論文 한국정밀공학회지 제 12 권 제 1 호 1995. 1
Journal of the Korean Society of Precision Engineering Vol. 12, No 1. January, 1995.

다주파수 입력을 갖는 비선형 시스템의 안정성 및 Chaos 해석

김 영 배*

Chaotic Response and Stability Analysis for Multi-input Nonlinear Systems

Young-Bae Kim*

ABSTRACT

다주파수 입력을 갖는 강한 비선형 시스템의 유사주기 (quasi-periodic) 해를 해석하기 위하여 개선된 고정점법(FPA : Fixed Point Algorithm)을 개발하였다. 안정성 및 천이 특성을 판별하기 위하여 사용되어지는 Floquet 지수인 해석적 자코비언을 구하기 위하여 Poincare 맵상에서 이산 적분법을 새로이 고안, 사용하였다. 본 방법의 우수성을 입증하기 위하여 2개의 주파수 입력을 갖는 선형 시스템과 비선형 시스템을 예로 사용하였다. 본 방법을 이용하여 비선형 시스템에서 발생한 복잡한 chaos 현상을 체계적으로 해석하였다.

Key Words : FPA(Fixed Point Algorithm), Floquet multipliers, Poincare map, quasi-periodic, Bifurcation, Chaos.

1. Introduction

Nonlinear dynamical systems have been analyzed recently by many researchers and various methods (theoretical and numerical or both) have been introduced for the solution of periodic or quasi-periodic responses of the nonlinear systems. In linear or nonlinear mechanical systems with multiple exciting frequnecies or nonlinearly coupled multi-degree-of-freedom systems with single exciting frequnecy, quasi-periodic motions can occur. However, it is very difficult to obtain their complete picture of response characteristics as well as to analyze their bifurcation types. Moreover, in many quasi-periodic motions the ratio of two or more exciting frequencies usually are not to be a simple rational number, therfore, the period of solution requires long observation of time by making difficult to obtain the periodicity of the system. As single frequency input to a nonlinear motions, it can be assumed that the nonlinear dynamical systems with multiple exciting freqquencies shows the bifurcation route through torus breaking to chaotic motion. For the nonlinear quasi-peri-

*전남대학교 공과대학 기계공학과

odic systems, Chua and Ushida[1] used the generalized harmonic balance method to obtain almost steady-state responses. The approach used by Chua and Ushida is to determine an initial point on the stedy-state solution, then use numerical integration to obtain the stable response. This method, however, is valid only for the system with smooth nonlinearity, with only a few number of frequency components of significant amplitude involved in the system response. If the system involved with a non-smooth nonlinearity, such as a piecewise-linear type nonlinearity, the method can not be applied to obtain the quasi-periodic responses. Moreover, the above method did not address the stability of the obtained solutions.

In contrast to the generalized harmonic balance method, the fixed point algorithm(FPA) is capable, in addition to identifying stable and unstable solutions, of performing a stability analysis of multi-periodically forced systems (see Kaas-Petersen[2,3]). Kaas-Peterson reformulated the problem of finding quasi-periodic solutions of a forced system with two forcing frequencies as a fixed point in the second order Poincare map. A forward difference approximation was used to estimate the Jacobian matrix which forward difference approximation was used to estimate the Jacobian matrix which could non-convergence problems due to improper choice of increments. Ling[4] presented a modified version of Kass-Peterson's method by employing analytical derivatives instead of the difference approximation and by employing all poincare points in the interpolation process. Both methods were applied to obtain the linear system responses with two input forcing frequencies. However, Ling did not describe how to analyze the stability of the quasi-periodic solutions. Recently Choi and Noah[5] modified FPA method to include analytical form for the piecewise-linear systems with multiple-input forcing frequnecies. They used interpolation technique to calculate the monodromy matrix whose eigenvalues determine the stability of the obtained quasi-periodic solution and its bifurcation characteristics. Although they presented new method to verify that FPA can be applied for the non-smooth type nonlinear system with multi-input frequencies, the interpolation technique they used for the stability analysis can lead to erroneous results since the response shape in the second order Poincare map has complex shape in most nonlinear dynamical systems. Also they did not mention the accuracy of the eigenvalues to determine the bifurcation characteristics.

In this paper, more elaborate and accurate FPA method for the calculation of the Jacobian matrix with discrete integral points selection scheme in the second order Poincare map will be presented. The accurate Jacobian matrix will be used to obtain the eigenvalues of the monodromy matrix which can tell the stability of the obtained solution as well as the bifurcation types if the solution is unstable. The accuracy and effectiveness for the propsed method will be verfied with two examples: 1) linear model with two forcing frequencies, 2) piecewise-linear model with strong nonlinearity.

2. The FPA Method

The dynamical system which has two input frequencies can be described as

$$\dot{x} = f(t, x, \omega_1, \omega_2) \tag{1}$$

where x is dimension n, ω_1 and ω_2 are two exciting frequencies and f can be a linear or nonlinear function. Let the solution of equation (1) be $q(\omega_1 t, \omega_2 t)$ which is quasi-periodic solution and 2π perioidc in ω_1t and ω_2

$$\tau_k = \begin{cases} 1+\theta_k & for \ \ 0 \le \theta_k < 0.5 \\ \theta_k & for \ \ 0.5 \le \theta_k < 1 \end{cases} \tag{2}$$

Note that τ_k is always in the range between 0 and 1. Lets is the stroboscopic function in the Poincare map, then

$$S(0) = q(0,\ 0)$$
$$S(\tau_k) = q(0,\ 2\pi\tau_k) \tag{3}$$

As q is quasi-perioidc and $\tau_k = 0$ and $\tau_k = 1$ are the solution of q which satisfies

$$s(0) = s(1) = q(0,2\pi) = q(0,0) \tag{4}$$

It is difficult to find k which makes $\tau_k = 1$, but according to Kaas-Peterson, intrpolation is used to locate $\tau_k = 1$ within some integer K, i.e.

$$1-\varepsilon_q \le \tau_k \le 1+\varepsilon_q \qquad k \in K \tag{5}$$

where ε_q is very small number. The interpolated point at $\tau_k=1$ will be the solution of s(0). Therefore, the problem of finding the quasi-periodic solution is reduced to the problem of finding the fixed point of (4) using Newton interative technique (therefore, it is called FPA method). Let x be a point close to $s(0)$ and P be the Poincare map of s, then the interation scheme is

$$x_{new} = x_{old} + h \tag{6}$$

where h is the solution of

$$D(P(x_{old}) - x_{old})h = -(P(x_{old}) - x_{old}) \tag{7}$$

or

$$h = -\frac{(P(x_{old}) - x_{old})}{D(P(x_{old}) - x_{old})} = -J^{-1}(P(x_{old}) - x_{old}) \tag{8}$$

where

$$J = D(P(x_{old}) - x_{old}) = \frac{\partial P(x_{old})}{\partial x_{old}} - I \tag{9}$$

in which J is a Jacobian matrix. Kass-Peterson used a forward difference approximation to estimate the Jacobian matrix for an interative process with an arbiarbitrary increment in x, which often leasds to poor estimation of the Jacobian matrix with erroneous results in some parameter ranges, and sometimes convergence problems appeared if improper incremental steps were selected. Ling presented analytic form of derivatives of the Jacobian matrix instead of the difference approximation, but detailed procedure for analyzing the stability of bifurcating quasi-periodic solutions with his method was not described. Choi and Noah extended Ling's method to obtain the analytical Jacobian matrix by interpolation technique, but the detailed procedure to apply the interpolation process were not mentioned. Also the interpolation process often leads to erroneous stability information unless careful interpolation points and interpolation methods(e.g. Lagrangian, spline etc.) should be chosen. In this paper, more accurate and effective method for the calculation of the Jacobian matrix as well as the monodromy matrix is proposed as shown in the next section.

If there are n input frequences in general case, the quasi-periodic solution could be obtained as a similar fashion. Consider the solution of $q(\omega_1 t, \omega_2 t, \cdots, \omega_n t)$ of nth order ODE

$$\dot{x} = f(t, x, \omega_1, \omega_2, \ldots, \omega_n) \tag{10}$$

If $t_k = kT_1$ and $\theta_k^{(i)}(t_k/T_{i+1})$ mod 1 ($i = 1, 2, \cdots, n-1$), then the following time series can be defined as before

$$\tau_k^{(i)} = \begin{cases} 1+\theta_k^{(i)} & for \ \ 0 \le \theta_k^{(i)} < 0.5 \\ \theta_k^{(i)} & for \ \ 0.5 \le \theta_k^{(i)} < 1, (i = 1, 2, \ldots, n-1) \end{cases} \tag{11}$$

When $t = t_k$, the solution of q yields

$$q(\omega_1 t_k, \omega_2 t_k, \cdots, \omega_n t_k) = q\left(0, \frac{2\pi}{T_2} t_k, \cdots, \frac{2\pi}{T_n} t_k\right)$$
$$= q(0, 2\pi\tau_k^{(1)}, \cdots, 2\pi\tau_k^{(n-1)}) \tag{12}$$

By defining the stroboscopic function $s(\cdot,\cdot,\cdots)$ such as

$$s(0,0,\cdots,0)=q(0,0,\cdots,0) \quad (13)$$
$$s(\tau_k^{(1)},\tau_k^{(2)},\cdots,\tau_k^{(n-1)})=q(0,2\pi\tau_k^{(1)},\cdots,2\pi\tau_k^{(n-1)})$$

or

$$s(0,0,\cdots,0)=s(1,1,\cdots,1)=q(0,2\pi,\cdots,2\pi)$$
$$=q(0,0,\cdots,0) \quad (14)$$

Therefore, the quasi-periodic problem is hereby again reduced to a fixed point problem. Note that nth order ODE is reduced to n-1th stroboscopic function in Poincare map. The Newton inerative technique can be utilized again to obtain the fixed solution as before.

3. Stability of the Fixed Point

Stability of periodic solutions of forced ODE with one forcing frequency is usually obtained from the Floquet multipliers (Iooss et al.[6] and Kim and Noah[7]), but could equally well have been applied as the stability of quasi-periodic solutions in the second order Poincare map. Let $x(t)=x(t+T)$ be a solution lying on the closed orbit Γ, based at $x(0)=p\in\Sigma_2$ where Σ_2 is the second order Poincare section. In equation[9], $\frac{\partial P(x_{old})}{\partial x_{old}}$ is obtained as a solution of the variational equation as

$$\dot{z}=\left[u(x_{old})\right]z \quad (15)$$

where $z=\{x,x',\ldots,x^{(n-1)}\}^T$ and

$$\left[u(x_{old})\right]=\begin{bmatrix} 0 & 1 & 0 & \mathrm{L} & 0 \\ 0 & 0 & 1 & \mathrm{L} & 0 \\ \mathrm{M} & \mathrm{M} & \mathrm{M} & \mathrm{O} & \mathrm{M} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial x'} & \frac{\partial f}{\partial x^{(2)}} & \mathrm{L} & \frac{\partial f}{\partial x^{(n-1)}} \end{bmatrix} \quad (16)$$

and prime denotes the time derivative. Then, $[u(x_{old})]$ is obtained from the following Floquet theory. If $[Z(\theta)]$ is the matrix solution satisfying

$$\left[\dot{Z}(\Theta)\right]=\left[u(x_{old})\right]\left[Z(\Theta)\right], \quad \left[Z(0)\right]=\left[I\right] \quad (17)$$

in which θ is 2π periodic then the monodromy matrix $[Z(\theta)]$ is equal to $[u(x_{old})]$. In periodic solution $[Z(\theta)]$ can be readily obtained by integrating equation (10) with analytical derivative form of $[u(x_{old})]$ (refer to Kim and Noah[7]), but discrete integral points should be selected for the calculation of the Jacobian matrix for the quasi-periodic solution. Let N be the total discrete intgral point number in H domain, then the selected integral points in H_k can be chosen according to the following equation.

$$\frac{k}{N}-\varepsilon\le\Theta_k<\frac{k}{N}+\varepsilon \quad (k=1,2,\cdots,N) \quad (18)$$

where ε represents very small number. After defining the θ_ks, the second order Poincare points $x(\theta_k)$, $x'(\theta_k)$,... are easily obtained for the calculation of analytic form of $[u(x_{old})]$.

The stability of the obtained quasi-periodic solution can be determined from the eigenvalues of the monodromy matrix $[Z(\theta)]$. If λ_i's are the Floquet multipliers of the $[Z(\theta)]$, then they decide the stability of the obtained fixed point. If all λ_i's are located inside of the unit disk, then the quasi-periodic solution is stable. If one of the multipliers leaves the unit disk at +1, then quasi-periodic solution becomes unstable through a saddle-node bifurcation, and one of the multipliers leaves the unit disk at -1, the system is known to the flip bifurcation and unstable quasi-periodic solution will bifurcate into a stable quasi-periodic solution of twice or triple of the period. Also flip bifurcation can lead to chaotic quasi-periodic motion through periodi doubling or tripling.

4. Examples

Two typical examples with a linear system and a piecewise-linear one with two input frequencies are considered for the application of the method.

Example 1. Let's consider the second order linear differential equation

$$\ddot{x} + 2\alpha\,\dot{x} + \beta x = (\beta - 2)\cos\sqrt{2}t - 2\sqrt{2}\alpha\sin\sqrt{2}t + (\beta - 1)\cos t - 2\alpha\sin t \qquad (19)$$

where $\omega_1 = \sqrt{2}$ and $\omega_2 = 1$. For interpolation process, Lagrangian interpolation is used with $\varepsilon_q = 0.1$. For $\alpha = 2$, $\beta = 1$ with $(x, \dot{x}) = (0, 0)$ as an initial condition, one can obtain the solution of Newton iteration of (1.996, -0.0001) as fixed point. The exact solution is known to (2.0, 0.0), therefore, the the method can give very favorable results. In applying equation (15) for the Jacobian and monodromy matrix, $[u(x_{old})] = \begin{bmatrix} 0 & 1 \\ -\beta & -2\alpha \end{bmatrix}$ with N = 16 is used to result in λ_1 = (0.18569, 0), λ_2 = (1.33×10^{-7}, 0), which clealy shows the stable quasi-periodic solution. The eigenvalues obtained by forward difference method by Kaas-Peterson(2) was (-0.00038, 0), (0, 0). The discrephancies between two eigenvalues are notable since Kaas-Peterson used approximated Jacobian matrix and the method of the present paper utilized analytical one. Therefore, it can be asumeed that the eigenvalues obtained by the forward difference scheme can lead to the erroneous results because of selecting inadequate increments. The discete integral point in H domain for the calculation of the monodromy matrix is shown in Fig. 1, and the quasi-periodic solution of the method in Poincare map is shown in Fig. 2. For further verifying accuracy of the proposed stability method, $\alpha = 2$ and $\beta = 0.1$ with very soft spring case was considered. One can observe that one of the multipliers leaves the unit circle through +1, which indicates unstable quasi-periodic solution with a saddle-node bifurcation. This type of bifurcation is well knwon as jumping phenomenon in mechanical sense which has two solutions depending on the initial condisions. However, in the linear system with multi-frequency inputs, the response form is very similar to the jumping phenomenon in multiple solutions can co-exist. Fig. 3 shows clearly the two solution responses depending on the initial conditions, which verifies the accuracy of the method. In the figure, dot lines represents the response with initial condition $(x, \dot{x})$ = (0.5, 0.5), while

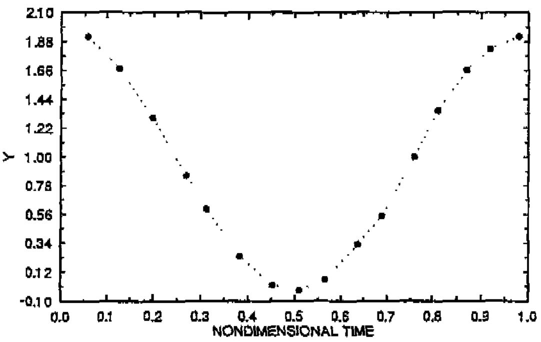

Fig. 1 Discrete integral points

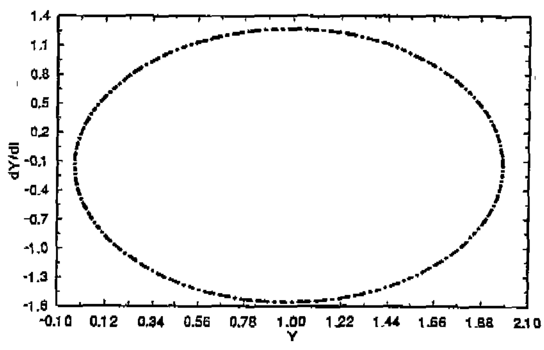

Fig. 2 Quasi-periodic solution with (1.996 -0.0001)

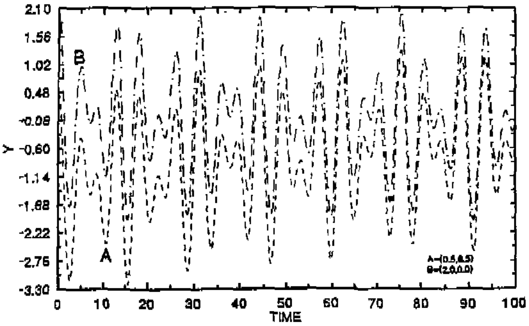

Fig. 3 Two quasi-periodic solutions (α=2, β=0.1)

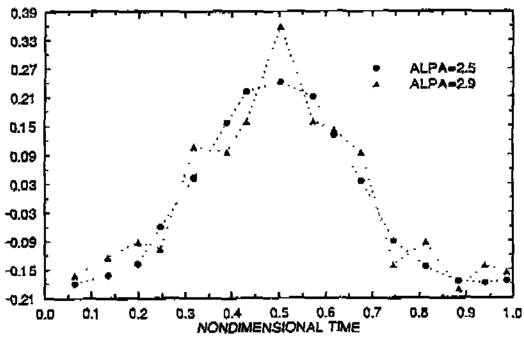

Fig. 4 Discrete integral points (α=2.5, β=2.9)

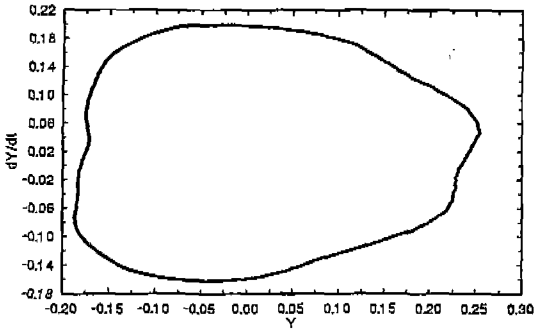

Fig. 5 Torus-one quasi-pe;riodic solution with α= 2.5(ζ=0.02, M_1=$m_2$0.5, Ω=2.5)

response B denoted by dash-dot line is obtained with $(x,\dot{x})$ =(2.0, 0.0). The two responses with same parameters can be explained as jumping between two responses.

Example 2. The folliwing nondimensinalized equation of an offshore articulated loading platform (ALP), considered as a piecewise-linear oscillator is employed (see Kim and Noah (8) and Choi and Lou (9)). The forcing function has two irrational frequencies as

$$y'' + \gamma y' + g(y) = F(\theta) \tag{20}$$

or, in the first order differential equation form,

$$\begin{aligned} y' &= z \\ z' &= -\gamma z - g(y) + F(\theta) \end{aligned} \tag{21}$$

where $F(\theta) = \frac{M_1}{\Omega^2}\cos(\theta) + \frac{M_2}{\Omega^2}\cos(p\theta)$ and

$$g(y) = \begin{cases} \beta_{1y} \ if \ y \geq 0 \\ \beta_{2y} \ if \ y < 0 \end{cases} \tag{22}$$

in which y is a nondimensionalized displacement, prime denotes differential with respect to nondimensional time θ, β_1 and β_2 are nondimensional equivalent stiffness, γ is nondimensional damping, Ω is nondimensional frequency, M_1 and M_2 are the amplitude corresponding to two wave frequencies. The proposed FPA is applied to study its solution and bifurcation characteristics. It is well reported that there exsits a chaotic motion through periodic dobling in single periodic input case(see Thompson(10)) in some parameter ranges. Therefore, it can be assumed that more complex or reach nonlinear responses are possible in a doubly periodic input system with the same parameter ranges. In applying the stability analysis of equation (16), the Jacobian matrix has the explicit form of

$$[U(x_{old})] = \begin{bmatrix} 0 & 1 \\ -g(y) & -\gamma \end{bmatrix} \tag{23}$$

where g(y) has the same expression in equation (22). In equation (23) it is apparent that the multipliers obtained by integrating

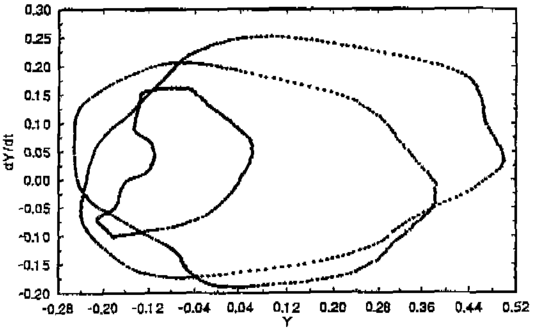

Fig. 6 Tours-three quasi-periodic solution with α=2.9

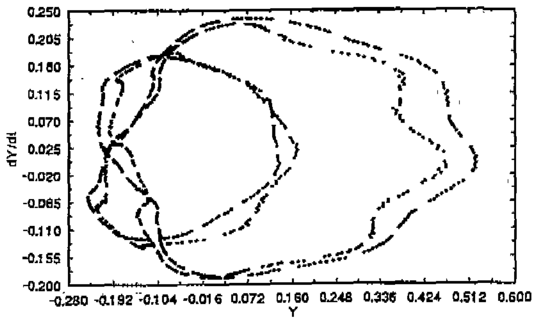

Fig. 8 Torus-four quasi-periodic solution with α=3.09

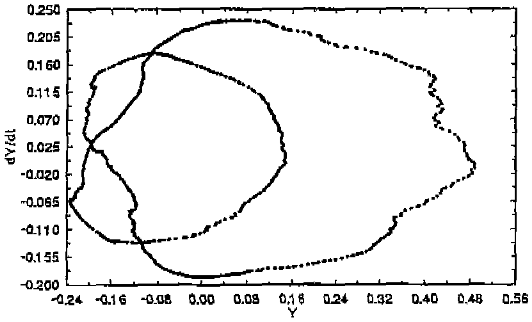

Fig. 7 Torus-two quasi-periodic solution with α=3.05

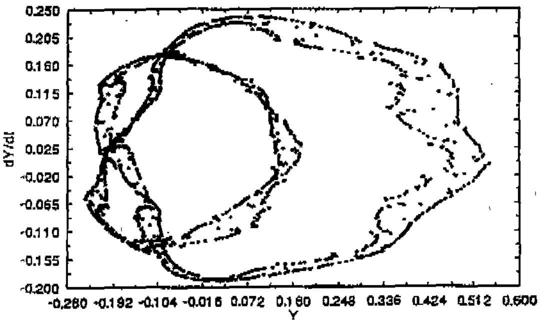

Fig. 9 Trous-eight quasi-periodic solution with α=3.1

(17) should consider exact form of (6) since g(y) is either a function of β_1 or β_2 depending on the y in the second order Poincare points. Unless exact value of g(y) is used in calculating the monodromy matrix [Z(θ)] of (17), the fixed point and its stability result can happen to erroneous ones. If Ling's method is used to calculate the Jacobian matrix by analytic derivatives, it is almost impossible to calculate the matching points explicitly since priori solutions shape is necessary in his method. In this example, the discrete data point of N = 16 are used and the obtained integral points are represented in Fig. 4. Note that for α = 2.9 the response has complex shape which requires more integral points for more accurate solution. However, during the calculation process in the process, N=16 can always offer convergent results with favourable accuracy. Fig. 5 shows the quasi-periodic response with fixed point of (-0.1614, 0.004). The eigenvalues obtained with the same parameters were (-0.7078, 0), (0.0102, 0) which shows a stable torus one solution. With increasing α value, one of the eigenvalues leaves the unit circle which results in periodic doubling or tripling in Poincare domain. For α = 2.9, one of the calculated eigenvalues were (-1.22622, 0) which clearly shows the flip bifurcation case. To verify the stability criteria, numerical integration was used to obtain torus three response in Poincare domain as shown in Fig. 6 with fixed point (-0.1811, 0.0737), which shows favorable agreement of the proposed stability method. By further increasing α, suddenly torus three quasi-periodic response

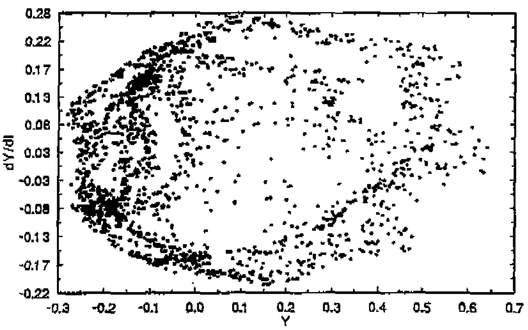

Fig. 10 Chaos motion with α=3.3

becomes a torus two solution with $\alpha = 3.05$. Fig. 7 shows a torus two two quasi-periodic motion with one of the multipliers is leaving the unit circle with value of (-1.3107, 0), which also shows a flip bifurcation case. However, it is almost impossible with the present method how to discern a torus two or torus three unstable Solution with only Floquet multipliers. More systematic research should be directed in this way to distinguish a period doubling or tripling. In a multi-degree-of freedom nonlinear rotor system similar period doubling and tripling can co-exist in a flip bifurcation (see Kim and Noah(11)). With further increasing α values of 3.08, 3.095, torus two quasi-periodic solution bifurcate further with torus four and tours eight solution as shown in Fig. 8 and 9. Fig. 10 shows chaotic response with α=3.1.

5. Conclusion

The modified but more accurate FPA (Fixed Point Algorithm) is developed with analytic form of Jacobian matrix in inerative process. The accurate Jacobian matrix is obtained in the Poincare domain by properly choosing discrete integral points. The eigenvalues of Jacobian matrix can offer quasi-periodic solution's stability as well as the bifurcation information. Two examples are shown to prove the accuracy and effectiveness of the method, one linear model and the other nonlinear one with piecewise-linear nonlinearity. The proposed method can be applied to general nonlinear systems with multi-input exciting frequencies in obtaining the quasi-periodic response and its stability.

ACKNOWLEDGEMENTS

This work was carried out as part of research projects supported by Chonnam National University. The author is grateful for the support.

REFERENCES

1. Chua, L.O. and Ushida, A., 1981. "Algorithms for Computing Almost-Periodic Circuits", IEEE Transactions on Circuits and Systems, CAS-28, pp.953-971.
2. Kaas-Petersen, C., 1985, "Computation of Quasi-Periodic Solutions of Forced Dissipative Systems", J. of Computational Physics, Vol.58, pp.395-403.
3. Kaas-Petersen, C., 1985, "Computation of Quasi-Periodic Solutions of Forced Dissipative Systems Ⅱ," J. of Computational Physics, Vol.64, pp.433-442.
4. Ling F.H., 1991, "Quasi-Periodic Solutions Calculated with Simple Shooting Technique", Journal of Sound and Vibration, Vol.144, pp.293-304.
5. Choi, S.K. and Noah, S.T., 1991, "Response and Stability Analysis of Piecewise-Linear Oscillators under Multi-forcing Frequencies", J. of Nonlinear Dynamics, Vol.3, pp.105-121.
6. Iooss, G. and Joseph, D.P., 1980, Elementary Stability and Bifurcation Theory, New York:Springer-Verlag.

7. Kim, Y.B. and Noah, S.T., 1991, "Stability and Biufurcation Analysis of Oscillators with Peicewise-linear Characteristics:A General Approach", ASME J. of Applied Mechanics, Vol.58, pp.545-553.
8. Kim, Y.B. and Noahc, S.T., 1992, "Periodic Response and Crisis Behavior for a System with Piecewise-Smooth Non-linearities", Int. J. of Non-Linear Mechanics, Vol.27, pp.833-843.
9. Choi, H.S. and Lou, J.Y.K., "The Onset of Chaotic Motions of a SDOF System with Piecewise-nonlinear Stiffness", to appear in Ocean Engineering.
10. Thompson, J.M.T., 1983. "Complex Dynamics of Compliant Offshore Structures", Proceedings of Royal Society London, Vol,1387, pp.407-427.
11. Kim, Y.B. and Noah, S.T., 1990, "Bifurcation Analysis for a Modified Jeffcott Rotor with Bearing Clearances", J. of Nonlinear Dynamics, Vol.1, pp.221-241.