side of (2.3). Then we easily see that $|K(\xi,\lambda)| \le \|\psi\|_1 b_m(|\lambda|)$, where $b_m(|\lambda|)$ is defined in (2.5). Because $K(\xi, \lambda)$ is expressed as a multiple with λ appearing in the integral only in an exponential, we may conclude that $K(\xi, \lambda)$ is continuous for all ξ and λ for Re$\lambda > 0$ and that for each real ξ, $K(\xi, \lambda)$ is analytic in λ for Re$\lambda > 0$.

Let $\phi \in L_1(\mathbb{R})$. We consider

$$g(\lambda) := \int_{-\infty}^{\infty} K(\xi, \lambda)\phi(\xi)\, d\xi \tag{2.11}$$

Then we can easily see that $g(\lambda)$ is analytic in λ for Re$\lambda > 0$.

By (2.6), for a positive real λ,

$$K(\xi, \lambda) = \big(I_\lambda(F)\psi\big)(\xi). \tag{2.12}$$

Hence by the definition, $I_\lambda^{an}(F)\psi$ exists for $\psi \in L_1(\mathbb{R})$ and Re$\lambda > 0$ and $\big(I_\lambda^{an}(F)\psi\big)(\xi) = K(\xi, \lambda)$. Thus Theorem 2.2 is proved. □

REMARK. If β, η are purely continuous measures or β, η are purely discrete and finitely supported, then we can easily obtain theorems analogues to Theorem 2.2.

Using the Chapman-kolmogorov's theorem and the Tonelli's theorem, we can obtain the following lemma.

LEMMA 2.3. *For $F \in A_{a,b}$, $\psi \in L_1(\mathbb{R})$ and $\lambda \in \mathbb{C}^+$, $I_\lambda^{an}(F)\psi$ exists as an elements of $L_1(\mathbb{R})$. In fact for such functionals F in $A_{a,b}$,*

$$I_\lambda^{an}(F) \in \tilde{\mathcal{L}} := \mathcal{L}\big((L_1(\mathbb{R}), \|\cdot\|_1),\ ((L_1(\mathbb{R}) \cap L_\infty(\mathbb{R})), \|\cdot\|_\infty)\big).$$

Lemma 2.4. *Let $F \in A_t$, then there exists $F_D \in A_t$ which dominates F in the sense that $|F(\lambda^{-\frac{1}{2}}x+\xi)| \le F_D(\lambda^{-\frac{1}{2}}x+\xi)$ for all $\lambda > 0$ and Leb. $\times\, m_t$ – a.e. $(\xi, x) \in \mathbb{R} \times C_0^t$. Further $I_\lambda^{an}(|F|)$ exists as an element of $\tilde{\mathcal{L}}$ for all $\lambda > 0$.*

Proof. Let a functional $F \in A_t$ for some m,

$$F(x) = \left[\int_{(0,t)} \int_{(0,t)} \theta(s,r,x(s),x(r))\, d\beta(s)\, d\eta(r)\right]^m .$$

Define F_D by

$$F_D(x) = \left[\int_{(0,t)} \int_{(0,t)} |\theta(s,r,x(s),x(r))|\, d|\beta|(s)\, d|\eta|(r)\right]^m .$$

Then the functional F_D dominates F in the above sense. Since $|\beta|, |\eta|$ are in $M^*(0,t), F_D \in A_t$. Thus by Lemma 2.3, $I_\lambda^{an}(F_D)$ exists as an element of $\tilde{\mathcal{L}}$ for all $\lambda \in \mathbb{C}^+$ and in particular for all $\lambda > 0$. It now follows that $I_\lambda^{an}(|F|)$ exists as an element of $\tilde{\mathcal{L}}$ for all $\lambda > 0$. □

Note that Lemma 2.4 does not assert that $|F| \in A_t$. It shows that $I_\lambda^{an}(|F|)$ exists as an element of $\tilde{\mathcal{L}}$ for all $\lambda > 0$.

Theorem 2.5. *If $F \in A_{t_1}, G \in A_{t_2}$ then for all $\lambda \in \mathbb{C}^+$, $I_\lambda^{an}(F * G)$ exists as an element of $\tilde{\mathcal{L}}$ and*

$$I_\lambda^{an}(F * G) = I_\lambda^{an}(F)\, I_\lambda^{an}(G). \tag{2.13}$$

Proof. It can be proved by the same method as in the proof of Theorem 4.1 in [6] by using Lemma 2.3 and Lemma 2.4. □

References

1. R. H. Cameron and D. A. Storvick, *An operator valued function space integral applied to integrals of functions of class L_1*, Proc. of the London. Math. Soc. **27** (1973), 345-360.
2. R. H. Cameron and D. A. Storvick, *An operator valued function space integral applied to multiple integrals of functions of class L_1*, Nagoya Math. J. **51** (1973), 91-122.

3. J. S. Chang and G. W. Johnson, *The Feynman integral and Feynman's operational calculus; the $\mathcal{L}(L_1(\mathbb{R}), C_0(\mathbb{R}))$ theory*, J. Korean Math. Soc. **28** (1991), 99-125.
4. G. W. Johnson, *Lecture Note (unpublished)*, Univ. of Nebraska.
5. G. W. Johnson and M. L. Lapidus, *Generalized Dyson series, generalized Feynman digrams, the Feynman integral and Feynman's operational calculus*, Mem. Amer. Soc. **62** (1986).
6. G. W. Johnson and M. L. Lapidus, *Noncommutative operations on Wiener functionals and Feynman's operational calculus*, Supplemento ai Rendiconti del Circolo Matematicodi Palermo (1987).
7. G. W. Johnson and D. L. Skoug, *The Cameron-Storvick function space integral; the L_1 theory*, J. Math. Analysis and Appl. **50** (1975), 647-667.
8. G. W. Johnson and D. L. Skoug, *The Cameron-Storvick function space integral; an $\mathcal{L}(L_p, L_{p'})$ theory*, Nagoya Math. J. (1976), 93-137.

Department of Mathematics
Dae Jin University
Pocheon 487-800, Korea