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Sampling Based Approach to Hierarchical Bayesian Estimation of Reliability Function[1]

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Abstract

For the stress-strengh function, hierarchical Bayes estimations are considered under squared error loss and entropy loss. In particular, the desired marginal postrior densities are obtained via Gibbs sampler, an iterative Monte Carlo method, and Normal approximation (by Delta method). A simulation is presented.

1. Introduction

In the statistical analysis of reliability theory, it is common to study the Rayleigh models. The density probability function(pdf), conditional on a parameter σ^2, is given by

$$f(x|\sigma^2) = \frac{x}{\sigma^2} \exp(-\frac{x^2}{2\sigma^2}),\ x>0. \tag{1.1}$$

Siddiqui(1962) mentioned the origin and properties of the Rayleigh distribution. Dyer and Whisenand(1973) discussed the use of the Rayleigh distribution in electrovacuum devices and communication engineering. Sinha and Howlander(1983) obtained a Bayes estimator of Rayleigh parameter and the associated reliability function using squared error loss function and Jeffreys' noninformative prior. Chung(1995) considered the Bayesian estimation of scale parameter of Rayleigh distribution under entropy loss funtion. In this paper, a prior distribution is chosen to reflect prior knowledge about σ^2. Throughout this paper we assume that the prior distribution of σ^2 is inverted gamma distribution with density

$$g(\sigma^2|\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}(\sigma^2)^{\alpha+1}} \exp(-\frac{1}{\beta\sigma^2}),\ \alpha^2>0, \alpha,\beta>0. \tag{1.2}$$

We shall denote this by $\sigma^2 \sim IG(\alpha,\beta)$.

A hierarchical Bayesian approach to inference in this problem has been impeded by the dificulties in computing required marginal posterior distributions. Recent developments in

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computation techniques helps us to implement such models. In particular, Carlin et. al(1992) have considered the hierarchical Bayesian analysis to the change-point problem using Gibbs sampler. In section 2, we briefly review the Gibbs sampler which underlies our iterative Monte Carlo method. In section 3, we formulate the hierarchical Bayesian stress-strength model. Fianally, section 4 has the simulation.

2. Gibbs Sampler

The Gibbs sampler is a Markovian updating scheme enabling one to obtaining sampling from full conditional distributions. It was developed formally by Geman and Geman(1984) in the context of image restorations. More recently, Gelfand and Smith(1990) showed its applicability to general parametric Bayesian Computations. Given a joint posterior density $f(\theta|X)$, functional forms of k univariate full conditional densities (i.e. the distributions of each individual component of θ conditional on specific values of the data x and all the other components) can be readily written down, at least up to propotionality. Given x, for convenience, the k univariate conditional densities are given by $f(\theta_1|\theta_2,\cdots,\theta_k)$, $f(\theta_2|\theta_1,\theta_3,\cdots,\theta_k)$, $\cdots$, $f(\theta_k|\theta_1,\cdots,\theta_{k-1})$. The Gibbs sanpling procedures are as follows : Choose starting point $\theta_1^{(0)},\theta_3^{(0)},\cdots,\theta_k^{(0)}$ and generate $\theta_1^{(1)}$ from $f(\theta_1|\theta_2^{(0)},\cdots,\theta_k^{(0)})$. Next, generate $\theta_2^{(1)}$ from $f(\theta_2|\theta_1^{(0)},\theta_3^{(0)},\cdots,\theta_k^{(0)})$, and continuing this procedure to get $\theta_k^{(1)}$ from $f(\theta_k|\theta_1^{(1)},\cdots,\theta_{k-1}^{(1)})$. After t such iteratives, we get $(\theta_1^{(t)},\cdots,\theta_k^{(t)})$. Under mild conditions, Geman and Geman(1984) showed that for each j, $\theta_j^{(t)} \xrightarrow{d} \theta_j \sim [\theta_j]$ as $t \to \infty$.

Independent parallel replications of the entire above process m times procedures m set of parameter vectors, $(\theta_{1j}^{(t)},\theta_{2j}^{(t)},\cdots,\theta_{kj}^{(t)})$ for $j=1,\cdots,m$. So we can estimate the marginal posterior density $f(\theta_1|X)$ and the expectation $E[g(\theta_i)|X]$ for any function $g(\theta_i)$,

$$\widehat{f(\theta_i|X)} = \frac{1}{m}\sum_{j=1}^{m} f(\theta_i|\theta_{1j}^{(t)},\cdots,\theta_{i-1,j}^{(t)},\theta_{i+1,j}^{(t)},\cdots,\theta_{kj}^{(t)})$$

and

$$\widehat{E}[g(\theta_i)] = \frac{1}{m}\sum_{i=1}^{m} g(\theta_{kj}^{(t)}).$$

3. Hierarchical Bayesian formulation for stress-strengh model

A definition of reliability function can be given as follows. Let X and Y be two random variables with cdf $F(x)$ and $G(y)$ respectively. Suppose that Y is the strength of a component subject to a random stress X. Then the component fails if at any moment the applied stress(or load) is greater than its strength or resistance. The reliability of the component in this case is given by

$$R = \Pr(X < Y) \ ,$$

which is called the stress-strength model. Johnson(1988) has the results based on the sampling theory approach. Also, Eni and Geisser(1971), Basu and Ebrahimi(1991) have considered the problem from Bayesian point of view.

Let $X_1, \cdots, X_m$ and $Y_1, \cdots, Y_n$ be independently and identically distributed as

$$f_1(x|\sigma_1^2) = \frac{x}{\sigma_1^2} \exp\left(-\frac{x}{2\sigma_1^2}\right), \tag{3.1}$$

and

$$f_2(y|\sigma_2^2) = \frac{y}{\sigma_2^2} \exp\left(-\frac{y}{2\sigma_2^2}\right), \tag{3.2}$$

respectively. If X and Y are independently distributed with densities given by (3.1) and (3.2), respectively, then it is easily shown

$$R = \Pr(Y - X > 0) = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} = \frac{\lambda}{1+\lambda}, \quad \lambda = \frac{\sigma_2^2}{\sigma_1^2}. \tag{3.3}$$

Assume that the parameters σ_1^2 and σ_2^2 are identically distributed the prior distribution of σ_i^2 being $IG(\alpha_i, \beta_i)$, $i = 1,2$ in (1.2). Let $\lambda = \sigma_2^2/\sigma_1^2$ and $v = \sigma_1^2$. Then the joint posterior density of λ, v given $\underline{X}$, $\underline{Y}$ is

$$p(\lambda, v|\underline{X}, \underline{Y}) = K\exp\left[-\left(\frac{t_1\beta_1+2}{2\beta_1}\right)+\left(\frac{t_2\beta_2+2}{2\beta_2}\right)\frac{1}{\lambda}\,\frac{1}{v}\right]$$

$$* \left(\frac{1}{\lambda}\right)^{n+\alpha_2+1} * \frac{1}{v^{m+n+\alpha_1+\alpha_2+1}} \tag{3.4}$$

where

$$K = \frac{1}{\Gamma(m+\alpha_1)\Gamma(n+\alpha_2)}\left(\frac{t_1\beta_1+2}{2\beta_1}\right)^{\alpha_1+m}\left(\frac{t_2\beta_2+2}{2\beta_2}\right)^{\alpha_2+n}.$$

Thus the marginal posterior density of $\lambda = \sigma_2^2/\sigma_1^2$ given $\underline{X}$ and $\underline{Y}$ is

$$p(\lambda|\underline{X},\underline{Y}) = \frac{1}{B(\alpha_1+m,\alpha_2+n)} u^{\alpha_1+m}\lambda^{\alpha-1+m-1}(1+u\lambda)^{-(m+n+\alpha_1+\alpha_2)} \tag{3.5}$$

where $u=\frac{\beta_2(2+t_1\beta_1)}{\beta_1(2+t_2\beta_2)}$ and $B(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$.

Theorem 3.1 Under square error loss, the Bayes estimate of R is given by

$$\widehat{R}=E[R|\underline{X},\underline{Y}] \quad = \int_0^\infty \frac{\lambda}{1+\lambda} p(\lambda|X,Y)d\lambda. \tag{3.6}$$

Next consider the entropy distance of Rayleigh scale parameter as follows;

$$L(\sigma^2,a)=E\left[\log \frac{\frac{X}{\sigma^2}\exp\left(-\frac{X}{2\sigma^2}\right)}{\frac{X}{a}\exp\left(-\frac{X}{2a}\right)}\right] = \frac{\sigma^2}{a} - \log\frac{\sigma^2}{a} - 1$$

This is called the entropy loss.

Theorem 3.2. Under the entropy loss, the Bayes estimate of R is given by

$$\widehat{R_1}=E[\frac{1}{R}|\underline{X},\underline{Y}]^{-1} = \left[\int_0^\infty \frac{1+\lambda}{\lambda} p(\lambda|X,Y);d\lambda\right]^{-1}. \tag{3.7}$$

From now, consider the hierarchical Bayesian analysis. Let $X_1,\ldots,X_m$ and $Y_1,\ldots,Y_m$ be independently and identically distributed as $f_1(x|\sigma_1^2)$ and $f_2(y|\sigma_2^2)$ given in (3.1) and (3.2) respectively. Assume that σ_1^2, given α_1 and β_1, and σ_2^2, given α_2 and β_2, are independent and distributed to $IG(\alpha_1,\beta_1)$ and $IG(\alpha_2,\beta_2)$ respectively. Furthermore, assume that β_1 and β_2 are independent and β_i has density $IG(a_i,b_i)$, $i=1,2$, where α_1, α_2, , a_1, a_2, b_1 and b_2 are known. Let $t_1=\sum_{i=1}^m x_i^2$ and $t_2=\sum_{i=1}^n y_i^2$. Then the posterior density of σ_1^2, σ_2^2, β_1 and β_2 is

$$\begin{aligned} p(\sigma_1^2,\sigma_2^2\beta_1,\beta_2|\underline{X},\underline{Y}) \propto & \frac{1}{(\sigma_1^2)^{m+\alpha_1+1}}\exp\left[-\frac{t_1\beta_1+2}{2\beta_1}\frac{1}{\sigma_1^2}\right] \\ & \times \frac{1}{(\sigma_2^2)^{n+\alpha_2+1}}\exp\left[-\frac{t_2\beta_2+2}{2\beta_2}\frac{1}{\sigma_2^2}\right] \\ & \times \frac{1}{\beta_1^{a_1+1}}\exp\left[-\frac{1}{\beta_1 b_1}\right]\frac{1}{\beta_2^{a_2+1}}\exp\left[-\frac{1}{\beta_2 b_2}\right] \end{aligned} \tag{3.8}$$

Thus the joint posterior density of σ_1^2 and σ_2^2 is

$$p(\sigma_1^2,\sigma_2^2|\underline{X},\underline{Y}) \propto \frac{1}{(\sigma_1^2)^{m+\alpha_1+1}(\sigma_2^2)^{n+\alpha_2+1}} \exp\left[-\frac{t_1}{2\sigma_1^2}-\frac{t_2}{2\sigma_2^2}\right]$$

$$\times\ \Gamma(a_1)\left(\frac{\sigma_1^2 b_1}{\sigma_1^2+b_1}\right)^{a_1}\Gamma(a_2)\left(\frac{\sigma_2^2 b_2}{\sigma_2^2+b_2}\right)^{a_2}$$

But $p(\sigma_1^2|\underline{X},\underline{Y})$ and $p(\sigma_2^2|\underline{X},\underline{Y})$ can not be expressed as the analytic form, and so we can not calculate the equations (3.6) nd (3.7) analytically. Thus we will approximately obtain these estimates using Gibbs sampler.

4. Simulation

The data in table 1 are drawn from the Rayleigh deviates with $\sigma_1^2=2$ and $\sigma_2^2=5$.

TABLE 1 Data generated from Rayleigh deviates :

X ($\sigma_1^2=2$, $m=30$)	Y ($\sigma_2^2=5$, n = 20)
2.43, 1.61, 0.47, 2.43, 1.63, 2.51 0.90, 1.18, 1.19, 1.77, 0.78, 3.35 2.57, 1.90, 1.78, 1.71, 0.80, 2.15 3.90, 0.55, 4.33, 1.52, 1.78, 3.00 1.39, 1.83, 1.84, 0.90, 0.89, 1.96	2.32, 1.78, 1.46, 0.63 4.69, 5.04, 3.09, 4.31 2.37, 1.55, 1.51, 2.62 3.24, 2.36, 5.46, 1.25 2.21, 7.94, 1.85, 2.95

For hierarchical Bayesian analysis, using Gibbs sampler, we need the univariate conditional densities which are as fellows;

$$p(\sigma_1^2|\sigma_2^2,\beta_1,\beta_2,\underline{X},\underline{Y}) = IG(m+\alpha_1, \frac{2\beta_1}{t_1\beta_1+2}), \tag{4.1}$$

$$p(\sigma_2^2|\sigma_1^2,\beta_1,\beta_2,\underline{X},\underline{Y}) = IG(n+\alpha_2, \frac{2\beta_2}{t_2\beta_2+2}), \tag{4.2}$$

$$p(\beta_1|\sigma_1^2\sigma_2^2,\beta_2\underline{X},\underline{Y}) = IG(a_1, \frac{\sigma_1^2 b_1}{\sigma_1^2+b_1}), \tag{4.3}$$

and

$$p(\beta_2|\sigma_1^2,\sigma_2^2,\beta_1,\underline{X},\underline{Y}) = IG(a_2, \frac{\sigma_2^2 b_2}{\sigma_2^2 + b_2}). \tag{4.4}$$

Actually we are interested in the distributions of $r=\sigma_2^2/\sigma_1^2$ and $R=\sigma_2^2/(\sigma_1^2+\sigma_2^2)$ and their expectations. So by Jacobian transformation method, the conditional density of r given σ_1^2, β_1, β_2, X, Y is

$$\begin{aligned} p(r|\sigma_1^2,\beta_1,\beta_2,X,Y) &= K\frac{\sigma_1^2}{(r\sigma_1^2)^{n+\alpha_2+1}} \exp\left[-\frac{t_2\beta_2+2}{2\beta_2}\frac{1}{r\sigma_1^2}\right] \\ &= K\frac{1}{r^{n+\alpha_2+1}}\frac{1}{(\sigma_1^2)^{n+\alpha_2}} \exp\left[-\frac{t_2\beta_2+2}{2\beta_2\sigma_1^2}\frac{1}{r}\right] \end{aligned}$$

and the conditional density of R given σ_1^2, β_1, β_2, X, Y is

$$p(R|\sigma_1^2,\beta_1,\beta_2,X,Y) = K\left(\frac{R}{1-R}\sigma_1^2\right)^{-n-\alpha_2-1} \exp\left[-\frac{t_2\beta_2+2}{2\beta_2}\frac{1-R}{R\sigma_1^2}\right]\frac{\sigma_1^2}{(1-R)^2}$$

where $K=\{\Gamma(n+\alpha_2)\}^{-1}(\frac{2\beta_2}{t_2\beta_2+2})^{-n-\alpha_2}$. Accordingly, the marginal density estimates of r and R given X and Y are respectively

$$\hat{p}(r=\frac{\sigma_2^2}{\sigma_1^2}|\underline{X},\underline{Y}) = \frac{1}{M}\sum_{i=1}^{M} p(r|\sigma_{1i}^2,\beta_{1i},\beta_{2i},\underline{X},\underline{Y}) \quad ;.$$

and

$$\hat{p}(R|X,Y) = \frac{1}{M}\sum_{i=1}^{M} p(R|\sigma_{1i}^2,\beta_{1i},\beta_{2i},\underline{X},\underline{Y}) \quad .$$

We choose vague second-stage priors for β_1 and β_2, letting $a_1=a_2=0$ and $b_1=b_2=1$. Convergence of the algorithm was obtained after 50 iterations and again M=80 by checking the stability of the quantiles of deviates.

Finally, we consider Normal approximation. Then take $\alpha_1=1$ and $\alpha_2=1$, and by empirical Bayes method (ML-II prior, Berger 1985), we can choose

$$\hat{\beta}_1 = \frac{2m}{\alpha_1\sum x_i^2}, \quad \hat{\beta}_2 = \frac{2n}{\alpha_2\sum y_i^2}.$$

For a normal approximation based on the posterior mode, the mean vector is a solution of $\frac{d}{d\sigma_i^2}\log p(\sigma_1^2,\sigma_2^2|data)=0$, and a measure of dispersion given by terms of Hessian matrix is

$$B = -\left[\frac{d^2 \log p(\sigma_1^2, \sigma_2^2 | data)}{d\sigma_i^2 d\sigma_j^2} \right]^{-1}$$

to be evaluated at the posterior mode. $(\sigma_1^2, \sigma_2^2) - (\widehat{\sigma_1^2}, \widehat{\sigma_2^2}) \sim N(0, B)$, where

$$\widehat{\sigma_1}^2 = \frac{t_1 \beta_1 + 2}{2(m + \alpha_1)\beta_1} = 2.0522406, \qquad \widehat{\sigma_2}^2 = \frac{t_2 \beta_2 + 2}{2(n + \alpha_2)\beta_2} = 5.5312395$$

and

$$B = \begin{vmatrix} 0.1568961 & -8.5 * 10^{-17} \\ -8.5 * 10^{-17} & 1.657783 \end{vmatrix}$$

By Delta Method $(f(\sigma_1^2, \sigma_2^2) = \sigma_2^2 / \sigma_1^2)$, $\sigma_2^2/\sigma_1^2 - \widehat{\sigma_2}^2 / \widehat{\sigma_1}^2 \sim N(0,\ 0.6642248)$.

Let $\lambda/(1+\lambda) = R$. Since

$$p(\lambda | \nu, \underline{X}, \underline{Y}) \propto \left(\frac{1}{\lambda} \right)^{n+\alpha_2+1} \exp\left[- \left(\frac{t_2 \beta_2 + 2}{2\beta_2} \right) \frac{1}{\lambda} \frac{1}{\upsilon} \right],$$

$$p(R | \nu, X, Y) \propto \left(\frac{1-R}{R} \right)^{n+\alpha_1+1} \exp\left\{ - \frac{t_2 \beta_2 + 2}{2\beta_2} \frac{1}{\nu} \frac{1-R}{R} \right\} \frac{1}{(1-R)^2}$$

and so, $\hat{p}(R | X, Y) \propto M^{-1} \sum_{i=1}^{M} p(R | \nu_i, X, Y)$.

Also by Delta method $(f(\sigma_1^2, \sigma_2^2) = \sigma_2^2 / (\sigma_1^2 + \sigma_2^2))$,

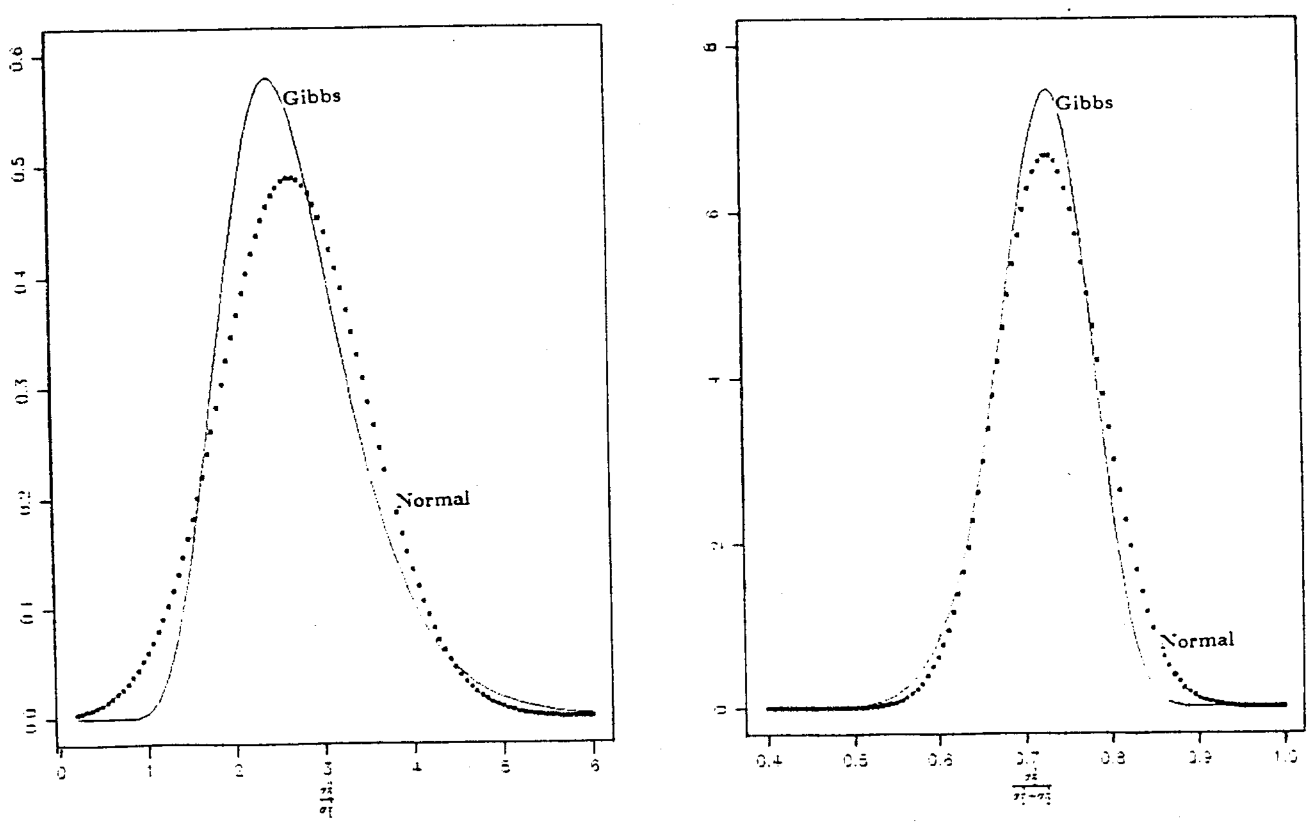
$$\frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} - \frac{\widehat{\sigma_2^2}}{\widehat{\sigma_1^2} + \widehat{\sigma_2^2}} \sim N(0, 0.7229)$$

and similarly

$$\frac{\sigma_1^2 + \sigma_2^2}{\sigma_2^2} - \frac{\widehat{\sigma_1^2} + \widehat{\sigma_2^2}}{\widehat{\sigma_2^2}} \sim N(0, 0.00644)$$

TABLE 2 : Approximate Moments

	Normal approximation	Gibbs Sampler	Exact
E(r)(std.)	2.6952(0.8013)	2.86378(0.4468)	2.5
E(R)(std.)	0.7294(0.8502)	0.721(0.0611)	0.714
$E(R^{-1})$(std.)	1.371027(0.0830)	1.39(0.1183)	1.4



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