Journal of the Korean
OR/ MS Society
Vol. 20, No. 1, April 1995

An Algorithm for Optimizing over the Efficient Set of a Bicriterion Linear Programming

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Abstract

In this paper a face optimization algorithm is developed for solving the problem (P) of optimizing a linear function over the set of efficient solutions of a bicriterion linear program. We show that problem (P) can arise in a variety of practical situations. Since the efficient set is in general a nonconvex set, problem (P) can be classified as a global optimization problem. The algorithm for solving problem (P) is guaranteed to find an exact optimal or almost exact optimal solution for the problem in a finite number of iterations. The algorithm can be easily implemented using only linear programming method.

1. Introduction

One of the more popular and practical models has been used to help make decisions involving multiple criteria is the multiple objective linear programming problem (MOLP) model. Let $X = \{x \in R^n | Ax \leqq b\}$ be a nonempty, compact polyhedron, where A is an $m \times n$ matrix and $b \in R^m$. Then the multiple objective linear programming problem may be written

(MOLP) "max" Cx, subject to $x \in X$,

where C is a $k \times n$ matrix whose ith row equals c^i, $i = 1,2,\cdots,k$.

Usually the most preferred compromise solution in the multiple criteria decision making (MCDM) problem is required to be an efficient (nondominated, Pareto-optimal) solution.

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Definition 1

A point $x^{\circ} \in X$ is an efficient solution of problem MOLP if and only if there exists no $x \in X$ such that $Cx \geqq Cx^{\circ}$ and $Cx \neq Cx^{\circ}$.

Let X_E denote the set of all efficient solutions of problem MOLP. The problem of optimizing a linear function over the set of efficient solutions for problem MOLP may be written

$\max \langle d,x \rangle$, subject to $x \in X_E$, where $d \in R^n$.

In general, when X is nonempty, X_E is a nonconvex set. This problem is generally a nonconvex programming problem. Such problems generally possess large number of local optima which need not be globally optimal.

The problem of optimizing a linear function over X_E can arise in a variety of practical situations including many types of transportation problems and production planning problems [5]. For instance, consider a typical problem a homogeneous product is to be transported from each of s sources to any of k destinations $(s+k=m)$. Let $x_{ji} \in R^n$ represent the unknown quantity to be shipped from source j to destination i. Assume that for any transportation plan x, the total cost is given by $\langle -d,x \rangle$, where $d \in R$. The overall goal of the decision maker is to find a minimum-transportation cost plan. However, to meet demand required in each of its k destinations, the decision maker also seeks to minimize total delivery time which takes from all sources to each of its k destinations. In this case, instead of minimizing $\langle -d,x \rangle$ over the set X of all feasible plans, the decision maker could minimize $\langle -d,x \rangle$ over the set of all efficient solutions of a MOLP problem. In this problem, for row c_i of C and each transportation plan $x \in X$, $\langle -c_i,x \rangle$ would give the total delivery time which takes from all sources to the destination i. The solution x^0 obtained from this approach would minimize total cost among all plans that are efficient in terms of total delivery time taken to each destination. The solution x^0 would guarantee that an efficient delivery time plan is achieved.

As shown in the above example, the computational burden of generating the entire efficient set is avoided by optimizing a linear function over the set of efficient solutions of problem MOLP. Furthermore, the decision maker is not required to choose a preferred solution from a potentially overwhelmingly-large set of efficient solutions.

Another important practical situation of the problem of optimizing a linear function over X_E can arise when one seeks to find a minimum criterion value over the efficient set of problem MOLP. Finding such a criterion value helps decision makers to set goals and to rank objective functions, and it may improve the performance of certain interactive algorithms for problem MOLP [9], [10], [13].

In spite of the potential benefits which can be obtained by optimizing a linear function over the efficient set, relatively few attempts have been made to solve the problem of optimizing a linear function over the efficient set. This is probably at least partially due to the inherent difficulties involved in solving this global optimization problem.

A few algorithms have been suggested for finding globally optimal solutions for the problem of optimizing a linear function over the efficient solutions of problem MOLP. However, these algorithms fail to be practically used since they require repeated use of difficult search procedures [10], [12], or of global optimization subroutines [6], [7].

In this paper we consider MOLP problem with two objectives, which is called a bicriterion linear programming problem (BLP). Thus the problem of central interest in this paper seeks to optimize a linear function over the set of all efficient solutions of problem BLP. It may be written

(P) max $\langle d,x \rangle$, subject to $x \in X_E$, where $d \in R^n$ and X_E denote the set of all efficient solutions of problem BLP.

In this paper a face optimization algorithm is developed for solving the problem P of optimizing a linear function over the set of efficient solutions of a bicriterion linear program. The goal of the algorithm is to find an exact optimal or almost exact optimal solution with a relatively small computational effort.

The plan of this article is as follows. In Section 2 we present the necessary theoretical prerequisites for developing our algorithm for solving problem P. In Section 3 the algorithm for solving problem P is presented. In Section 4 an example is solved to illustrate the face optimization algorithm and its implementation. Concluding remarks are given in Section 5.

2. Theoretical Background

One of the more attractive features of problem P is described in the following theorem, which follows immediatedly from [4].

Theorem 1: Problem P has an optimal solution which is an extreme point of X.

The algorithm we shall present for solving problem P will find a new efficient extreme point at each iteration and terminate with an efficient extreme point which yields the maximum $\langle d, x \rangle$ among all of the extreme points of X thereby found.

Definition 2

Let $[l_i, u_i]$, i=1,2 be the range of values that $c_i^T x$ achieves over the efficient solution set of problem (BLP). For each i=1,2, $[l_i, u_i]$ is called the range of nondominance for $c_i^T x$ for problem (BLP).

In the algorithm, theorem 2 [1] is used to determine the range of nondominance for each objective function and theorem 3 [3] is used to find a new efficient point which belongs to a new efficient face.

Theorem 2: Consider the following problems for i, $j \in \{1,2\}$ and $i \neq j$.

u_i = Maximize $\langle c_i, x\rangle$ subject to $x \in X$, (U_i)

l_j = Maximize $\langle c_j, x\rangle$ subject to $x \in X_i^*$, (L_j)

where X_i^* denotes the set of optimal solutions for problem (U_i). Then for i=1,2, the range of nondominance for $c_i^T x$ for problem (BLP) is given by $[l_i, u_i]$.

Theorem 3: Let $[l_i, u_i]$ be the range of nondominance for $c_2^T x$ for problem (BLP). Then $x^0 \in X_E$ if and only if x^0 is an optimal solution for problem (P_b) given by

maximize $\langle c_1, x\rangle$ subject to $c_2^T x \geqq b$, $x \in X$,

for some $b \in [l_i, u_i]$.

In the algorithm, the following two results, theorem 4 and theorem 5, are used to find a new efficient face which contains a new efficient point. The first result follows easily from [2]. The second result is derived from the first result by using duality theory [8].

First, consider the following linear program $P_{x^\circ, \lambda}$ with $x^\circ \in X$.

$(P_{x^\circ,\lambda})$ max $\langle \lambda^T C, x\rangle$

subject to

$$Cx \geqq Cx^\circ \quad (1)$$

$$x \in X$$

Theorem 4: Let $x^\circ \in X$. Then $x^\circ \in X_E$ if and only if for any $\lambda > 0$, x° is an optimal solution of the linear program $(P_{x^\circ,\lambda})$.

Theorem 5: Assume that $\lambda^\circ > 0$ and $x^\circ \in X_E$. Let $(u^{\circ T}, w^{\circ T})$ be any optimal solution to the linear programming dual $D_{x^\circ,\lambda^\circ}$ of problem $P_{x^\circ,\lambda^\circ}$, where u° represents the dual variables corresponding to the constraints (1). Then x° belongs to the efficient face $X_{\hat{\lambda}^\circ}$ of X, where $\hat{\lambda}^\circ = u^\circ + \lambda^\circ$ and $X_{\hat{\lambda}^\circ}$ denotes the optimal solution set of the weighted sum problem(P_λ) with $\lambda = \hat{\lambda}^\circ$: $\max\langle \lambda^T C, x\rangle$ subject to $x \in X$.

PROOF: To prove the desired result, we need to show that x° is an optimal solution to problem P_λ with $\lambda = \hat{\lambda}^\circ$. The dual linear proram to problem $P_{x^\circ, \lambda^\circ}$ is given by

$D_{x^\circ, \lambda^\circ}$:

$$\min \; -\langle x^{\circ\,T} C^T, u\rangle + \langle b, w\rangle,$$

subject to

$$-C^T u + A^T w = C^T \lambda^\circ,$$

$$u, \; w \geqq 0.$$

By Theorem 4, since $x^\circ \in X_E$. x° is an optimal solution for problem $P_{x^\circ, \lambda^\circ}$. By duality theory of linear programming [11],

$$\lambda^{\circ T} C x^\circ = -\langle x^{\circ T} C^T, u^\circ\rangle + \langle b, w^\circ\rangle.$$

Rearranging this equation, we obtain

$$(u^\circ + \lambda^\circ)^T C x^\circ = \langle b, w^\circ\rangle \qquad (2)$$

Now consider the dual linear program to problem P_λ with $\lambda = \hat{\lambda}^\circ$. This dual program is given by

D:

$$\min \langle b, w\rangle,$$

subject to

$$A^T w = C^T (u^\circ + \lambda^\circ)^T,$$

$$w \geqq 0.$$

Since($u^{\circ T}$, $w^{\circ T}$) is an optimal solution to problem $D_{x^\circ, \lambda^\circ}$, w° is a feasible solution for problem D. Let w be any feasible solution for problem D. Then it is easily seen that ($u^{\circ T}$, $w^{\circ T}$) is a feasible solution for problem $D_{x^\circ, \lambda^\circ}$. Since($u^{\circ T}$, $w^{\circ T}$) is an optimal solution for problem $D_{x^\circ, \lambda^\circ}$, this implies that

$$-\langle x^{\circ T} C^T, u^\circ\rangle + \langle b, w\rangle \geqq -\langle x^{\circ T} C^T, u^\circ\rangle + \langle b, w^\circ\rangle,$$

or, equivalently, $\langle b, w\rangle \geqq \langle b, w^\circ\rangle$. It follows that w° is an optimal solution for problem D. Notice also that since $x^\circ \in X_E$, x° is a feasible solution for problem P_λ with $\lambda = \hat{\lambda}^\circ$.

To summarize, we have shown that with $\lambda = \hat{\lambda}^\circ$, x° is a feasible solution to the linear program P_λ, w° is an optimal solution to the dual linear program D of problem P_λ, and, by (2), that the objective function value of x° in problem P_λ equals the objective function value of w° in problem D. From linear programming duality theory [11], this implies that x° is an optimal solution to problem P_λ with $\lambda = \hat{\lambda}^\circ$, and the proof is complete.

The following corollary of Theorem 5 is immediate.

Corollary 1: Assume that $\lambda^\circ > 0$ and $x^\circ \in X_E$. Let $(u^{\circ T}, w^{\circ T})$ be any optimal solution to the linear programming dual $D_{x^\circ,\lambda^\circ}$ of problem $P_{x^\circ,\lambda^\circ}$, where u° represents the dual variables corresponding to the constrants (1). Let $\hat{\lambda}^\circ = u^\circ + \lambda^\circ$, and let $v_o = (\hat{\lambda}^\circ)^T C x^\circ$. Then the efficient face $X_{\hat{\lambda}^\circ}$ of X can be represented as

$$X_{\hat{\lambda}^\circ} = \{x \in X \mid (\hat{\lambda}^\circ)^T Cx = v_o\}$$

From corollary 1, it can be easily seen that the following linear programming $F_{x^\circ,\lambda^\circ}$ finds an efficient extreme solution shich maximizes $\langle d, x \rangle$ over the set of the efficient face $X_{\hat{\lambda}^\circ}$ which contains x°

$(F_{x^\circ,\lambda^\circ})$ max $\langle d, x \rangle$

subject to

$$(\lambda^\circ)^T Cx = (\hat{\lambda}^\circ)^T Cx^\circ$$
$$x \in X.$$

3. The Algorithm

Let $x \in X$ and $e \in R^k$ is a whose entries each equal to one. Let v denote the optimal objective function value for problem P. The algorithm can be described as follows.

Initialization

(a) Choose arbitrary small positive number ε.

(b) Specify the objective function which would be used in reducing the range of nondominance. Without loss of generality, assume that this function is $c_2^T x$.

(c) Determine the range of nondominance $[l_2, u_2]$ for $c_2^T x$ by solving the following problems for i, $j \in \{1,2\}$ and $i \neq j$.

u_i = Maximize $\langle c_i, x \rangle$ subject to $x \in X$, (U_i)

l_j = Maximize $\langle c_j, x \rangle$ subject to $c_i^T x \geqq c_i^T x^*$, $x \in X$, (L_j)

where x^* is the optimal solution found for problem (U_i).

(d) Let x^c be the optimal solution found for problem (L_2) and $v^c = l_2$.

(e) Solve the linear program given by maximize $\langle d, x \rangle$, subject to $x \in X$.

Let x^r be the optimal solution and $m = \langle d, x^r \rangle$.

If $v^c = m$, then stop and conclude that v is equal v^c and x^c is an exact optimal solution for problem P.

(f) Let $b_1 = l_2 + \varepsilon$ and $k = 1$.

At iteration $k (k \geqq 1)$,

Step 1.(Face optimization)

Step 1.1

Find a new efficient point x° by solving the following problem(P_b) given by

$$\text{maximize } \langle c_1, x \rangle, \quad \text{subject to } c_2^T x \geqq b_k, \quad x \in X.$$

Step 1.2

Solve the linear program($D_{x^\circ, e}$) given by

$$D_{x^\circ, e} : \min \; -\langle x^{\circ T} C^T, u \rangle + \langle b, w \rangle$$

subject to

$$-C^T u + A^T w = C^T e$$

$$u, \; w \geqq 0.$$

Let($u^{\circ T}$, $w^{\circ T}$) be an optimal solution to problem $D_{x^\circ, e}$.

Step 1.3

Solve the linear program ($F_{x^\circ, e}$) given by

$$F_{x^\circ, e} : \max \; \langle d, x \rangle$$

subject to

$$(u^\circ + e)^T Cx = (u^\circ + e)^T Cx^\circ$$

$$x \in X$$

Let x^k be an optimal solution to problem $F_{x^\circ, e}$.

If $\langle d, x^k \rangle > v^c$, then $x^c = x^k$ and $v^c = \langle d, x^k \rangle$.

If $v^c = m$, then go to Step 5.

Step 2. (Update the value of b_k)

Solving the following linear program given by

$$\max \quad \langle c_2, x \rangle$$

subject to

$$(u^\circ + e)^T Cx = (u^\circ + e)^T Cx^\circ$$

$$x \in X.$$

Let b_k be the optimal objective function value.

If $b_k = u_2$, then go to Step 5.

Step 3. (Update the value of x^r)

If $c_2^T x^r \rangle b_k$, go to Step 4.

Update the value of x^r by solving the following linear program given by maximize $\langle d, x\rangle$, subject to $c_2^T x \geq b_k$, $x \in X$.

Let x^r be the optimal solution and $m=\langle d, x^r\rangle$.

If $v^c \geq m$, then go to Step 5.

Step 4.

Go to Step 1 with $b_{k+1}=b_k+\varepsilon$. Let $k=k+1$

Step 5.

Conclude that v is almost equal to v^c and that x^c is an almost exact optimal solution for problem P, and terminate.

The algorithm uses a face optimization procedure to find an exact optimal or almost exact optimal solution for problem P in a finite number of iterations. Note that the algorithm can be efficiently implemented using only linear programming methods.

At each iteration, with a little perturbation of b_k in Step 4, the algorithm finds a new efficient face of X and a new efficient extreme point which maximizes $\langle d,x\rangle$ over each face found. With the properly chosen value of ε, the efficient faces the algorithm finds would be adjacent and they include an exact optimal solution of problem P. Whenever the algorithm terminates at Step 5, the point x^c is chosen so as to maximize $\langle d,x\rangle$ over all of the efficient extreme points of X thereby found. Since there are finite number of efficient extreme point in X, it is obvious that the algorithm always terminates in a finite number of iterations. Although the algorithm searchs all efficient faces of X, in many cases it terminates before it finds all efficient faces of X due to the execution of Step 3 in the algorithm. This would improve the performance of the algorithm in terms of computational efficiency. In Step 5, x^c is an exact optimal or almost exact optimal solution for problem P in the following two senses: (1) Problem P has an optimal solution which is an efficient extreme point. (2) With the properly chosen value of ε, there is little possibility that the algorithm fails to find an efficient extreme point which is an optimal solution of problem P.

4. An Example

To illustrate the suggested implementation of the face optimization algorithm, consider the MOLP given by

$$\max \quad 2x_1 + x_2$$
$$\max \quad -x_1 \qquad + x_3$$
subject to
$$5x_1 + 6x_2 + 3x_3 \leqq 30$$
$$x_1 + x_2 + x_3 \leqq 6$$
$$5x_1 + 3x_2 + 6x_3 \leqq 30$$
$$x_1, x_2, x_3 \geqq 0$$

Let $d=(3, -1, 2)$.

Then the problem P: $\max \langle d, x\rangle$

subject to $x \in X_E$

, where $X_E = \gamma(C, D) \cup \gamma(B, C, F, G) \cup \gamma(G, H)$.

The sets X and X_E are shown in Figure 1. Table 1 lists the extreme points of X. The maximum value of $\langle d, x\rangle$ over X equals 15.33 and is achieved at the extreme point E, which does not belong to X_E.

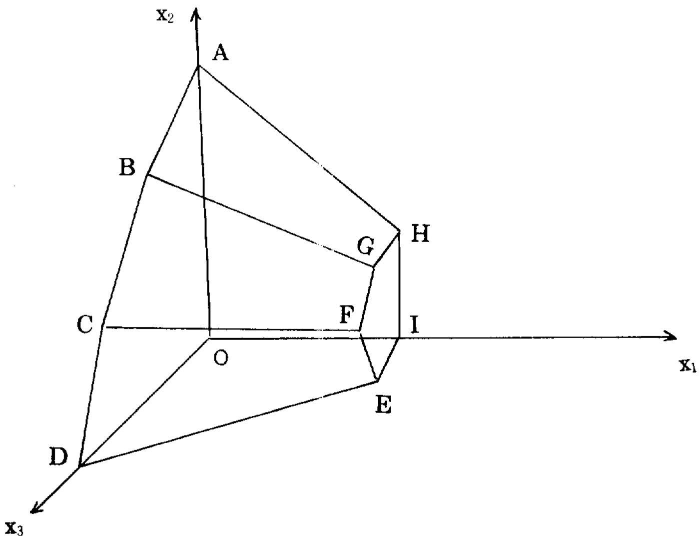

Figure 1. The Sets X and X_E

Point	coordinates		
A	(0,	5,	0)
B	(0,	4,	2)
C	(0,	2,	4)
D	(0,	0,	5)
E	(4,	0,	1.667)
F	(4,	0.667,	1.333)
G	(4,	1.333,	0.667)
H	(4,	1.667,	0)
I	(4,	0,	0)
O	(0,	0,	0)

Table 1. Extreme Points of X

Initialization

(a) Let $\varepsilon=0.05$

(b) Let the objective function which would be used in reducing the range of nondominance be $c_2^T x$.

(c) $[l_2, u_2]=[-4, 5]$

(d) $x^c=(4, 1.667, 0)$, $v^c=10.33$.

(e) $x^r=(4, 0, 1.667)$, $m=15.33$. Since $v^c \neq m$, continue.

(f) $b_1=-3.95$ and $k=1$.

Iteration 1

Step 1.

Step 1.1.

$x^o=(4, 1.642, 0.05)$ is found.

Step 1.2.

$(u^{oT}, w^{oT})=(u_1, u_2, w_1, w_2, w_3, w_4)=(1, 0, 0.33, 0, 0, 1.33)$ is found.

Step 1.3.

$x^1=(4, 1.33, 0.667)$ is found. So $\langle d, x^1\rangle=12$.

Since $\langle d, x^1\rangle > v^c$, then $x^c=(4, 1.33, 0.667)$ and $v^c=12$.

Since $v^c \neq m$, continue.

Step 2.

$b_1=-3.33$. Since $b_1 \neq u_2$, continue.

Step 3.

Since $c_2^T x^r = -2.33 > b_1$, go to Step 4.

Step 4.

Go to Step 1 with $b_2 = -3.28$ and $k=2$.

Iteration 2

Step 1.

Step 1.1.

$x^o = (4, 1.28, 0.72)$ is found.

Step 1.2.

$(u^{oT}, w^{oT}) = (u_1, u_2, w_1, w_2, w_3, w_4) = (0, 0, 0, 1, 0, 0)$ is found.

Step 1.3.

$x^2 = (4, 0.667, 1.33)$ is found. So $\langle d, x^2 \rangle = 14$.

Since $\langle d, x^2 \rangle > v^c$, then $x^c = (4, 0.667, 1.33)$ and $v^c = 14$.

Since $v^c \neq m$, continue.

Step 2.

$b_2 = 4$. Since $b_2 \neq u_2$, continue.

Step 3.

Since $c_2^T x^r = -2.33 \leqq b_2$, update the value of x^r.

$x^r = (0, 2, 4)$ and $m = 6$ are found.

Since $v^c \geqq m$, go to Step 5.

Step 5.

Conclude that v is almost equal to $v^c (=14)$ and that $x^c = (4, 0.667, 1.33)$ is an almost exact optimal solution for problem P, and terminate.

5. Concluding Remarks

We have considered the problem (P) of optimizing a linear function over the set of efficient solutions of a bicriterion linear programming. This problem is a case of problem of optimizing a linear function over the set of efficient solutions of a multiple objective linear program with two objectives. In this paper, we have developed a face optimization algorithm for solving problem (P). The algorithm finds an exact optimal or almost exact optimal solution for the problem (P) in a finite number of iterations. It can be practically used in solving problem

(P), since it can be easily implemented using only linear programming method and does not use difficult search procedures or global optimization subroutines.

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