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Some Partial Orderings of Life Distributions[1]

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Abstract

The concept of positive ageing describes the adverse effects of age on the lifetime of units. Various aspects of this concept are described in terms of conditional probability distribution of residual life times, failure rates, equilibrium distributions, etc. In this paper, we will consider some partial ordering relations of life distributions under residual life functions and equilibrium distributions. Under residual life distributions, we study the relationships of IFR, NBU and NBUFR classes and that of DMRL and NBUE classes. By using WLR ordering comparison between F and its equilibrium H_F, we can decide if F belongs to NBUFR class.

1. Introduction

By the ageing of a mechanical unit, component, or some other physical or biological systems, we mean the phenomenon by which an older system has a shorter remaining lifetime, in some stochastic sense, than a newer or younger one.

Suppose that X and Y are nonnegative absolutely continuous random variables with probability density functions $f(x)$ and $g(x)$, respectively. Let F and G be the cumulative distribution functions of X and Y, and $\overline{F}(x)=1-F(x)$ and $\overline{G}(x)=1-G(x)$ be the corresponding survival functions. Partial orderings, namely, likelihood ratio ordering, weak likelihood ratio ordering, failure rate ordering, stochastic ordering, variable ordering, mean residual life ordering, harmonic average mean residual life ordering, expectation ordering, and initial failure rate ordering between two random variables X and Y are known in the literatures.

Deshpande, Kochar, and Singh(1986) introduced various aspects of this concept which are described in terms of conditional probability distributions of residual life times, failure rates, equilibrium distributions. Gupta(1987) studied how the ageing properties, IFR, NBU, NBUE and DMRL of the original distribution were transformed into the ageing properties of the distribution of the residual life. Kochar and Wiens(1987) defined new partial orderings of life distributions and studied the relationships of some partial orderings. Singh(1989) defined two new partial orderings and discussed relevance of these partial orderings for comparing life of

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a new unit with residual life of a used unit. Kochar(1989) studied some properties of partial orderings of life distributions.

Deshpande, Singh, Bagai, and Jain(1990) investigated partial ordering relations with existing partial orders of probability distributions for describing the phenomenon of ageing. Gupta and Kirmani(1990) studied the relationship between the weighted distributions and parent distributions in the context of reliability and life testing. Fagiuoli and Pellerey(1993) introduced new concepts of partial stochastic ordering and studied relations between them and the classical partial orderings.

The residual life of a component of age t and the equilibrium distribution are of great interest in actuarial studies, survival analysis and reliability. So, in Section 2, we consider some partial ordering relations of life distribution under the residual life function. First, LR ordering and WLR ordering are equivalent. Secondly, ST ordering and FR ordering and Initial Failure Rate ordering are equivalent. Finally MR ordering and HAMR ordering and Expectation ordering are equivalent. In Section 3, using above stochastic order relations, we study the relationships of IFR and NBU and NBUFR classes and that of DMRL and NBUE classes. In Section 4, we consider some partial ordering relations of life distributions under the equilibrium distributions. First, stochastic comparison between the equilibrium distributions corresponding to life distributions F and G can be considered. Secondly, by using WLR ordering comparison between F and its equilibrium H_F, we can decide if F belongs to NBUFR class.

2. Relations among some partial orderings under residual life distributions

First of all, we briefly discuss some concepts of the well-known partial orderings.

Definition 2.1. X is said to be larger than Y in likelihood ratio ordering, written as $X \overset{LR}{\geq} Y$, if $\frac{f(x)}{g(x)}$ is nondecreasing in $x \geq 0$.

Definition 2.2. X is said to be larger than Y in weak likelihood ratio ordering, written as $X \overset{WLR}{\geq} Y$, if $\frac{f(x)}{g(x)} \geq \frac{f(0)}{g(0)}$, where $\frac{f(0)}{g(0)}$ is assumed to belong to $(0,\infty)$.

Definition 2.3. X is said to be larger than Y in failure rate ordering, written as $X \overset{FR}{\geq} Y$, if $r_F(x) \leq r_G(x)$ for all $x \geq 0$, where $r_F(x)[=f(x)/\overline{F}(x)]$ and $r_G(x)$ are failure rate functions of X and Y, respectively.

It is known that $X \overset{FR}{\geq} Y$ if and only if $\frac{\overline{F}(x)}{\overline{G}(x)}$ is nondecreasing in $x \geq 0$.

Definition 2.4. X is said to be larger than Y in stochastic ordering, written as $X \overset{ST}{\geq} Y$, if $\overline{F}(x) \geq \overline{G}(x)$ for all $x \geq 0$.

Definition 2.5. X is said to be larger than Y in mean residual life ordering, written as $X \overset{MR}{\geq} Y$, if $e_F(x) \geq e_G(x)$ for all $x \geq 0$, where $e_F(x)[= \frac{1}{\overline{F}(x)} \int_x^\infty \overline{F}(u)du]$ and e_G are mean residual life functions of X and Y, respectively.

It is shown that $X \overset{MR}{\geq} Y$ if and only if $\dfrac{\int_x^\infty \overline{F}(u)du}{\int_x^\infty \overline{G}(u)du}$ is nondecreasing in $x \geq 0$.

Definition 2.6. X is said to be larger than Y in harmonic average mean residual life ordering, written as $X \overset{HAMR}{\geq} Y$, if $(\frac{1}{x}\int_0^x \frac{1}{e_F(u)} du)^{-1} \geq (\frac{1}{x}\int_0^x \frac{1}{e_G(u)} du)^{-1}$ for all $x \geq 0$.

It is observed that $X \overset{HAMR}{\geq} Y$, if and only if $\frac{1}{\mu_F}\int_x^\infty \overline{F}(u)du \geq \frac{1}{\mu_G}\int_x^\infty \overline{G}(x)du$ for all $x \geq 0$.

Definition 2.7. X is said to be larger than Y in variance residual life ordering, written as $X \overset{VR}{\geq} Y$, if $\upsilon_F(x) \leq \upsilon_G(x)$ for all $x \geq 0$. where $\upsilon_F(x)[= \frac{d}{dx}[-\log \int_x^\infty \int_u^\infty \overline{F}(\upsilon)d\upsilon du]$.

It is observed that $X \overset{VR}{\geq} Y$ if and only if $\dfrac{\int_x^\infty \int_u^\infty \overline{F}(\upsilon)d\upsilon du}{\int_x^\infty \int_u^\infty \overline{G}(\upsilon)d\upsilon du}$ is nondecreasing in $x \geq 0$.

Definition 2.8. X is said to be larger than Y in expectation ordering, written as $X \overset{E}{\geq} Y$, if $e_F(0) \geq e_G(0)$.

Definition 2.9. X is said to be larger than Y in initial failure rate ordering, written as $X \overset{r(0)}{\geq} Y$, if $r_F(0) \leq r_G(0)$.

The following chains of implications hold for these known orderings.

$$
\begin{array}{ccccccc}
X \overset{LR}{\geq} Y & \Rightarrow & X \overset{FR}{\geq} Y & \Rightarrow & X \overset{MR}{\geq} Y & \Rightarrow & X \overset{VR}{\geq} Y \\
\Downarrow & & \Downarrow & & \Downarrow & & \\
X \overset{WLR}{\geq} Y & & X \overset{ST}{\geq} Y & & X \overset{HAMR}{\geq} Y & & \\
 & & \Downarrow & & \Downarrow & & \\
 & & X \overset{r(0)}{\geq} Y & & X \overset{E}{\geq} Y. & &
\end{array}
$$

Let $X_t = (X-t) \mid X>t$ be the random variable with probability density function $f_t(x)$ and survival function $\overline{F}_t(x) = P[X > x+t \mid X > t] = \overline{F}(x+t)/\overline{F}(t)$, the conditional probability of a unit of age t .

Theorem 2.1. The following conditions are equivalent.

(1) $X_t \overset{WLR}{\geq} Y_t$ for all $t \geq 0$,

(2) $X_t \overset{LR}{\geq} Y_t$ for all $t \geq 0$,

(3) $X \overset{LR}{\geq} Y$.

Proof. For $\dfrac{\overline{G}(t)}{\overline{F}(t)} > 0$,

$$
\begin{aligned}
X_t \overset{WLR}{\geq} Y_t,\ t \geq 0 \ &\Leftrightarrow \frac{f_t(0)}{g_t(0)} \leq \frac{f_t(x)}{g_t(x)}, \quad x \geq 0,\ t \geq 0 \\
&\Leftrightarrow \frac{\overline{G}(t)}{\overline{F}(t)} \frac{f(t)}{g(t)} \leq \frac{\overline{G}(t)}{\overline{F}(t)} \frac{f(t+x)}{g(t+x)}, \quad x \geq 0,\ t \geq 0 \\
&\Leftrightarrow \frac{f(t)}{g(t)} \leq \frac{f(t+x)}{g(t+x)}, \quad x \geq 0,\ t \geq 0 \\
&\Leftrightarrow \frac{f(x)}{g(x)} \text{ is nondecreasing in } x \geq 0 \\
&\Leftrightarrow X \overset{LR}{\geq} Y
\end{aligned}
$$

$$
\begin{aligned}
X \overset{LR}{\geq} Y \ &\Leftrightarrow \frac{f(x+t)}{g(x+t)} \text{ is nondecreasing in } x \geq 0,\ t \geq 0 \\
&\Leftrightarrow \frac{\overline{G}(t)}{\overline{F}(t)} \frac{f(t+x)}{g(t+x)} \text{ is nondecreasing in } x \geq 0,\ t \geq 0
\end{aligned}
$$

$$\Leftrightarrow \frac{f_t(x)}{g_t(x)} \text{ is nondecreasing in } x \geq 0,\ t \geq 0$$

$$\Leftrightarrow X_t \overset{LR}{\geq} Y_t,\ t \geq 0 .$$

Example 2.1. We consider two random variables X and Y with probability density functions f and g, respectively. Suppose that

$$f(x) = x + \frac{1}{2},\quad 0 \leq x \leq 1 \qquad \text{and}$$

$$g(x) = 3(x - \frac{1}{2})^2 + \frac{3}{4},\quad 0 \leq x \leq 1.$$

It is clear that $X \overset{WLR}{\geq} Y$, but $X_t \overset{WLR}{\not\geq} Y_t$ for all $t > 0$.

A basic problem is to compare reliability characteristics of an equipment (system) with those of another equipment (system). The stochastic ordering is extremely useful in stochastic modeling. The failure rate ordering states that, at the same age, the system whose life is Y is more likely to instantaneously fail than the one whose life is X. Besides reliability applications, this ordering is useful in comparing counting processes.

It is easily seen that $r_{F_t}(x) = r_F(t+x)$ and $e_{F_t}(x) = e_F(t+x)$.

Theorem 2.2. The following conditions are equivalent.

(1) $X_t \overset{ST}{\geq} Y_t$ for all $t \geq 0$,

(2) $X_t \overset{FR}{\geq} Y_t$ for all $t \geq 0$,

(3) $X_t \overset{r(0)}{\geq} Y_t$ for all $t \geq 0$,

(4) $X \overset{FR}{\geq} Y$.

Proof. For $\dfrac{\overline{F}(t)}{\overline{G}(t)} > 0$,

$$X_t \overset{ST}{\geq} Y_t,\ t \geq 0 \Leftrightarrow \overline{F}_t(x) \geq \overline{G}_t(x),\quad x \geq 0,\ t \geq 0$$

$$\Leftrightarrow \frac{\overline{F}(t+x)}{\overline{F}(t)} \geq \frac{\overline{G}(t+x)}{\overline{G}(t)},\quad x \geq 0,\ t \geq 0$$

$$\Leftrightarrow \frac{\overline{F}(x)}{\overline{G}(x)} \text{ is nondecreasing in } x \geq 0$$

$$\Leftrightarrow X \overset{FR}{\geq} Y$$

$$X \overset{FR}{\geq} Y \quad \Leftrightarrow \quad \frac{f(x)}{\overline{F}(x)} \leq \frac{g(x)}{\overline{G}(x)}, \quad x \geq 0$$

$$\Leftrightarrow r_F(x) \leq r_G(x), \quad x \geq 0$$

$$\Leftrightarrow r_F(t+x) \leq r_G(t+x), \quad x \geq 0, \quad t \geq 0$$

$$\Leftrightarrow r_{F_t}(x) \leq r_{G_t}(x), \quad x \geq 0, \quad t \geq 0$$

$$\Leftrightarrow X_t \overset{FR}{\geq} Y_t, \quad t \geq 0$$

$$X_t \overset{FR}{\geq} Y_t, \quad t \geq 0 \Leftrightarrow r_F(t+x) \leq r_G(t+x), \quad x \geq 0, \quad t \geq 0$$

$$\Leftrightarrow r_F(t) \leq r_G(t), \quad t \geq 0$$

$$\Leftrightarrow r_{F_t}(0) \leq r_{G_t}(0), \quad t \geq 0$$

$$\Leftrightarrow X_t \overset{r(0)}{\geq} Y_t, \quad t \geq 0.$$

Example 2.2. Let F denote the life distribution of a parallel system of two independent components having survival functions $\overline{G}(x) = \exp(-x)$, $x \geq 0$ and $\overline{H}(x) = \exp(-2x)$, $x \geq 0$. Then $F(x) = G(x)H(x)$ so that $\overline{F}(x) \geq \overline{G}(x)$. *i.e.,* $F \overset{ST}{\geq} G$, but $\frac{\overline{F}(x)}{\overline{G}(x)} = 1 - \exp(-2x) + \exp(-x)$ which is increasing on $(0, \ln 2)$ and decreasing on $(\ln 2, \infty)$. It follows that F_t and G_t are not in stochastic ordering.

mean residual life ordering is particularly important in analyzing maintenance policies and renewal processes.

Theorem 2.3. The following conditions are equivalent.

(1) $X_t \overset{HAMR}{\geq} Y_t$ for all $t \geq 0$,

(2) $X_t \overset{MR}{\geq} Y_t$ for all $t \geq 0$,

(3) $X_t \overset{E}{\geq} Y_t$ for all $t \geq 0$,

(4) $X \overset{MR}{\geq} Y$.

Proof. For $\frac{\overline{F}(t)}{\overline{G}(t)} > 0,$

$$X_t \overset{HAMR}{\geq} Y_t,\ t\geq 0 \Leftrightarrow \frac{\int_x^\infty \overline{F}_t(u)du}{\int_x^\infty \overline{G}_t(u)du} \geq \frac{\int_o^\infty \overline{F}_t(u)du}{\int_o^\infty \overline{G}_t(u)du},\quad x\geq 0,\ t\geq 0$$

$$\Leftrightarrow \frac{\int_x^\infty \overline{F}(t+u)du}{\int_x^\infty \overline{G}(t+u)du} \geq \frac{\int_o^\infty \overline{F}(t+u)du}{\int_o^\infty \overline{G}(t+u)du},\quad x\geq 0,\ t\geq 0$$

$$\Leftrightarrow \frac{\int_{x+t}^\infty \overline{F}(u)du}{\int_{x+t}^\infty \overline{G}(u)du} \geq \frac{\int_t^\infty \overline{F}(u)du}{\int_t^\infty \overline{G}(u)du},\quad x\geq 0,\ t\geq 0$$

$$\Leftrightarrow \frac{\int_x^\infty \overline{F}(u)du}{\int_x^\infty \overline{G}(u)du} \text{ is nondecreasing in } x\geq 0$$

$$\Leftrightarrow X \overset{MR}{\geq} Y$$

$$X \overset{MR}{\geq} Y \Leftrightarrow \frac{1}{\overline{F}(x)}\int_x^\infty \overline{F}(u)du \geq \frac{1}{\overline{G}(x)}\int_x^\infty \overline{G}(u)du,\quad x\geq 0$$

$$\Leftrightarrow \frac{1}{\overline{F}(t+x)}\int_{t+x}^\infty \overline{F}(u)du \geq \frac{1}{\overline{G}(t+x)}\int_{t+x}^\infty \overline{G}(u)du,\quad x\geq 0,\quad t\geq 0$$

$$\Leftrightarrow \frac{1}{\overline{F}(t+x)}\int_x^\infty \overline{F}(t+u)du \geq \frac{1}{\overline{G}(t+x)}\int_x^\infty \overline{G}(t+v)dv,\quad x\geq 0,\quad t\geq 0$$

$$\Leftrightarrow \frac{1}{\overline{F}_t(x)\,\overline{F}(t)}\int_x^\infty \overline{F}_t(v)\cdot \overline{F}(t)dv \geq \frac{1}{\overline{G}_t(x)\,\overline{G}(t)}\int_x^\infty \overline{G}_t(v)\cdot \overline{G}(t)dv,$$

$x\geq 0,\quad t\geq 0$

$$\Leftrightarrow \frac{1}{\overline{F}_t(x)}\int_x^\infty \overline{F}_t(v)dv \geq \frac{1}{\overline{G}_t(x)}\int_x^\infty \overline{G}_t(v)dv,\quad x\geq 0,\quad t\geq 0$$

$$\Leftrightarrow X_t \overset{MR}{\geq} Y_t,\quad t\geq 0$$

$$
\begin{aligned}
X_t \overset{MR}{\geq} Y_t,\ t\geq 0 \quad &\Leftrightarrow\ e_F(t+x)\geq e_G(t+x),\quad x\geq 0\ ,\ t\geq 0 \\
&\Leftrightarrow\ e_F(t)\geq e_G(t),\ t\geq 0 \\
&\Leftrightarrow\ e_{F_t}(0)\geq e_{G_t}(0),\ t\geq 0 \\
&\Leftrightarrow\ X_t \overset{E}{\geq} Y_t,\ t\geq 0.
\end{aligned}
$$

In this section, we obtain the following chains of implications under residual life distributions.

$$
\begin{array}{ccccc}
X \overset{LR}{\geq} Y & \Rightarrow & X \overset{FR}{\geq} Y & \Rightarrow & X \overset{MR}{\geq} Y \\
\Updownarrow & & \Updownarrow & & \Updownarrow \\
X_t \overset{LR}{\geq} Y_t,\ t\geq 0 & \Rightarrow & X_t \overset{FR}{\geq} Y_t,\ t\geq 0 & \Rightarrow & X_t \overset{MR}{\geq} Y_t,\ t\geq 0 \\
\Updownarrow & & \Updownarrow & & \Updownarrow \\
X_t \overset{WLR}{\geq} Y_t,\ t\geq 0 & \Rightarrow & X_t \overset{ST}{\geq} Y_t,\ t\geq 0 & \Rightarrow & X_t \overset{HAMR}{\geq} Y_t,\ t\geq 0 \\
 & & \Updownarrow & & \Updownarrow \\
 & & X_t \overset{r(0)}{\geq} Y_t,\ t\geq 0 & \Rightarrow & X_t \overset{E}{\geq} Y_t,\ t\geq 0
\end{array}
$$

3. Relations of life distributions classes under residual life distributions

The notion of ageing plays an important role in reliability and maintenance theory. Many partial orders are utilized for making comparisons between probability distributions of residual life times at different ages in order to describe positive ageing. Therefore several classes of life distributions have been proposed in order to model different aspects of ageing. These classes are the followings:

1. Increasing Likelihood Ratio(ILR)
2. Increasing Failure Rate(IFR)
3. Increasing Failure Rate Average(IFRA)
4. Decreasing Mean Residual Life(DMRL)
5. New Better than Used(NBU)
6. New Better than Used in Expectation(NBUE)
7. New Better than Used in Failure Rate(NBUFR)

8. New Better than Used in Failure Rate Average(NBUFRA)
9. Harmonic New Better than Used in Expectation(HNBUE)

The relations between the classes 1. - 9. are as follows :

$$\begin{array}{ccccccccccc} \text{ILR} & \Rightarrow & \text{IFR} & \Rightarrow & \text{IFRA} & \Rightarrow & \text{NBU} & \Rightarrow & \text{NBUFR} & \Rightarrow & \text{NBUFRA} \\ & & \Downarrow & & & & \Downarrow & & & & \\ & & \text{DMRL} & & \Rightarrow & & \text{NBUE} & & \Rightarrow & & \text{HNBUE} \end{array}$$

Theorem 3.1. X is ILR if and only if X_t is ILR for every $t \geq 0$.

Proof. By definition, X is ILR iff $X \overset{LR}{\geq} X_a$ for all $a \geq 0$. Using Theorem 2.1, $X \overset{LR}{\geq} X_a$ for all $a \geq 0$ iff $X_t \overset{LR}{\geq} X_{t+a}$ for all $a,t \geq 0$, *i.e.*, X is ILR iff X_t is ILR for every $t \geq 0$.

Theorem 3.2. The following conditions are equivalent.

(1) X is IFR,

(2) X_t is IFR for every $t \geq 0$,

(3) X_t is NBU for every $t \geq 0$,

(4) X_t is NBUFR for every $t \geq 0$.

Proof. By definition, X is IFR iff $X \overset{FR}{\geq} X_a$ for all $a \geq 0$. Using Theorem 2.2, we obtain that $X \overset{FR}{\geq} X_a$ for all $a \geq 0$ and $X_t \overset{FR}{\geq} X_{t+a}$ for all $a,t \geq 0$ and $X_t \overset{ST}{\geq} X_{t+a}$ for all $a,t \geq 0$ and $X_t \overset{r(0)}{\geq} X_{t+a}$ for all $a,t \geq 0$ are equivalent. *i.e.*, X is IFR, X_t is IFR for every $t \geq 0$, X_t is NBU for every $t \geq 0$, and X_t is NBUFR for every $t \geq 0$ are equivalent.

Theorem 3.3. The following conditions are equivalent.

(1) X is DMRL,

(2) X_t is DMRL for every $t \geq 0$,

(3) X_t is NBUE for every $t \geq 0$.

Proof. By definition, X is DMRL iff $X \overset{MR}{\geq} X_a$ for all $a \geq 0$. Using Theorem 2.3, $X \overset{MR}{\geq} X_a$ for all $a \geq 0$ and $X_t \overset{MR}{\geq} X_{t+a}$ for all $a,t \geq 0$ and $X_t \overset{E}{\geq} X_{t+a}$ for all $a,t \geq 0$ are equivalent. *i.e.,* X is DMRL, X_t is DMRL for every $t \geq 0$, and X_t is NBUE for every $t \geq 0$ are equivalent.

Therefore, we obtain relationships among these classes under the residual life distribution.

For X_t, $t \geq 0$,

$$\begin{array}{ccccccc} \text{ILR} & \Rightarrow & \text{IFR} & \Rightarrow & \text{DMRL} & & \\ & & \Updownarrow & & \Updownarrow & & \\ & & \text{NBU} & \Rightarrow & \text{NBUE} & \Rightarrow & \text{HNBUE} \\ & & \Updownarrow & & & & \\ & & \text{NBUFR} & & & & \end{array}$$

4. Equilibrium distributions

We have a unit in operation whose life distribution is F. As soon as this unit fails, another new unit, which acts independently of the first and has the same life distribution, is activated. This renewal of the system is continued indefinitely. Then the residual life of the unit under operation at time t as $t \to \infty$ is given by $H_F(t)$, the equilibrium distribution. Hence, obviously, stochastic comparisons between the equilibrium distributions corresponding to life distributions, F and G, can be considered.

The equilibrium distribution has been used to describe the concepts of positive ageing in Deshpande, Kochar and Singh(1986).

Let equilibrium distribution $H_F(x) = \frac{1}{\mu_F} \int_0^x \overline{F}(t)dt$, and equilibrium survival function

$$\overline{H_F}(x) = \frac{1}{\mu_F} \int_x^\infty \overline{F}(t)dt, \quad 0 \leq x < \infty, \text{ where } \mu_F = \int_0^\infty \overline{F}(t)dt.$$

Deshpande, Singh, Bagai, and Jain(1990) proved below results(in Theorem 4.3.) under the assumption having the same mean, but we can obtain the followings without the assumption.

Proposition 4.1.

1) $F \overset{FR}{\geq} G$ if and only if $H_F \overset{LR}{\geq} H_G$.

2) $F \overset{MR}{\geq} G$ if and only if $H_F \overset{FR}{\geq} H_G$.

3) $F \overset{HAMR}{\geq} G$ if and only if $H_F \overset{ST}{\geq} H_G$.

Theorem 4.2. $F \overset{ST}{\geq} G$ if and only if $H_F \overset{WLR}{\geq} H_G$.

Proof. Since $h_F(x) = \dfrac{\overline{F}(x)}{\mu_F}$ and $h_G(x) = \dfrac{\overline{G}(x)}{\mu_G}$,

$$\begin{aligned} F \overset{ST}{\geq} G &\Leftrightarrow \overline{F}(x) \geq \overline{G}(x), \quad x \geq 0 \\ &\Leftrightarrow \mu_F h_F(x) \geq \mu_G h_G(x), \quad x \geq 0 \\ &\Leftrightarrow \frac{h_F(0)}{h_G(0)} = \frac{1/\mu_F}{1/\mu_G} \leq \frac{h_F(x)}{h_G(x)}, \quad x \geq 0 \\ &\Leftrightarrow H_F \overset{WLR}{\geq} H_G. \end{aligned}$$

Theorem 4.3. $F \overset{VR}{\geq} G$ if and only if $H_F \overset{MR}{\geq} H_G$.

Proof.

$$F \overset{VR}{\geq} G \Leftrightarrow \frac{\int_x^\infty \int_u^\infty \overline{F}(v)\,dv\,du}{\int_x^\infty \int_u^\infty \overline{G}(v)\,dv\,du} \text{ is nondecreasing in } x \geq 0$$

$$\Leftrightarrow \frac{\int_x^\infty \int_u^\infty \frac{\overline{F}(v)}{\mu_F}\,dv\,du}{\int_x^\infty \int_u^\infty \frac{\overline{G}(v)}{\mu_G}\,dv\,du} \text{ is nondecreasing in } x \geq 0 \text{ for } \frac{\mu_G}{\mu_F} > 0$$

$$\Leftrightarrow \frac{\int_x^\infty \overline{H_F}(u)\,du}{\int_x^\infty \overline{H_G}(u)\,du} \text{ is nondecreasing in } x \geq 0$$

$$\Leftrightarrow H_F \overset{MR}{\geq} H_G.$$

Similarly, the following result brings out the fact that the comparison between the distribution F and its equilibrium distribution H_F also leads to recognizable positive ageing classes of distributions.

Theorem 4.4. $F \overset{WLR}{\geq} H_F$ if and only if F is NBUFR class.

Proof.

$$F \overset{WLR}{\geq} H_F \Leftrightarrow \frac{f(x)}{h_F(x)} \geq \frac{f(0)}{h_F(0)}, \quad x \geq 0$$

$$\Leftrightarrow \frac{f(x)}{\overline{F}(x)} \geq \frac{f(0)}{\overline{F}(0)}, \quad x \geq 0$$

$$\Leftrightarrow r_F(x) \geq r_F(0), \quad x \geq 0$$

$\Leftrightarrow$ F is NBUFR.

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