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COMPACT MANIFOLDS WITH THE MINIMAL ENTROPY

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ABSTRACT. On a compact manifold without conjugate points, the volume entropy can be obtained as the average mean curvature of the horospheres in the universal covering space. In the case when the volume entropy is zero, we prove that the universal covering space is diffeomorphic to a product space with a line factor. This fact can be considered as a surporting evidence for the Mañé's conjecture, which claims the flatness of the manifold.

1. Introduction

Let M be a compact manifold without conjugate points. On the unit tangent bundle SM of M, the Liouville measure dv is invariant under the geodesic flow, and the measure entropy h_μ of the geodesic flow with respect to the normalized Liouville measure $\mu = \overline{dv} = Vol(SM)^{-1}dv$ can be defined. It was shown in [9] that this is the average of the mean curvature, $u(v)$ (the trace of the second fundamental form), of the horospheres in the universal covering space $\widetilde{M}$. For manifolds without conjugate points it is clear that $h_\mu \geq 0$, and it was conjectured by Mañé that either $h_\mu > 0$ or M is flat. This conjecture is not yet answered. It has been known that this conjecture is closely related to the so-called Hopf conjecture, which states that an n-torus without conjugate points must be flat. Recently it was shown that the Hopf conjecture is true[cf. 3, 5], and Mañé's conjecture looks more plausible. In this paper we provide another convincing evidence for the conjecture. In fact, we will show that the universal covering space $\widetilde{M}$ is foliated by the bi-asymptotic

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lines orthogonal to horospheres, which is what happens in the case of torus without conjugate points.

For any $v \in SM$ let $\gamma_v : [0,\infty) \to M$ be a geodesic such that $\gamma'(0) = v$, and denote by N_vM the subspace of T_pM orthogonal to v and by $R_v(t)w = R(w, \gamma_v'(t))$ the linear map defined on $N_{\gamma_v'(t)}M$ by the curvature tensor R. The Ricatti equation of M is:

$$U' + U^2 + R_v = 0.$$

Let $U_s^{\pm}$ denote the solution such that $U_s(\mp s) = \pm\infty$. Then it is easy to see that $U_s^+(t)$ is the shape operator at $\gamma_v(t)$ of the geodesic sphere of radius $t+s$ centered at $\gamma_v(-s)$. Put $U^{\pm} = \lim_{s\to\infty} U_s^{\pm}$. Then it is clear that $U_v^+(t)$ defined by γ_v for some $v \in SM$ is same as $-U_{-v}^-(-t)$ defined by γ_{-v}. It is also known that $U^{\pm}$ are measurable functions, and we have

$$h_\mu = \int_{SM} \mathrm{tr} U^+ d\mu.$$

It turns out that $-\mathrm{tr}U^-$ is the mean curvature of the horosphere defined by the ray γ_{-v} in $\widetilde{M}$, and its behavior can be studied in terms of horofunctions defined on $\widetilde{M}$. We will provide some necessary details in the following section.

2. Lagrange tensors and horospheres

In this section we review some well known facts about the Lagrange tensors and the geometry of horospheres. We refer the readers to [7],[8] for more detailed description of the Lagrange tensors.

Let $\gamma_v : [0,\infty) \to M$ be a ray in M and $R_v = R_v(t)$ be the tensor field along γ_v defined by $R_v(t)V = R(V, \gamma_v'(t))\gamma_v'(t)$ for any vector field $V(t)$ orthogonal to $\gamma_v'(t)$, where $R(\cdot,\cdot)$ is the curvature tensor of M. Then a matrix solution to the differential equation,

$$Z'' + R_v Z = 0\ ,$$

is called a *Jacobi tensor*. Each Jacobi tensor, applied to a parallel normal vector field along γ_v, gives rise to an $(n-1)$-dimensional space of Jacobi fields along γ_v when the dimension of M is n. A Jacobi tensor $Z(t)$ is called a *Lagrange tensor* if the Wronskian $W(Z,Z) = Z'Z^* - Z^*Z'$ vanishes, where * denotes the adjoint with respect to the Riemannian metric. If Z is nowhere singular, this is equivalent to saying that the tensor $Z'Z^{-1}$ and Z^*Z' are symmetric. Furthermore, the tensor field $U = Z'Z^{-1}$ is a solution of the Ricatti equation

$$U' + U^2 + R_v = 0.$$

Let H be an oriented C^2-hypersurface in M. The normal bundle has a canonical trivialization using the oriented unit normal vectors: $NH = H \times \mathbb{R}$. There exists a neighborhood B of the zero-section such that the mapping $\exp|_B : B \to M$ is a diffeomorphism. Let V be the unit vector field over B defined by the velocity vectors of the geodesics normal to H. Then a solution of the tensor equation,

$$Z' = (\nabla V)Z,$$

is uniquely determined by its initial value, and $Z(t)$ is nonsingular along a geodesic in B. Furthermore, $Z'Z^{-1}(0) = \nabla V(\gamma(0))$ is the shape operator of H at $\gamma(0)$, and hence it is clearly symmetric. Therefore we see that Z is a Lagrange tensor. In fact, it can be shown that each Lagrange tensor along any geodesic arises in this way, and we will say that Z is *related* to H if $Z'Z^{-1}(t)$ is the shape operator of H for some t. From this fact, we easily see that the solutions $U_s^{\pm}$ of the Ricatti equation have the property we mentioned in the introduction.

Since M has no conjugate points, for any geodesic $\gamma : \mathbb{R} \to M$, the Lagrange tensor A given by the initial conditions, $A(0) = 0,\ A'(0) = Id$, is nonsingular for all $s \neq 0$. Let D_s be the Lagrange tensor along γ satisfying the boundary condition $D_s(0) = Id,\ D_s(s) = 0$. Then D_s uniquely exists for each $s \in \mathbb{R}$, and nonsingular for all $t \neq s$. It is in fact related to the metric sphere centered at $\gamma(s)$, which is clearly C^∞ because there are no conjugate points in M. Furthermore, from the basic property of the Lagrange tensor, D_s can be computed in terms of A by the formula

$$D_s(t) = A(t) \int_t^s (A^*A)^{-1}(u)du$$

for all t between 0 and s. Since $D_s(0) = Id$, we see that $D'_s D_s^{-1}$ is the tensor U_s^- and $U_s^-(0) = D'_s(0)$.

From the theory of differential equations it follows that as $s \to \infty$ the field D_s converges to some Lagrange tensor D called the *stable Jacobi tensor* along γ if and only if $\lim_{s\to\infty} D'_s(0)$ exists. It is well known [8],[10] that for manifolds without conjugate points this limit always exists, and we have $D(0) = Id$, $D'(0) = \lim_{s\to\infty} D'_s(0)$ and $D(t) = A(t)\int_t^\infty (A^*A)^{-1}(u)du$. In terms of the Ricatti equation, it is clear that $U^- = D'D^{-1}$ and $U^-(0) = D'(0)$. For this reason, we will use $D'(0)$ instead of U^- to compute the entropy. From the above integral formula we also see that for any $s > t > 0$ and $\alpha > 0$,

$$D'_t(0) < D'_s(0) < D'(0), \quad D'_s(0) < D'_{-\alpha}(0),$$

where $T_1 < T_2$ means $T_2 - T_1$ is positive definite for tensors T_1 and T_2.

A Jacobi field Z_v defined for each $v \in SM$ is called *continuous* if the initial values $Z_v(0)$ and $Z'_v(0)$ are continuous as tensors of the vector bundle $\{(x, v) \in TM \times SM |\ x \perp v\}$ over SM. For each unit vector $v \in SM$ let D_v be the stable Jacobi tensor defined by the geodesic γ_v.

For a ray $\gamma : [0, \infty) \to M$, the *Busemann function* b_γ of γ is defined by

$$b_\gamma(x) = \lim_{t\to\infty}(d(x, \gamma(t)) - t).$$

For each unit vector $v \in S_pM$ let $\gamma_v = \exp_p tv$ be the unit speed geodesic, and denote by b_v the Busemann function defined by γ_v. In fact, on each compact subset of M, b_v is a uniform limit of the distance function $d_{vt}(\cdot) \stackrel{\text{def}}{=} d(\gamma_v(t), \cdot)$. If b_v is normalized so that $b_v(p) = 0$, we call $H_v = b_v^{-1}(0)$ the *horosphere*. A vector $w \in S_qM$ for an arbitrary $q \in M$ is called *asymptotic* to v if it is a limit of $-\nabla d_{vt}(q)$ as $t \to \infty$. Then it is known that $w \in S_qM$ is asymptotic to v if and only if $w = -\nabla b_v(q)$. If the horosphere H_v is C^2-differentiable, then the asymptotic vectors form the normal bundle of H_v and the stable Jacobi fields D_v is related to H_v. Furthermore, since $D_v(0) = Id$, we know that $D'_v(0)$ is the shape operator of H_v.

For a complete, simply connected manifold without conjugate points, it is well known that the Busemann function is C^1-differentiable [7], and if the stable Jacobi field is continuous, it is C^2-differentiable [16]. For each $v \in SM$ let $D_{t,v}$ be the unique Jacobi tensor along γ_v satisfying $D_{t,v}(0) = Id$, $D_{t,v}(t) = 0$. Since $D_{t,v}$ is a solution of an ordinary differential equation, it is easy to see that $D'_{t,v}(0)$ depends continuously on v for a fixed number $t > 0$, and as $t \to \infty$, $D_{t,v}$ converges to the stable Jacobi tensor D_v.

3. Compact manifold with zero entropy

In this section, M will be a compact complete Riemannian manifold without conjugate points, and the measure entropy h_μ of M is zero.

PROPOSITION 3.1. *Let M be a compact Riemannian manifold without conjugate points. Then the measure entropy h_μ of M is nonnegative, and if $h_\mu = 0$, then for almost every $v \in SM$ we have*

$$D'_v(0) = -D'_{-v}(0).$$

PROOF. For any $v \in SM$ we lift this vector to $S\widetilde{M}$ and denote it by v again by abuse of notation. Since $S\widetilde{M}$ has no conjugate points, it is clear that the tensor $D'_v(0) + D'_{-v}(0)$ is nonpositive. In fact, otherwise, it would produce Jacobi fields $J_\pm$ along $\gamma_v(t) = \exp tv$ such that $J_+(0) = J_-(0)$, $\lim_{t\to\pm\infty} J_\pm(t) = 0$ and $\langle J'_+(0) - J'_-(0), J(0)\rangle > 0$, and from these we can construct a smooth nonvanishing Jacobi field which vanishes at two different points on γ_v. Therefore we have $\operatorname{tr}(D'_v(0) + D'_{-v}(0)) \le 0$. Since every vector in SM can be paired with the opposite direction, we have

$$\begin{aligned} h_\mu &= \int_{SM} \operatorname{tr}(U^+(0))d\mu \\ &= -\int_{SM} \operatorname{tr}(U^-(0))d\mu \\ &= -\frac{1}{2}\int_{SM} \operatorname{tr}(D'_v(0) + D'_{-v}(0))d\mu \ge 0. \end{aligned}$$

Furthermore, if $h_\mu = 0$, then we see that $\operatorname{tr}(D'_v(0) + D'_{-v}(0)) = 0$ for almost every $v \in SM$. Since $D'_v(0) + D'_{-v}(0)$ is a symmetric negative semi-definite tensor, we can conclude that the tensor itself must be zero. □

With the help of the monotonicity of the convergence $D'_{t,v}(0) \to D'_v(0)$, we can in fact show that the above result holds for every $v \in SM$, and the stable Jacobi tensor is continuous.

LEMMA 3.2. *Let M be a compact Riemannian manifold without conjugate points. If the measure entropy of M is zero, then for every $v \in SM$ we have*

$$D'_v(0) = -D'_{-v}(0).$$

PROOF. Since the measure entropy of M is zero, we know that $D'_v(0) = -D'_{-v}(0)$ for almost every $v \in SM$. Furthermore, for each fixed $v \in SM$ the convergence $D'_{t,v}(0) \to D'_v(0)$ is monotone increasing with respect to t, and for each fixed t $D'_{t,v}(0)$ is a C^∞-differentiable in SM. Consider for each t a tensor field $E_t(v) = D'_{t,v}(0) + D'_{t,-v}(0)$. E_t is a monotone increasing smooth tensor and converges to 0 for almost every $v \in SM$ as $t \to \infty$.

For any compact subset $K \subset SM$ and any positive number ε, we will show that there exists $T > 0$ such that $||E_t(v)|| < \varepsilon$ for all $t \geq T$ and $v \in K$. Since D_t is C^∞ for each t, for any $v \in K$ for which $E_t(v) \to 0$ as $t \to \infty$, there exist a number $T_v > 0$ and an open neighborhood B_v such that for any $w \in B_v$ we have $||D_{T_v}(w)|| < \varepsilon$. Since $E_t(v) \to 0$ for almost every $v \in K$, the compact set K can be covered by finitely many of these open neighborhoods. Let T be the maximum of those finite T_v's for which the open neighborhoods B_v's can cover K. Then by the monotonicity of E_t, this T will have the desired property. □

LEMMA 3.3. *If M is as above, then on the universal covering space $\widetilde{M}$ the Jacobi tensor D is continuous on $v \in S\widetilde{M}$.*

PROOF. We first claim that $D'_v(0)$ is upper semi-continuous on $v \in S\widetilde{M}$. Let $\{v_i\}$ be a sequence of unit vectors converging to $v \in T_p\widetilde{M}$ as $i \to \infty$. For each i let b_i be the Busemann function defined by the ray $\gamma_{v_i}(t) = \exp_p tv_i$ such that $b_i(p) = 0$. For any compact subset K of $\widetilde{M}$ containing p and any number $\varepsilon > 0$ there exists a positive number T such that the map $f(q) = d(q, \gamma_v(T)) - T$ and the Busemann function b_γ has uniform distance less than ε. Since v_i converges to v we have $\gamma_{v_i}(T) \to \gamma_v(T)$ as $i \to \infty$, and by the triangle inequality we have for each $q \in K$

$$\limsup_{i\to\infty} b_i(q) \leq \limsup_{i\to\infty}(d(q, \gamma_{v_i}(T)) + b_i(\gamma_{v_i}(T)))$$
$$= d(q, \gamma_v(T)) - T = f(q).$$

Since f is ε close to b_γ we can conclude that for any $q \in K$ we have

$$\limsup_{i\to\infty} b_i(q) \leq b_\gamma(q).$$

From the fact that $\gamma_{v_i}(p) = b_\gamma(p)$, we now have

$$\limsup_{i\to\infty} D'_{v_i}(0) \leq D'_v(0).$$

We now apply the same argument to the opposite directions $-v_i$ and $-v$ to obtain that

$$-D'_{-v}(0) \leq -\liminf_{i\to\infty} D'_{-v_i}(0)$$
$$= \limsup_{i\to\infty} D'_{v_i}(0) \leq D'_v(0).$$

However, the two end terms in the above inequality are same, and therefore we have equalities everywhere, which implies the continuity of the Jacobi tensor D' and also D. □

If the stable Jacobi tensor is continuous, then the Busemann functions are C^2-differentiable, and hence the horospheres are also C^2-differentiable. We therefore have shape operators of these horospheres and they coincide with $D'(0)$ at each point. Furthermore, the normal vectors, $V = -\nabla b$, satisfy the differential equation

$$\nabla V = D'_V(0).$$

Since $\widetilde{M}$ has no conjugate points, in general, the Busemann functions b_v and b_{-v} satisfy $b_v(p) + b_{-v}(p) \geq 0$ for all $p \in \widetilde{M}$. In the following lemma, we will show that this inequality becomes equality in the case of zero entropy.

LEMMA 3.4. *If M is above, then for any $v \in S_p\widetilde{M}$ we have*

$$\nabla b_v = -\nabla b_{-v}.$$

In particular, we have $b_v(q) + b_{-v}(q) = 0$ for all $q \in \widetilde{M}$.

PROOF. For each compact subset K of $\widetilde{M}$ the tensor field E_t uniformly converges to zero on SK as $t \to \infty$, and the Busemann functions b_v and b_{-v} can be uniformly approximated by the distance functions d_{vt} and d_{-vt} respectively. We will show that the uniform distance $||d_{vt} - d_{-vt}||$ on K can be made arbitrarily small by taking t large enough. Put $V = -\nabla d_{vt}$ and $W = -\nabla d_{-vt}$. Then they are solutions to the equation,

$$\nabla V = D'_{t,V}.$$

Since $D'_{t,V} + D'_{t,-V}$ can be bounded by an arbitrarily small number on SK and $V(p) + W(p) = v - v = 0$, we see that the vector field $V + W$ is also bounded by an arbitrarily small number on K. Hence, by integration on the compact set K, we see that $||d_{vt} - d_{-vt}||$ is also arbitrarily small because $d_{vt}(p) = d_{-vt}(p) = t$. Therefore, we have that $\nabla b_v = -\nabla b_{-v}$ on K. Then by the connectedness of $\widetilde{M}$ we can obtain the equality on $\widetilde{M}$. Furthermore, since $b_v(p) = b_{-v}(p) = 0$, we have $b_v(q) + b_{-v}(q) = 0$ for all $q \in \widetilde{M}$. □

COROLLARY 3.5. *If M is as above, then on the universal covering space $\widetilde{M}$, the asymptotic relation is an equivalence relation. Furthermore, every asymptotic lines are bi-asymptotic.*

PROOF. Suppose a geodesic $\sigma : (-\infty, +\infty) \to \widetilde{M}$ is asymptotic to a geodesic $\gamma : (-\infty, +\infty) \to \widetilde{M}$. Then $\nabla b_\gamma(\sigma(t)) = \sigma'(t)$ for each $t \in \mathbb{R}$. If we assume that $b_\gamma(\sigma(0)) = 0$, by the same triangle inequality argument as in the proof of Lemma 3.3, we see that $b_\gamma(p) \le b_\sigma(p)$ for all $p \in \widetilde{M}$. Put $\bar{\gamma}(t) = \gamma(-t)$ and $\bar{\sigma}(t) = \sigma(-t)$. Since b_γ and $b_{\bar{\gamma}}$ have same level sets, we have $\nabla b_{\bar{\gamma}}(\sigma(t)) = \bar{\sigma}'(t)$ for each t. We now apply the same argument as above to obtain $b_{\bar{\gamma}}(p) \le b_{\bar{\sigma}}(p)$. By the lemma, we have $b_{\bar{\sigma}}(p) + b_\sigma(p) = 0$ and $b_{\bar{\gamma}}(p) + b_\gamma(p) = 0$. Therefore, we have

$$b_\gamma(p) \le b_\sigma(p) = -b_{\bar{\sigma}}(p) \le -b_{\bar{\gamma}}(p) = b_\gamma(p),$$

and hence we obtain $b_\gamma = b_\sigma$ and $b_{\bar\gamma} = b_{\bar\sigma}$, which clearly implies that the asymptotic relation is an equivalence relation and the geodesics are bi-asymptotic. □

With all of the above results combined, we can now state our main theorem.

THEOREM 3.6. *Let M be a compact Riemannian manifold without conjugate points. If the measure entropy of M is zero, then for any $v \in S\widetilde{M}$, there exists a C^2-diffeomorphism $F : N \times \mathbb{R} \to \widetilde{M}$ such that for each fixed $x \in N$, $F(x,t)$ is a line bi-asymptotic to $\gamma(t) = \exp tv$ and for each fixed $t \in \mathbb{R}$, $F(N,t)$ is a level set of the Busemann function b_γ.*

PROOF. For any $v \in S_p\widetilde{M}$ let γ and b_γ be the line $\exp tv$ and the corresponding Busemann function such that $b_\gamma(p) = 0$. Put $N = b_\gamma^{-1}(0)$. Then N is a C^2-differentiable submanifold of $\widetilde{M}$. We now define the map $F : N \times \mathbb{R} \to \widetilde{M}$ by $F(q,t) = \exp_q t\nabla b_\gamma(q)$ for each $(q,t) \in N \times \mathbb{R}$. Then clearly F is a C^2-diffeomorphism and has the claimed property. □

If we can show that this F is in fact an isometry, we have a splitting of $\widetilde{M}$ by a line, and the totally geodesic submanifold N will have the same property as $\widetilde{M}$. We can therefore apply the same argument to N, and then by induction we can prove that $\widetilde{M}$ is isometric to $\mathbb{R}^n$, which is the claim of Mañé's conjecture. At this moment, however, it looks quite difficult to show directly that F is an isometry.

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