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ON GROUP EXTENSIONS OF MINIMAL HOMEOMORPHISMS II

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ABSTRACT. We define a group extension and characterized some properties of the group extension. In particular, we show that the quotient map v is a continuous group isomorphism and subgroup $H_1(H_2)$ is normal in $G_1(G_2)$.

Let X be a compact monothetic group. Assume, that $T : X \to X$ is defined by the formula

$$T(x) = a + x,$$

where a is an element of X such that the set of all powers na, n integers, is dense in X. Then T is a minimal homeomorphism of X. Denote by $C(T)$ the *centralizator* of T i.e. the set of all continuous transformations of X commuting with T. Then
(A)
$S \in C(T)$ *iff there exists* $b \in X$ *such that* $S(x) = b + x$ *for all* $x \in X$.

Let G be a compact metric group (not necessarily abelian). For a continuous $\varphi : X \to G$ we define a homeomorphism $T_\varphi : X \times G \to X \times G$ setting

$$T_\varphi(x, g) = (T(x), \varphi(x)g).$$

Such homeomorphism is not necessarily minimal. We will call T_φ a *group extension*, or, indicating the group, a *G-extension* of T. If F is a closed subgroup of G then we can consider the action of T_φ on $X \times F$ and will denote it by $T_{\varphi,F}$. We will call $T_{\varphi,F}$ a *natural factor* of T_φ and an *isometric extension* of T. If F is normal in G, then we call $T_{\varphi,F}$,

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a *normal natural factor* of T_φ. There is a natural right action of G on $X \times G$ given by

$$(x, g)h = (x, gh).$$

Let M be a T_φ-minimal subset of $X \times G$. Let $\pi : X \times G \to X$ be the natural projection.

LEMMA 1. $$\pi(M) = X.$$

PROOF. We have $T(\pi(M)) = \pi(T_\varphi(M)) = \pi(M)$. Hence $\pi((M)) = X$. □

Put

$$H = \{g \in G : Mg = M\} = \{g \in G : \forall (x, h) \in M,\ (x, hg) \in M\}.$$

Observe that if $g \in G$, then either $Mg = M$ or $Mg \cap M = \phi$. Therefore

(B) $$h \in H \text{ iff } \exists (x, g) \in M \text{ such that } (x, gh) \in M.$$

LEMMA 2.
i) *H is a closed subgroup of G.*
ii) *If $(x, g), (x, h) \in M$ then $hH = gH$.*

PROOF. i) Because H is obviuosly a group, it is enough to show that H is a closed set. To do this assume that $h_n \in H$, $n \geq 1$ and $h_n \to h \in G$. Take $(x, g) \in M$. Then $(x, g)h_n = (x, gh_n) \to (x, gh) \in M$ since M is closed. Thus $(x, g)h \in M$ and $h \in H$.

ii) Let $(x, g), (x, h) \in M$. Then $(x, g)g^{-1}h = (x, h) \in M$. This (see (B)) implies that $g^{-1}h \in H$ which finishes the proof of ii). □

For $x \in X$ let

$$M_x = \{g \in G : (x, g) \in M\}.$$

As an immediate consequence of Lemma 2 ii) we have

LEMMA 3. *For each $x \in X$ there exists a $g = g_x \in G$ such that*

$$M_x = gH.$$

Let us define a function $\tau : X \to G/H$ by

$$\tau(x) = g_x H = M_x.$$

LEMMA 4.
i) *τ is a continuous map.*
ii) *For all $x \in X$, $\tau(T(x)) = \varphi(x)\tau(x)$.*

PROOF. i) Take $x_n \in X$, $n \geq 1$, $x_n \to x$. We may assume (choosing a subsequence) that there are $g_n \in G$ such that $g_n \in M_{x_n}$ i.e. $(x_n, g_n) \in M$ and $g_n \to g \in G$. Then $(x_n, g_n) \to (x, g) \in M$ since M is a closed set. Thus $g \in M_x$ which implies $M_{x_n} \to M_x$ in G/H. Hence τ is a continuous map.

ii) Let $x \in X, g \in M_x$. Then $M_x = gH$. By T_φ-invariance of M, $M \ni T_\varphi(x,g) = (T(x), \varphi(x)g)$ which implies $M_{T(x)} = \varphi(x)gH$. Thus $\tau(T(x)) = \varphi(x)\tau(x)$. □

We intend to describe minimal subsets of $(X \times G_1) \times (X \times G_2)$ for given minimal homeomorphisms T_{φ_1} and T_{φ_2} acting on $X \times G_1$ and $X \times G_2$ respectively. First we recall a description of $T \times T$-minimal subsets of $X \times X$. Fix $b \in X$. Then the set

$$A_b = \{(x, b + x) : x \in X\}. \tag{C}$$

is $T \times T$-minimal because T is minimal. Moreover.

$$\bigcup_{b \in X} A_b = X \times X.$$

Thus all $T \times T$-minimal subsets of $X \times X$ are of the form $A_b, b \in X$. In view of (A), each $T \times T$-minimal subset of $X \times X$ is a graph of some $S \in C(T)$.

Let G_1, G_2 be compact metric groups. Assume, that $\varphi_i : X \to G_i$ is a continuous map such that $T_{\varphi_i} : X \times G_i \to X \times G_i$ is a minimal homeomorphism, $i = 1, 2$. Let M be a $T_{\varphi_1} \times T_{\varphi_2}$-minimal subset of $(X \times G_1) \times (X \times G_2)$. Then the projection of M into $X \times G_i$ is a T_{φ_i}-minimal set, $i = 1, 2$. By the minimality of T_{φ_1} and T_{φ_2}, the projections are equal to $X \times G_1$ and $X \times G_2$ respectively.

Denote by π the map $\pi : (X \times G_1) \times (X \times G_2) \to X \times X, \pi(x, g, y, h) = (x, y)$. Then by the above remarks,

$$\pi(M) = A_b \text{ for some } b \in X$$

where A_b is defined by (C) for $G = G_1 \times G_2$.

Let $\pi_i : G_1 \times G_2 \to G_i, \pi_i(g_1, g_2) = g_i, i = 1, 2$.

LEMMA 5. *Let* $H = \{g \in G : M_g = M\} = \{g \in G : \forall (x,h) \in M, (x,hg) \in M\}$. *Then*

$$\pi_1(H) = G_1, \quad \pi_2(H) = G_2.$$

PROOF. We will only show that $\pi_1(H) = G_1$. Take $g_1 \in G_1$. We will find $g_2 \in G_2$ such that $M(g_1, g_2) = M$. It is enough to show that there exists a $g_2 \in G_2$ such that $M(g_1, g_2) \cap M \neq \phi$ (see (B)).

Fix $x \in X$. Then there are $g', g'' \in G_2$ such that $(x, g_1, b + x, g') \in M$ and $(x, e, b + x, g'') \in M$. Put $g_2 = (g'')^{-1} g'$. Then

$$M(g_1, g_2) \ni (x, e, b + x, g'')(g_1, g_2) = (x, g_1, b + x, g') \in M.$$

Thus $M(g_1, g_2) \cap M \neq \phi$ which implies $M(g_1, g_2) = M$ and therefore $(g_1, g_2) \in H$.□

Let H_1, H_2 be defined by

$$H_1 = \{g_1 \in G_1 : (g_1, e) \in H\},$$
$$H_2 = \{g_2 \in G_2 : (e, g_2) \in H\},$$

where e denotes the unit elements of the groups G_1, G_2.

Clearly H_i is a closed subgroup of $G_i, i = 1,2$. As an immediate consequence of Lemma 5 we have the following lemma:

LEMMA 6. *The subgroup* $H_1(H_2)$ *is normal in* $G_1(G_2)$.

THEOREM 7.
a) *If* $(g_1, g_2) \in H, (g_1, \tilde{g}_2) \in H$ *then* $\tilde{g}_2^{-1} g_2 \in H_2$.
b) *If* $(g_1, g_2) \in H, (\tilde{g}_1, g_2) \in H$ *then* $\tilde{g}_1^{-1} g_1 \in H_1$.
c) $(g_1, g_2) \in H$ *iff* $g_1 H_1 \times g_2 H_2 \subset H$.

PROOF.
a) Assume that $(g_1, g_2), (g_1, \tilde{g}_2) \in H$. Then $(g_1^{-1}, g_2^{-1}) \in H$ and $H \ni (g_1^{-1}, \tilde{g}_2^{-1})(g_1, g_2) = (e_1, \tilde{g}_2^{-1} g_2)$. Therefore $\tilde{g}_2^{-1} g_2 \in H_2$.
The proof of b)is similar to the proof of a).
c) Assume that $(g_1, g_2) \in H$. Take $h_1 \in H_1, h_2 \in H_2$. Then $(h_1, e_2) \in H, (e_1, h_2) \in H$ and $(h_1, h_2) = (h_1, e_2)(e_1, h_2) \in H$. Therefore $H \ni (g_1, g_2)(h_1, h_2) = (g_1 h_1, g_2 h_2)$.

Thus we have proved that $g_1H_1 \times g_2H_2 \subset H$.□

We define a map $v : G_1/H_1 \to G_2/H_2$ by the following formula

$$v(g_1H_1) = \Pi_2((g_1H_1 \times G_2) \cap H),$$

where $\Pi_2 : G_1 \times G_2 \to G_2, \Pi_2(g_1, g_2) = g_2$.

THEOREM 8. *The map v is a continuous group isomorphism.*

PROOF. By Theorem 7, v is well defined. The continuity of v is evident. Obviously v is bijective. We will prove that v is a group homomorphism.

Since $H_1 \times H_2 \subset H$, $v(H_1) = H_2$. Take $gH_1, \bar{g}H_1 \in G_1/H_1$. Denote $v(gH_1\bar{g}H_1) = \tilde{g}H_2, v(gH_1) = g_1H_2, v(\bar{g}H_1) = \bar{g}_1H_2$. Then $g\bar{g}H_1 \times \tilde{g}H_2 \subset H$. Moreover $gH_1 \times g_1H_2 \subset H, \bar{g}H_1 \times \bar{g}_1H_2 \subset H$ which implies $gH_1\bar{g}H_1 \times g_1H_2\bar{g}_1H_2 \subset H$. Thus $g_1\bar{g}_1H_2 = \tilde{g}H_2$ i.e. $v(gH_1\bar{g}H_1) = v(gH_1)v(\bar{g}H_1)$.□

As an immediate consequence of Theorem 7 and Theorem 8 we have

LEMMA 9.

$$H = \bigcup_{g \in G} gH_1 \times v(gH_1).$$

Recall, that we consider a $T_{\varphi_1} \times T_{\varphi_2}$-minimal subset M of $(X \times G_1) \times (X \times G_2)$, where $T : X \to X$, $T(x) = a + x$ is a minimal rotation on a compact monothetic group X, φ_i is a continuous map defined on X with values in G_i, such that $T_{\varphi_i} : X \times G_i \to X \times G_i$, $T_{\varphi_i}(x, g) = (T(x), \varphi_i(x)g)$ is minimal, $i = 1, 2$.

THEOREM 10. *Let M be a $T_{\varphi_1} \times T_{\varphi_2}$-minimal subset of $(X \times G_1) \times (X \times G_2)$. There exist closed normal subgroups $H_1 \subset G_1, H_2 \subset G_2$, a continuous group isomorphism $v : G_1/H_1 \to G_2/H_2, a, b \in X$ and a continuous map $f : X \to G_2/H_2$ such that*

$$M = \bigcup_{\substack{x \in X \\ g \in G_1}} \{x\} \times gH_1 \times \{b + x\} \times f(x)v(gH_1).$$

PROOF. First we will show the following formula :

(D) If $(h_1, h_2) \in H$ then $h_2 v(h_1^{-1} H_1) = H_2$.

Indeed, by Lemma 7, $h_1 H_1 \times h_2 H_2 \subset H$. Therefore $v(h_1 H_1) = h_2 H_2$ or, which is the same, $h_2 v(h_1^{-1} H_1) = H_2$.

Let $\alpha : (G_1 \times G_2)/H \to G_2/H_2$ be given by the formula

$$\alpha((g_1, g_2)H) = g_2 v(g_1^{-1} H_1).$$

By virtue of (D), α is well-defined. Let

$$f(x) = \alpha(\tau(x, b + x)),$$

where τ satisfies Lemma 4 ii) for $\varphi : X \to G_1 \times G_2, \varphi(x) = (\varphi_1(x), \varphi_2(b + x))$. Clearly, f is continuous. Moreover it satisfies

(E) $$f(Tx) = \varphi_2(b + x) f(x) v(\varphi(x)^{-1} H_1),$$

because, denoting $(g_1, g_2)H = \tau(x, b + x)$, we have

$$\begin{aligned} f(Tx) &= \alpha(\tau(Tx)) \\ &= \alpha((\varphi_1(x), \varphi_2(b + x)) \\ &= \alpha((\varphi_1(x) g_1, \varphi_2(b + x) g_2)H) \\ &= \varphi_2(b + x) g_2 v(g_1^{-1} \varphi_1(x)^{-1} H_1) \\ &= \varphi_2(b + x) g_2 v(g_1^{-1} h_1) v(\varphi_1(x)^{-1} H_1) \\ &= \varphi_2(b + x) \alpha(\tau(x, b + x)) v(\varphi_1(x)^{-1} H_1) \\ &= \varphi_2(b + x) f(x) v(\varphi_1(x)^{-1} H_1). \end{aligned}$$

Using (E) we can describe the set $\tau(x, b + x) = M_{(x, b+x)}$.

(F) For each $x \in X$, $M_{(x,b+x)} = \bigcup_{g \in G_1} gH_1 \times f(x) v(gH_1)$.

Now we are in the position to prove our theorem. Denote by M' the set on the right hand side of the equality in this theorem. First we will show that

(G) $$(T_{\varphi_1} \times T_{\varphi_2})(M') \subset M'.$$

Indeed, take $(x, g, b+x, g') \in M'$. Then

$$\begin{aligned}(T_{\varphi_1} \times T_{\varphi_2})(x, g, b+x, g') &= (T(x), \varphi_1(x)g, t(b+x), \varphi_2(b+x)g') \\ &= (T(x), \varphi_1(x)g, b+T(x), \varphi_1(b+x)g')\end{aligned}$$

All we have to prove is

$$\varphi_2(b+x)g' \in f(T(x))\upsilon(\varphi_1(x)gH_1).$$

By virtue of (E),

$$f(T(x)) = \varphi_2(b+x)f(x)\upsilon(\varphi_1(x)^{-1}H_1).$$

Because $g' \in f(x)\upsilon(gH_1)$,

$$\begin{aligned}\varphi_2(b+x)g' &\in \varphi_2(b+x)f(x)\upsilon(gH_1) \\ &= \varphi_2(b+x)f(x)\upsilon(\varphi_1(x)^{-1}H_1)\upsilon(\varphi_1(x)H_1)\upsilon(gH_1) \\ &= (\varphi_2(b+x)f(x)\upsilon(\varphi_1(x)^{-1}H_1))\upsilon(\varphi_1(x)gH_1) \\ &= f(T(x))\upsilon(\varphi_1(x)gH_1).\end{aligned}$$

Thus $(T_{\varphi_1} \times T_{\varphi_2})(x, g, b+x, g') \in M'$. We have proved (G).

Now we will show that the following inclusion :

(H) $$M' \subset M.$$

Take $(x, g, b+x, g') \in M'$. Then $g' \in f(x)\upsilon(gH_1)$. By virtue of (F), $g, g' \in M_{(x,b+x)}$. Therefore $(x, g, b+x, g') \in M$ and (H) is proved.

The last formula we need to prove theorem is the following

(I) *M' is a closed set.*

To show (I) we define a map $S : X \times G_1/H_1 \to X \times G_2/H_2$ setting

$$S(x, gH_1) = (T(x), f(x)\upsilon(gH_1)).$$

Then S is a homeomorphism, that implies that its graph is a closed set. Clearly the graph of S is just M' which gives (I).

By virtue of (G),(H), (I), $M = M'$. The proof of Theorem 1 is complete. □

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