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CHARACTERIZATIONS OF DISCRETE CLASSICAL ORTHOGONAL POLYNOMIALS

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ABSTRACT. We give a simple unified proof of various characterizations of discrete classical orthogonal polynomials including two new ones.

1. Introduction

Classical orthogonal polynomials of Jacobi, Laguerre, Hermite, and Bessel have many properties common to all of them. For example, we have :

(a) (Bochner [3]) they all satisfy a second order differential equation of the form

$$\alpha(x)y''(x) + \beta(x)y'(x) = \lambda_n y(x), \tag{1.1}$$

where $\alpha(x) = ax^2 + bx + c \not\equiv 0$ and $\beta(x) = dx + e$ are polynomials independent of n and $\lambda_n = an(n-1) + dn$, $n \geq 0$;

(b) (Hahn [6]) their derivatives also form orthogonal polynomials ;

(c) (Hildebrandt [7]) they are all orthogonal relative to a quasi-definite moment functional u satisfying a functional differential equation

$$(\alpha(x)u)' - \beta(x)u = 0 \tag{1.2}$$

for some polynomials $\alpha(x) = ax^2 + bx + c \not\equiv 0$ and $\beta(x) = dx + e$;

(d) (Al-Salam and Chihara [2]) they all satisfy a difference-differential equation of the form

$$\alpha(x)P_n'(x) = (r_n x + s_n)P_n(x) + t_n P_{n-1}(x) \tag{1.3}$$

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for some polynomial $\alpha(x) = ax^2 + bx + c \not\equiv 0$ and constants r_n, s_n, and t_n.

Conversely, any one of the above four properties characterizes the classical orthogonal polynomials. We refer to Al-Salam [1] for the history and the list of contributors of these characterizations of classical orthogonal polynomials. A simple unified proof of the above characterizations as well as some others can be found in [8].

Replacing the differential operator $D = d/dx$ in (1.1) by the difference operator Δ defined by

$$\Delta f(x) = f(x+1) - f(x), \tag{1.4}$$

we obtain the following second-order difference equation

$$\alpha(x)\Delta^2 y(x-1) + \beta(x)\Delta y(x-1) = \lambda_n y(x). \tag{1.5}$$

Lancaster [9] (see also [10] and [11]) showed that there are essentially only four distinct orthogonal polynomials that arise as eigenfunctions of the difference equation (1.5). They are discrete classical orthogonal polynomials of Charlier, Meixner, Krawtchouk, and Hahn.

It is well known that these discrete classical orthogonal polynomials can also be characterized by the properties analogous to those listed in the beginning for classical orthogonal polynomials.

In this work, we shall give simple unified proofs of these characterizations and two new characterizations of discrete classical orthogonal polynomials.

2. Main Theorems

In this work, all polynomials are assumed to be real polynomials in the real variable x. We denote the degree of a polynomial $\phi(x)$ by $\deg(\phi)$ with the convention that $\deg(0) = -1$. By a polynomial system (PS), we mean a sequence of polynomials $\{P_n(x)\}_{n=0}^{\infty}$ with $\deg(P_n) = n$, $n \geq 0$. We let $\mathcal{P}$ be the space of all polynomials and call any linear functional on $\mathcal{P}$ a moment functional. For a moment functional u, we denote its action on a polynomial $\phi(x)$ by

$$\langle u, \phi \rangle$$

and call

$$\{u_n := \langle u, x^n \rangle\}_{n=0}^{\infty}$$

the moments of u.

DEFINITION. A PS $\{P_n(x)\}_{n=0}^{\infty}$ is called a weak Tchebychev polynomial system (WTPS) (respectively, a Tchebychev polynomial system (TPS)) if there is a non-zero moment functional u such that

$$(2.1) \qquad \langle u, P_m P_n \rangle = K_n \delta_{mn} \qquad (\, m \text{ and } n \geq 0\,),$$

where K_n are real (respectively, non-zero real) constants and δ_{mn} is the Kronecker delta function. If $K_n > 0$ for $n \geq 0$, we usually refer $\{P_n(x)\}_{n=0}^{\infty}$ as an orthogonal polynomial system (OPS). In either case, we say that $\{P_n(x)\}_{n=0}^{\infty}$ is a WTPS (respectively, a TPS) relative to u and call u an orthogonalizing moment functional of $\{P_n(x)\}_{n=0}^{\infty}$.

A moment functional u is called to be quasi-definite (respectively, positive-definite) if its moments $\{u_n\}_{n=0}^{\infty}$ satisfy the Hamburger condition

$$(2.2) \qquad \Delta_n(u) := \det[u_{i+j}]_{i,j=0}^{n} \neq 0 \qquad (\text{respectively}, \Delta_n(u) > 0)$$

for $n \geq 0$. It is well known (see chapter 1 in [5]) that a moment functional u is quasi-definite (respectively, positive-definite) if and only if there is a TPS (respectively, an OPS) relative to u.

We let ∇ be the backward difference operator defined by

$$(2.3) \qquad \nabla f(x) = f(x) - f(x-1).$$

For a moment functional u and a polynomial $\psi(x)$, we let Δu, ∇u, and ϕu be the moment functionals defined by

$$(2.4) \qquad \langle \Delta u, \phi \rangle = -\langle u, \nabla \phi \rangle;$$

$$(2.5) \qquad \langle \nabla u, \phi \rangle = -\langle u, \Delta \phi \rangle;$$

$$(2.6) \qquad \langle \psi u, \phi \rangle = \langle u, \psi \phi \rangle,$$

for any polynomial $\phi(x)$.

LEMMA 2.1. *For any moment functional u and any polynomial $\psi(x)$, we have*

(i) $\Delta u = 0$ (*or* $\nabla u = 0$) *if and only if* $u = 0$;
(ii) *if u is quasi-definite, then* $\psi(x)u = 0$ *if and only if* $\psi(x) \equiv 0$;
(iii) $\Delta(\psi(x)u) = \psi(x+1)\Delta u + (\Delta\psi(x))u$;
(iv) $\nabla(\psi(x)u) = \psi(x-1)\nabla u + (\nabla\psi(x))u$.

PROOF. (i) If $\Delta u = 0$, then $\langle \Delta u, x^n\rangle = -\langle u, \nabla x^n\rangle = 0$, $n \geq 0$. Hence $u = 0$ since $\{\nabla x^n\}_{n=1}^{\infty}$ is a PS. The converse is trivial and the same proof works for ∇u.

(ii) Assume that u is quasi-definite and let $\{P_n(x)\}_{n=0}^{\infty}$ be a TPS relative to u. Assume that $\psi(x)u = 0$ but $\psi(x) \not\equiv 0$. Then, we may write $\psi(x)$ as $\psi(x) = \sum\limits_{j=0}^{k} c_j P_j(x)$ where $k = \deg(\psi) \geq 0$ and c_j are constants with $c_k \neq 0$. Then

$$0 = \langle \psi(x)u, P_k(x)\rangle = \sum_{j=0}^{k} c_j \langle u, P_j P_k\rangle = c_k\langle u, P_k^2\rangle$$

so that $c_k = 0$, which is a contradiction. Hence, $\psi(x) \equiv 0$. The converse is trivial.

(iii) Let $\phi(x)$ be a polynomial. Then we have

$$\begin{aligned}\langle \Delta(\psi(x)u), \phi(x)\rangle =& -\langle u, \psi(x)\nabla\phi(x)\rangle \\ =&\langle u, -\psi(x+1)\phi(x) + \psi(x)\phi(x-1) \\ &+ \psi(x+1)\phi(x) - \psi(x)\phi(x)\rangle \\ =&\langle u, -\nabla[\psi(x+1)\phi(x)] + (\Delta\psi(x))\phi(x)\rangle \\ =&\langle \psi(x+1)\Delta u + (\Delta\psi(x))u, \phi(x)\rangle.\end{aligned}$$

(iv) Let $\phi(x)$ be a polynomial. Then we have

$$\begin{aligned}\langle \nabla(\psi(x)u), \phi(x)\rangle =&\langle u, -\psi(x)(\Delta\phi(x))\rangle \\ =&\langle u, -\psi(x)\phi(x+1) + \psi(x-1)\phi(x) \\ &+ \psi(x)\phi(x) - \psi(x-1)\phi(x)\rangle \\ =&\langle u, -\Delta(\psi(x-1)\phi(x)) + (\nabla\psi(x))\phi(x)\rangle \\ =&\langle \psi(x-1)\nabla u + (\nabla\psi(x))u, \phi(x)\rangle. \qquad \square\end{aligned}$$

LEMMA 2.2. *Let $\{P_n(x)\}_{n=0}^\infty$ be a TPS relative to u. Then for any moment functional v and any integer $k \geq 0$, $\langle v, P_n(x)\rangle = 0$ for $n > k$ if and only if there is a polynomial $\phi(x)$ of degree $\leq k$ such that $v = \phi(x)u$.*

PROOF. Assume $\langle v, P_n(x)\rangle = 0$ for $n > k$ and consider a moment functional $\tilde{v} = (\sum_{j=1}^{k} c_j P_j(x))u$, where c_j are constants to be determined later. Then

$$\langle \tilde{v}, P_n(x)\rangle = \sum_{j=1}^{k} c_j \langle u, P_j P_n\rangle = \begin{cases} 0 & , n > k \\ c_n\langle u, P_n^2\rangle & , 0 \leq n \leq k. \end{cases}$$

Hence $\langle \tilde{v}, P_n(x)\rangle = \langle v, P_n(x)\rangle$, $n \geq 0$, that is, $v = \tilde{v}$ if and only if $c_j = \langle v, P_j\rangle / \langle u, P_j^2\rangle$, $0 \leq j \leq n$. The converse follows immediately from the orthogonality of $\{P_n(x)\}_{n=0}^\infty$ relative to u. □

Any PS $\{P_n(x)\}_{n=0}^\infty$ determines a moment functional u, called a canonical moment functional of the PS $\{P_n(x)\}_{n=0}^\infty$, uniquely up to a non-zero constant multiple by the conditions

$$\langle u, P_0\rangle \neq 0 \quad \text{and} \quad \langle u, P_n\rangle = 0, \qquad n \geq 1. \tag{2.7}$$

Note that if $\{P_n(x)\}_{n=0}^\infty$ is a TPS relative to u, then u is a canonical moment functional of $\{P_n(x)\}_{n=0}^\infty$.

We call a TPS $\{P_n(x)\}_{n=0}^\infty$ a discrete classical TPS if for each $n \geq 0$, $P_n(x)$ satisfies a fixed second order difference equation of the form (1.5).

LEMMA 2.3. *If the difference equation (1.5) has a PS $\{P_n(x)\}_{n=0}^\infty$ of solutions, then any canonical moment functional u of $\{P_n(x)\}_{n=0}^\infty$ satisfies the functional equation*

$$\nabla(\alpha(x)u) = \beta(x)u. \tag{2.8}$$

PROOF. Suppose that $\{P_n(x)\}_{n=0}^\infty$ is a PS of solutions of the difference equation (1.5) and let u be a canonical moment functional of $\{P_n(x)\}_{n=0}^\infty$. Then we have for $n \geq 1$

$$\begin{aligned} 0 = \lambda_n\langle u, P_n\rangle &= \langle u, \lambda_n P_n\rangle \\ &= \langle u, \alpha\Delta^2 P_n(x-1) + \beta\Delta P_n(x-1)\rangle \\ &= \langle \beta u - \nabla(\alpha u), \Delta P_n(x-1)\rangle, \end{aligned}$$

which implies (2.8) since $\{\Delta P_n(x-1)\}_{n=1}^{\infty}$ is also a PS. □

COROLLARY 2.4. *If the difference equation (1.5) has a TPS $\{P_n(x)\}_{n=0}^{\infty}$ of solutions, then $\lambda_n \neq 0$, $n \geq 1$.*

PROOF. Assume that the difference equation (1.5) has a TPS $\{P_n(x)\}_{n=0}^{\infty}$ of solutions and let u be a canonical moment functional of $\{P_n(x)\}_{n=0}^{\infty}$. Then $\{P_n(x)\}_{n=0}^{\infty}$ is a TPS relative to u and by Lemma 2.3, u satisfies the equation (2.8). Assume $\lambda_n = 0$ for some $n \geq 1$. Then we have by (2.8)

$$\begin{aligned}0 &= \lambda_n P_n u \\ &= [\alpha(x)\Delta^2 P_n(x-1) + \beta(x)\Delta P_n(x-1)]u \\ &= \alpha(x)[\Delta^2 P_n(x-1)]u + \Delta P_n(x-1)\nabla(\alpha(x)u) \\ &= \nabla[(\Delta P_n(x))\alpha(x)u]\end{aligned}$$

so that $(\Delta P_n(x))\alpha(x)u = 0$. Hence $(\Delta P_n(x))\alpha(x) \equiv 0$ by Lemma 2.1 (ii) and so $\Delta P_n(x) \equiv 0$ since $\alpha(x) \not\equiv 0$, which implies $n = 0$ contradicting the fact that $n \geq 1$. □

Now we are ready to give our main results.

THEOREM 2.5. *For any TPS $\{P_n(x)\}_{n=0}^{\infty}$ relative to u, the following statements are all equivalent.*

(a) *$\{P_n(x)\}_{n=0}^{\infty}$ is a discrete classical TPS.*
(b) *$\{\Delta P_{n+1}(x)\}_{n=0}^{\infty}$ is also a TPS.*
(c) *$\{\Delta P_{n+1}(x)\}_{n=0}^{\infty}$ is a WTPS.*
(d) *The moment functional u satisfies a functional equation (2.8) for some polynomials $\alpha(x) = ax^2 + bx + c \not\equiv 0$ and $\beta(x) = dx + e$.*
(e) *For each $n \geq 0$, $P_n(x)$ satisfies a functional-difference equation*

$$\alpha(x)\Delta P_n(x) = (r_n x + s_n)P_n(x) + t_n P_{n-1}(x), \qquad n \geq 1, \tag{2.9}$$

for some polynomial $\alpha(x) \not\equiv 0$, independent of n, and constants r_n, s_n, and t_n.

PROOF. (a) $\Rightarrow$ (d) : It is just a special case of Lemma 2.3.

(a) $\Rightarrow$ (b) : Assume that $\{P_n(x)\}_{n=0}^{\infty}$ is a TPS satisfying the equation (1.5). Since (a) implies (d), we have by Lemma 2.1 (iii)

$$\begin{aligned}\lambda_n P_n(x)u =&\alpha(x)[\Delta^2 P_n(x-1)]u + \beta(x)[\Delta P_n(x-1)]u\\ =&\nabla[(\Delta P_n(x))\alpha(x)u].\end{aligned}$$

Hence

$$\begin{aligned}\langle\alpha(x)u, \Delta P_{m+1}(x)\Delta P_{n+1}(x)\rangle &= -\langle\nabla[(\Delta P_{n+1}(x))\alpha(x)u], P_{m+1}(x)\rangle\\ &= -\lambda_{n+1}\langle u, P_{m+1}(x)P_{n+1}(x)\rangle.\end{aligned}$$

Therefore, $\{\Delta P_{n+1}(x)\}_{n=0}^{\infty}$ is a TPS relative to $\alpha(x)u$ since $\lambda_{n+1} \neq 0$, $n \geq 0$ by Corollary 2.4 and $\{P_n(x)\}_{n=0}^{\infty}$ is a TPS relative to u.

(b) $\Rightarrow$ (c) : It is trivial by definition.

(c) $\Rightarrow$ (d) : Assume that $\{\Delta P_{n+1}(x)\}_{n=0}^{\infty}$ is a WTPS relative to v so that

$$\langle v, \Delta P_{m+1}(x)\Delta P_{n+1}(x)\rangle = 0 \qquad \text{for } m \neq n,\ m \text{ and } n \geq 0. \tag{2.10}$$

Set $m = 0$ in (2.10). Then we have for every $n > 0$

$$0 = \langle v, \Delta P_1(x)\Delta P_{n+1}(x)\rangle = -\Delta P_1(x)\langle\nabla v, P_{n+1}(x)\rangle$$

so that $\langle\nabla v, P_{n+1}(x)\rangle = 0$. Hence Lemma 2.2 implies

$$\nabla v = \beta(x)u \tag{2.11}$$

for some polynomial $\beta(x)$ of degree ≤ 1. Set $m = 1$ in (2.10). Then we have, by Lemma 2.1 (iv) and (2.11), for every $n > 1$

$$\begin{aligned}0 &= \langle v, \Delta P_2(x)\Delta P_{n+1}(x)\rangle = -\langle\nabla[(\Delta P_2(x))v], P_{n+1}(x)\rangle\\ &= -\langle[\nabla\Delta P_2(x)]v, P_{n+1}(x)\rangle - \langle[\Delta P_2(x-1)\nabla v, P_{n+1}(x)\rangle\\ &= -[\nabla\Delta P_2(x)]\langle v, P_{n+1}(x)\rangle - \langle\nabla v, [\Delta P_2(x-1)]P_{n+1}(x)\rangle\\ &= -[\nabla\Delta P_2(x)]\langle v, P_{n+1}(x)\rangle - \langle u, \beta(x)[\Delta P_2(x-1)]P_{n+1}(x)\rangle.\end{aligned}$$

Since $\langle u, \beta(x)[\Delta P_2(x-1)]P_{n+1}(x)\rangle = 0$ for $n > 1$ and $\nabla\Delta P_2(x) \not\equiv 0$, $\langle v, P_{n+1}(x)\rangle = 0$ for $n > 1$ so that by Lemma 2.2,

$$v = \alpha(x)u \tag{2.12}$$

for some polynomial $\alpha(x)$ of degree ≤ 2. The equation (2.8) follows from (2.11) and (2.12) and $\alpha(x) \not\equiv 0$ since $v \neq 0$.

(d) $\Rightarrow$ (a) : Assume that the condition (d) holds. Then we have from Lemma 2.1 (iv) and (2.8)

$$\begin{aligned}\langle \alpha u, \Delta P_m \Delta P_n\rangle &= -\langle \nabla[(\Delta P_m)\alpha u], P_n\rangle \\ &= -\langle (\nabla\Delta P_m)\alpha u + (\Delta P_m(x-1))\nabla(\alpha u), P_n\rangle \\ &= -\langle u, [(\nabla\Delta P_m)\alpha + \beta(\Delta P_m(x-1))]P_n\rangle \\ &= 0\end{aligned}$$

for $1 \leq m < n$ since $\deg[(\nabla\Delta P_m)\alpha + \beta(\Delta P_m(x-1))] \leq m$. Hence, $\{\Delta P_n(x)\}_{n=1}^{\infty}$ is a WTPS relative to αu. Since $\alpha(x)\Delta^2 P_n(x-1) + \beta(x)\Delta P_n(x-1)$ is a polynomial of degree $\leq n$, we may write it as

$$\alpha(x)\Delta^2 P_n(x-1) + \beta(x)\Delta P_n(x-1) = \sum_{j=0}^{n} c_j P_j(x) \tag{2.13}$$

for some constants c_j, $j = 0, 1, \ldots, n$. Multiplying (2.13) by $P_k(x)$ and then applying u, we obtain for $k = 0, 1, \ldots, n-1$

$$\begin{aligned}c_k\langle u, P_k^2\rangle &= \langle u, P_k \sum_{j=0}^{n} c_j P_j\rangle \\ &= \langle u, P_k[\alpha\Delta^2 P_n(x-1) + \beta\Delta P_n(x-1)]\rangle \\ &= \langle P_k\beta u - \nabla(P_k\alpha u), \Delta P_n(x-1)\rangle \\ &= \langle P_k\nabla(\alpha u) - \nabla(P_k\alpha u), \Delta P_n(x-1)\rangle \\ &= \langle \nabla P_k\nabla(\alpha u) - (\nabla P_k)\alpha u, \Delta P_n(x-1)\rangle \\ &= -\langle \alpha u, \Delta(\nabla P_k\Delta P_n(x-1)) + \nabla P_k\Delta P_n(x-1)\rangle \\ &= -\langle \alpha u, \Delta P_n\Delta P_k\rangle = 0\end{aligned}$$

since $\{\Delta P_n(x)\}_{n=1}^{\infty}$ is a WTPS relative to αu. Hence, we have $c_j = 0$, $j = 0, 1, \ldots, n-1$ and so

$$\alpha(x)\Delta^2 P_n(x-1) + \beta(x)\Delta P_n(x-1) = c_n P_n(x)$$

and $c_n = \lambda_n$ by comparing the coefficients of x^n from both sides.

(a) $\Rightarrow$ (e) : Assume that $\{P_n(x)\}_{n=0}^{\infty}$ is a TPS satisfying the equation (1.5). Since $\alpha(x)\Delta P_n(x) - \frac{1}{2}n\alpha''(x)xP_n(x)$ is a polynomial of degree $\leq n$, we may write it as

$$\alpha(x)\Delta P_n(x) - \frac{1}{2}n\alpha''(x)xP_n(x) = \sum_{j=0}^{n} c_j P_j(x) \tag{2.14}$$

for some constants c_j, $j = 0, 1, \ldots, n$. Multiplying (2.14) by $P_k(x)$ and then applying u, we obtain for $k = 0, 1, \ldots, n-2$

$$\begin{aligned} c_k\langle u, P_k^2\rangle &= \langle u, P_k \sum_{j=0}^{n} c_j P_j\rangle = \langle P_k u, \alpha(x)\Delta P_n(x) - \frac{1}{2}n\alpha''(x)xP_n(x)\rangle \\ &= -\langle \nabla(P_k \alpha u), P_n(x)\rangle - \langle u, \frac{1}{2}n\alpha''(x)xP_kP_n\rangle \\ &= -\langle P_k(x-1)\beta(x)u + (\nabla P_k)\alpha u, P_n\rangle - \langle u, \frac{1}{2}n\alpha''(x)xP_kP_n\rangle \\ &= -\langle u, [P_k(x-1)\beta(x) + (\nabla P_k)\alpha(x) + \frac{1}{2}n\alpha''(x)xP_k(x)]P_n(x)\rangle \\ &= 0 \end{aligned}$$

since $\deg[P_k(x-1)\beta(x) + (\nabla P_k)\alpha(x) + \frac{1}{2}n\alpha''(x)xP_k(x)] \leq k+1 < n$. Hence, we have $c_j = 0$, $j = 0, 1, \ldots, n-2$ and the equation (2.14) becomes (2.9).

(e) $\Rightarrow$ (c) : Assume that the condition (e) holds. Then we have

$$\begin{aligned} \langle \alpha u, \Delta P_m \Delta P_n\rangle &= \langle u, (\Delta P_m)\alpha\Delta P_n\rangle \\ &= \langle u, \Delta P_m[(r_n x + s_n)P_n + t_n P_{n-1}]\rangle = 0 \end{aligned}$$

for $1 \leq m < n$ since $\deg[(r_n x + s_n)\Delta P_m] \leq m$ and $\deg(\Delta P_m) \leq m-1$. Hence, $\{\Delta P_{n+1}(x)\}_{n=0}^{\infty}$ is a WTPS relative to $\alpha(x)u$. $\square$

The equivalence of the statements (a) and (c) in Theorem 2.5 is new, which can be restated in terms of Sobolev orthogonality as follows.

THEOREM 2.6. *Let $\{P_n(x)\}_{n=0}^{\infty}$ be a TPS relative to u. Then $\{P_n(x)\}_{n=0}^{\infty}$ is a discrete classical TPS if and only if there are two moment functionals $v_1 \neq 0$ and v_0 such that*

$$\langle v_1, \Delta P_m \Delta P_n\rangle + \langle v_0, P_m P_n\rangle = 0 \qquad \text{for } m \neq n,\ m \text{ and } n \geq 0. \tag{2.15}$$

PROOF. Assume that $\{P_n(x)\}_{n=0}^{\infty}$ is a discrete classical TPS relative to u and satisfy the equation (1.5).
Then, by Theorem 2.5, $\{\Delta P_{n+1}(x)\}_{n=0}^{\infty}$ is also a TPS relative to $\alpha(x)u$. Then we have (2.15) with $v_1 = \alpha(x)u$ and $v_0 = u$. Conversely, assume that we have (2.15). If we set $m = 0$ in (2.15), then we have $\langle v_0, P_n\rangle = 0$ for $n \geq 1$. Hence, $v_0 = cu$ for some constant c by Lemma 2.2 so that $\langle v_0, P_m P_n\rangle = 0$ for $m \neq n$. Then we have from (2.15) $\langle v_1, \Delta P_m \Delta P_n\rangle = 0$ for $m \neq n$, that is, $\{\Delta P_{n+1}(x)\}_{n=0}^{\infty}$ is a WTPS relative to v_1. By Theorem 2.5, $\{P_n(x)\}_{n=0}^{\infty}$ is a discrete classical TPS. □

Recently, Branquinho and Petronilho [4] found another characterization of classical orthogonal polynomials : a TPS $\{P_n(x)\}_{n=0}^{\infty}$ is classical if and only if there are constants a_n, b_n, and c_n for $n \geq 2$ such that

$$P_n(x) = a_n Q_n(x) + b_n Q_{n-1}(x) + c_n Q_{n-2}(x), \qquad n \geq 2,$$

where $Q_n(x) = P'_{n+1}(x)$, $n \geq 0$.
We finally give the discrete version of the above result.

LEMMA 2.7. *If a PS $\{P_n(x)\}_{n=0}^{\infty}$ satisfy a three-term recurrence relation*

$$xP_n(x) = \alpha_n P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x), \qquad n \geq 1 \tag{2.16}$$

with constants α_n, β_n, and γ_n, $n \geq 1$, then $\{P_n(x)\}_{n=0}^{\infty}$ is a WTPS.

PROOF. Let u be a canonical moment functional of $\{P_n(x)\}_{n=0}^{\infty}$. Then, by definition, $\langle u, P_n\rangle = 0$, $n \geq 1$. Applying u to (2.16), we obtain $\langle u, xP_n\rangle = 0$, $n \geq 2$. Multiplying (2.16) by x and then applying u, we obtain $\langle u, x^2 P_n\rangle = 0$, $n \geq 3$. Continuing the same process, we have

$$\langle u, x^m P_n\rangle = 0, \qquad 0 \leq m < n$$

so that $\{P_n(x)\}_{n=0}^{\infty}$ is a WTPS relative to u. □

THEOREM 2.8. *A TPS* $\{P_n(x)\}_{n=0}^{\infty}$ *is a discrete classical TPS if and only if there are constants* a_n, b_n, *and* c_n, *for* $n \geq 2$ *such that*

$$P_n(x) = a_n Q_n(x) + b_n Q_{n-1}(x) + c_n Q_{n-2}(x), \qquad n \geq 2, \tag{2.17}$$

where $Q_n(x) = \Delta P_{n+1}(x)$, $n \geq 0$.

PROOF. Assume that $\{P_n(x)\}_{n=0}^{\infty}$ is a discrete classical TPS satisfying the difference equation (1.5). Then by Theorem 2.5 (b), $\{Q_n(x)\}_{n=0}^{\infty}$ is also a TPS relative to $v = \alpha(x)u$. On the other hand, since $\{Q_n(x)\}_{n=0}^{\infty}$ is a PS, we may write $P_n(x)$ as

$$P_n(x) = \sum_{j=0}^{n} c_j Q_j(x),$$

where c_j, $0 \leq j \leq n$, are constants depending on n. Then we have for $k = 0, 1, \ldots, n$

$$c_k \langle v, Q_k^2 \rangle = \langle v, Q_k \sum_{j=0}^{n} c_j Q_j \rangle = \langle v, Q_k P_n \rangle = \langle u, \alpha Q_k P_n \rangle.$$

Since $\{P_n(x)\}_{n=0}^{\infty}$ is a TPS relative to u, $\langle u, \alpha Q_k P_n \rangle = 0$ if $k + 2 < n$. Hence, we have (2.17). Conversely, assume that the condition (2.17) holds. As a TPS, $\{P_n(x)\}_{n=0}^{\infty}$ satisfy a three-term recurrence relation (2.16) (with $\gamma_n \neq 0$, $n \geq 1$). Applying the difference operator Δ to (2.16), we have by Lemma 2.1 (iii) and (2.17)

$$\begin{aligned} &xQ_{n-1}(x) \\ &= (\alpha_n - a_n)Q_n(x) + (\beta_n - b_n - 1)Q_{n-1}(x) + (\gamma_n - c_n)Q_{n-2}(x), \\ &\qquad n \geq 2. \end{aligned}$$

Hence, $\{Q_n(x)\}_{n=0}^{\infty}$ is a WTPS by Lemma 2.7 and so $\{P_n(x)\}_{n=0}^{\infty}$ is a discrete classical TPS by Theorem 2.5. $\square$

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