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Admissible Estimation for Parameters in a Family of Non-regular Densities[1)]

Byung Hwee Kim[2)] and In Hong Chang[2)]

Abstract

Consider an estimation problem under squared error loss in a family of non-regular densities with both terminals of the support being decreasing functions of an unknown parameter. Using Karlin's(1958) technique, sufficient conditions are given for generalized Bayes estimators to be admissible for estimating an arbitrarily positive, monotone parametric function and then treat some examples which illustrate our results.

Keyword : Admissible Estimation, generalized Bayes, scale invariance, squared error loss.

1. Introduction

Let X be a random variable whose density is given by

$$f(x;\theta) = \begin{cases} r(x)c(\theta), & a(\theta) < x < b(\theta),\ \theta \in (\underline{\theta}, \overline{\theta}) \\ 0 & \text{otherwise} \end{cases} \tag{1.1}$$

with respect to Lebesgue measure where $(\underline{\theta}, \overline{\theta})$ is a nondegenerate interval in the real line which may be an infinite interval, $r(x)$ is a positive measurable function of x, $c^{-1}(\theta) = \int_{a(\theta)}^{b(\theta)} r(x)\,dx < \infty$ for all $\theta \in (\underline{\theta}, \overline{\theta})$, and both $a(\theta)$ and $b(\theta)$ are functions of θ such that $a(\theta) < b(\theta)$ for all $\theta \in (\underline{\theta}, \overline{\theta})$.

Karlin(1958) produced admissible estimators of $c^{-\alpha}(\theta)$, $\alpha > 0$ under squared error loss in cases when $a(\theta) = \underline{\theta}$ and $b(\theta) = \theta$, or $a(\theta) = \theta$ and $b(\theta) = \overline{\theta}$. Later, these results were

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1) This paper was supported by Hanyang University Research Fund, 1994.
2) Department of Mathematics, Hanyang University, Seoul 133-791, KOREA

slightly extended to $\alpha > -\frac{1}{2}$ by Sharma(1975) and Singh(1971), and were further extended by Sinha and Das Gupta(1984) and Kim(1994a). Recently, Kim(1994b) provided admissible estimators of an arbitrarily positive, monotone parametric function $h(\theta)$ under squared error loss in the case when both $a(\theta)$ and $b(\theta)$ are monotone increasing function of θ.

In this paper we deal with the case when both $a(\theta)$ and $b(\theta)$ are monotone decreasing functions of θ.

Let $X_1, X_2, \cdots, X_n$ be a random sample of size $n\ (\geq 2)$ from the density (1.1). Consider the problem of estimating an arbitrarily positive, monotone function $h(\theta)$ under squared error loss. Let Y_1 and Y_n be respectively the minimum and the maximum in the sample $X_1, X_2, \cdots, X_n$. Then the joint density of $X_1, X_2, \cdots, X_n$ is given by

$$f(x_1, \cdots, x_n; \theta) = c(\theta)\, u(y_1 - a(\theta))\, u(b(\theta) - y_n) \prod_{i=1}^{n} r(x_i),$$

where $u(y) = 1$ if $y \geq 0$, and $u(y) = 0$ otherwise. It follows from the Factorization Criterion that Y_1 and Y_n are a pair of (minimal) sufficient statistics of θ. Moreover, the strict convexity of the loss function guarantees from the viewpoint of risk that only nonrandomized estimators based on Y_1 and Y_n need be considered (see Berger (1985), p40-41).

Consider the (possibly improper) prior of the form

$$\pi_g(\theta) = \frac{|h'(\theta)|\, g[h(\theta)]}{c^n(\theta)} \tag{1.2}$$

for almost all $\theta \in (\underline{\theta}, \overline{\theta})$ where g is a nonnegative function defined on the range of h. The prior under consideration is assumed to be absolutely continuous with respect to Lebesgue measure with the density (1.2).

In Section 2 we provide, using Karlin's(1958) technique, sufficient conditions for admissibility of the (nonrandomized), generalized Bayes Estimator, say, $\delta_g(Y_1, Y_n)$, of $h(\theta)$ with respect to the prior (1.2). Finally, Section 3 contains some examples which illustrate our results.

2. Admissibility of generalized Bayes estimators

Let $X_1, X_2, \cdots, X_n$ be a random sample from the density (1.1) where both $a(\theta)$ and $b(\theta)$ are monotone decreasing functions of θ. Assume that there is a unique value $\eta\ (\geq \overline{\theta})$

of θ such $a(\theta) = b(\theta)$. Note that η may be $+\infty$. Also, we assume that the parameter space $(\underline{\theta}, \overline{\theta})$ is rich enough so that $b(\overline{\theta}) < a(\underline{\theta})$.

First, consider the problem of estimating a positive, monotone increasing function $h(\theta)$ of θ under squared error loss. Then, the generalized Bayes estimator δ_g of $h(\theta)$ with respect to the prior (1.2) is given by

$$\delta_g(Y_1, Y_n) = \frac{\int_{\max\{a^{-1}(Y_1), \underline{\theta}\}}^{\min\{b^{-1}(Y_n), \overline{\theta}\}} h(\theta)\, p(Y_1, Y_n; \theta)\, \pi_g(\theta)\, d\theta}{\int_{\max\{a^{-1}(Y_1), \underline{\theta}\}}^{\min\{b^{-1}(Y_n), \overline{\theta}\}} p(Y_1, Y_n; \theta)\, \pi_g(\theta)\, d\theta}$$

$$= \frac{\int_{\max\{a^{-1}(Y_1), \underline{\theta}\}}^{\min\{b^{-1}(Y_n), \overline{\theta}\}} h(\theta)\, h'(\theta)\ \ g[h(\theta)]\, d\theta}{\int_{\max\{a^{-1}(Y_1), \underline{\theta}\}}^{\min\{b^{-1}(Y_n), \overline{\theta}\}} h'(\theta)\ \ g[h(\theta)]\, d\theta}$$

$$= \frac{\int_{h(\max\{a^{-1}(Y_1), \underline{\theta}\})}^{h(\min\{b^{-1}(Y_n), \overline{\theta}\})} u\, g(u)\, du}{\int_{h(\max\{a^{-1}(Y_1), \underline{\theta}\})}^{h(\min\{b^{-1}(Y_n), \overline{\theta}\})} g(u)\, du}$$

$$= \frac{\int_{\max\{h(a^{-1}(Y_1)), h(\underline{\theta})\}}^{\min\{h(b^{-1}(Y_n)), h(\overline{\theta})\}} u\, g(u)\, du}{\int_{\max\{h(a^{-1}(Y_1)), h(\underline{\theta})\}}^{\min\{h(b^{-1}(Y_n)), h(\overline{\theta})\}} g(u)\, du} \tag{2.1}$$

under certain integrability conditions imposed on g for δ_g to be well-defined, where $p(y_1, y_n; \theta)$ is the joint density of Y_1 and Y_n given by

$$p(y_1, y_n; \theta) = n(n-1)c^n(\theta)\left[\int_{y_1}^{y_n} r(x)dx\right]^{n-2} r(y_1) r(y_n)$$

for $a(\theta) < y_1 \le y_n < b(\theta)$, $\underline{\theta} < \theta < \overline{\theta}$.

The following theorem provides sufficient conditions for admissibility of δ_g in (2.1).

Theorem 2.1 Let $g \ge 0$ defined on $(0, \infty)$ be such that

$$\int_a^b g(u)\, du < \infty \quad \text{and} \quad \int_a^b u\, g(u)\, du < \infty \tag{2.2}$$

for every $0 < a < b < \infty$. Then, the generalized Bayes estimator δ_g of $h(\theta)$ in (2.1) is admissible if

$$\int_b^{\overline{\theta}} \frac{h'(\theta)\, g[h(\theta)]\, c^n(\theta)}{\left[\int_{\max\{h(a^{-1}(b(\theta))),h(\underline{\theta})\}}^{\min\{h(b^{-1}(a(\theta))),h(\overline{\theta})\}} u\, g(u)\, du\right]^2} d\theta = \infty$$
$$= \int_{\underline{\theta}}^{a} \frac{h'(\theta)\, g[h(\theta)]\, c^n(\theta)}{\left[\int_{\max\{h(a^{-1}(b(\theta))),h(\underline{\theta})\}}^{\min\{h(b^{-1}(a(\theta))),h(\overline{\theta})\}} u\, g(u)\, du\right]^2} d\theta \tag{2.3}$$

for $\underline{\theta} < a, b < \overline{\theta}$.

Proof. Suppose that δ_g is not admissible. Then there exists another estimator δ' such that

$$E_\theta[\delta'(Y_1, Y_n) - h(\theta)]^2 \le E_\theta[\delta_g(Y_1, Y_n) - h(\theta)]^2 \tag{2.4}$$

for all $\theta \in (\underline{\theta}, \overline{\theta})$ with strict inequality for at least one θ. In (2.4) the expectation operates through the joint density $p(y_1, y_n; \theta)$ of Y_1 and Y_n. Now, (2.4) implies

$$\iint_{\{a(\theta) < y_1 \le y_n < b(\theta)\}} (\delta_g - \delta')^2 c^n(\theta) \left[\int_{y_1}^{y_n} r(x)\, dx\right]^{n-2} r(y_1)\, r(y_n)\, dy_1\, dy_n$$
$$\le 2 \iint_{\{a(\theta) < y_1 \le y_n < b(\theta)\}} [\delta_g - h(\theta)]^2 (\delta_g - \delta')\, c^n(\theta) \left[\int_{y_1}^{y_n} r(x)\, dx\right]^{n-2} r(y_1)\, r(y_n)\, dy_1\, dy_n \tag{2.5}$$

for all $\underline{\theta} < \theta < \overline{\theta}$, where $\delta_g \equiv \delta_g(Y_1, Y_n)$ and $\delta' \equiv \delta'(Y_1, Y_n)$. Let θ_1 and θ_2 be such that $\underline{\theta} < \theta_1 < \theta_2 < \overline{\theta}$ and $b(\theta_2) < a(\theta_1)$. Here, without loss of generality, we can assume $b(\theta_2) < a(\theta_1)$ since if $b(\theta_2) \ge a(\theta_1)$ and $\theta_1 \to \underline{\theta}$, $\theta_2 \to \overline{\theta}$, then $b(\overline{\theta}) \ge a(\underline{\theta})$ which contradicts our assumption $b(\overline{\theta}) < a(\underline{\theta})$ in the beginning of this section. Now, integrating both sides of (2.5) over (θ_1, θ_2) with respect to $\pi_g(\theta)$ in (1.2) and then applying Fubini's theorem yield, after some algebra,

$$\int_{\theta_1}^{\theta_2} \iint_{\{a(\theta) < y_1 \le y_n < b(\theta)\}} (\delta_g - \delta')^2 c^n(\theta) \left[\int_{y_1}^{y_n} r(x)\, dx\right]^{n-2} r(y_1)\, r(y_n)\, \pi_g(\theta)\, dy_1\, dy_n\, d\theta$$
$$\le 2 \int_{\theta_1}^{\theta_2} \iint_{\{a(\theta) < y_1 \le y_n < b(\theta)\}} [\delta_g - h(\theta)]^2 (\delta_g - \delta')\, c^n(\theta) \left[\int_{y_1}^{y_n} r(x)\, dx\right]^{n-2} r(y_1)\, r(y_n)\, \pi_g(\theta)\, dy_1\, dy_n\, d\theta$$

$$= 2\left\{ \iint_{\{a(\theta_1) < y_1 \le y_n < b(\theta_1)\}} \left[\int_{\theta_1}^{b^{-1}(y_n)} [\delta_g - h(\theta)]\, c^n(\theta)\pi_g(\theta)\,d\theta\right]\left[\int_{y_1}^{y_n} r(x)\,dx\right]^{n-2} (\delta_g - \delta')\, r(y_1)\, r(y_n)\, dy_1\, dy_n \right.$$

$$\left. + \iint_{\{a(\theta_2) < y_1 \le y_n < b(\theta_2)\}} \left[\int_{\theta_1}^{b^{-1}(y_n)} [\delta_g - h(\theta)]\, c^n(\theta)\pi_g(\theta)\,d\theta\right]\left[\int_{y_1}^{y_n} r(x)\,dx\right]^{n-2} (\delta_g - \delta')\, r(y_1)\, r(y_n)\, dy_1\, dy_n \right\} \tag{2.6}$$

By using the condition (2.2) and the fact that a Bayes estimator of a monotone function is monotone, it can be shown after simple algebra that the absolute values of inner integrals in the right-hand side of (2.6) have, respectively, the following upper bounds :

$$\left| \int_{\theta_1}^{b^{-1}(y_n)} [\delta_g - h(\theta)]\, c^n(\theta)\pi_g(\theta)\,d\theta \right| < A(\theta_1), \quad a(\theta_1) < y_1 \le y_n < b(\theta_1) \tag{2.7}$$

and

$$\left| \int_{a^{-1}(y_1)}^{\theta_2} [\delta_g - h(\theta)]\, c^n(\theta)\pi_g(\theta)\,d\theta \right| < A(\theta_2), \quad a(\theta_2) < y_1 \le y_n < b(\theta_2). \tag{2.8}$$

where $A(\theta) = \int_{\max\{h(a^{-1}(b(\theta_1))),\, h(\underline{\theta})\}}^{\min\{h(b^{-1}(a(\theta))),\, h(\overline{\theta})\}} u\, g(u)\, du$.

Define

$$B(\theta) = \iint_{\{a(\theta) < y_1 \le y_n < b(\theta)\}} (\delta_g - \delta')^2 c^n(\theta) \left[\int_{y_1}^{y_n} r(x)\,dx\right]^{n-2} r(y_1)\, r(y_n)\, dy_1\, dy_n$$

Then, from (2.6), (2.7), and (2.8), we have, using Cauchy-Schwartz Inequality

$$\int_{\theta_1}^{\theta_2} B(\theta)\pi_g(\theta)\,d\theta$$

$$\le 2\left\{ B(\theta_1) \iint_{\{a(\theta_1) < y_1 \le y_n < b(\theta_1)\}} \left[\int_{y_1}^{y_n} r(x)\,dx\right]^{n-2} |\delta_g - \delta'|\, r(y_1)\, r(y_n)\, dy_1\, dy_n \right.$$

$$\left. + B(\theta_2) \iint_{\{a(\theta_2) < y_1 \le y_n < b(\theta_2)\}} \left[\int_{y_1}^{y_n} r(x)\,dx\right]^{n-2} |\delta_g - \delta'|\, r(y_1)\, r(y_n)\, dy_1\, dy_n \right\}$$

$$\leq 2B(\theta_1)\left\{\iint_{\{a(\theta_1)<y_1\leq y_n<b(\theta_1)\}} |\delta_g-\delta'|^2\left[\int_{y_1}^{y_n} r(x)\,dx\right]^{n-2} r(y_1)r(y_n)\,dy_1\,dy_n\right\}^{\frac{1}{2}}$$

$$\cdot\left\{\iint_{\{a(\theta_1)<y_1\leq y_n<b(\theta_1)\}} \left[\int_{y_1}^{y_n} r(x)\,dx\right]^{n-2} r(y_1)r(y_n)\,dy_1\,dy_n\right\}^{\frac{1}{2}}$$

$$+\,2B(\theta_2)\left\{\iint_{\{a(\theta_2)<y_1\leq y_n<b(\theta_2)\}} |\delta_g-\delta'|^2\left[\int_{y_1}^{y_n} r(x)\,dx\right]^{n-2} r(y_1)r(y_n)\,dy_1\,dy_n\right\}^{\frac{1}{2}}$$

$$\cdot\left\{\iint_{\{a(\theta_2)<y_1\leq y_n<b(\theta_2)\}} \left[\int_{y_1}^{y_n} r(x)\,dx\right]^{n-2} r(y_1)r(y_n)\,dy_1\,dy_n\right\}^{\frac{1}{2}}$$

$$\leq 2[n(n-1)]^{-\frac{1}{2}}\left\{A(\theta_1)B^{\frac{1}{2}}(\theta_1)C^{-n}(\theta_1)+A(\theta_2)B^{\frac{1}{2}}(\theta_2)C^{-n}(\theta_2)\right\}$$

$$=2[n(n-1)]^{-\frac{1}{2}}\left\{B^{\frac{1}{2}}(\theta_1)\pi_g^{\frac{1}{2}}(\theta_1)\frac{A(\theta_1)}{[h'(\theta_1)g\{h(\theta_1)\}c^n(\theta_1)]^{\frac{1}{2}}}\right.$$

$$\left.+\,B^{\frac{1}{2}}(\theta_2)\pi_g^{\frac{1}{2}}(\theta_2)\frac{A(\theta_2)}{[h'(\theta_2)g\{h(\theta_2)\}c^n(\theta_2)]^{\frac{1}{2}}}\right\}. \tag{2.9}$$

Now, the analysis proceeds by examining the following two cases :

Case 1 : $\lim_{\theta_1\to\underline{\theta}} B^{\frac{1}{2}}(\theta_1)\pi_g^{\frac{1}{2}}(\theta_1)\dfrac{A(\theta_1)}{[h'(\theta_1)g\{h(\theta_1)\}c^n(\theta_1)]^{\frac{1}{2}}} > 0.$

Case 2 : $\lim_{\theta_1\to\underline{\theta}} B^{\frac{1}{2}}(\theta_1)\pi_g^{\frac{1}{2}}(\theta_1)\dfrac{A(\theta_1)}{[h'(\theta_1)g\{h(\theta_1)\}c^n(\theta_1)]^{\frac{1}{2}}} = 0.$

It could be shown that Case 1 cannot occur while Case 2 entails $B(\theta)=0$ a.e. which in turn implies that $\delta'(x)=\delta_g(x)$ a.e.. The details omitted since they are similar to those of Karlin(1958), Ralescu and Ralescu(1981), and Puri and Ralescu(1988).

Next, consider the problem of estimating a positive monotone decreasing function $h(\theta)$ of θ under squared error loss. Then the generalized Bayes estimator of $h(\theta)$ with respect to the prior (1.2) is given by

$$\delta_g(Y_1, Y_n) = \frac{\int_{\max\{h(b^{-1}(Y_n)),h(\bar{\theta})\}}^{\min\{h(a^{-1}(Y_1)),h(\underline{\theta})\}} u\, g(u)\, du}{\int_{\max\{h(b^{-1}(Y_n)),h(\bar{\theta})\}}^{\min\{h(a^{-1}(Y_1)),h(\underline{\theta})\}} g(u)\, du} \tag{2.10}$$

assuming some integrability conditions imposed on g for δ_g to be well-defined.

The following theorem gives sufficient conditions for admissibility of δ_g in (2.10). The proof is omitted because it is exactly the same as that of Theorem 2.1 with obvious simple modifications.

Theorem 2.2 Let $g \ge 0$ defined on $(0, \infty)$ be such that

$$\int_a^b g(u)\, du < \infty \text{ and } \int_a^b u\, g(u)\, du < \infty \tag{2.11}$$

for every $0 < a, b < \infty$. Then the generalized Bayes estimator δ_g of $h(\theta)$ in (2.10) is admissible if

$$\int_b^{\bar{\theta}} \frac{|h'(\theta)|\, g[h(\theta)]\, c^n(\theta)}{\left[\int_{\max\{h(b^{-1}(a(\theta))),h(\bar{\theta})\}}^{\min\{h(a^{-1}(b(\theta))),h(\underline{\theta})\}} u\, g(u)\, du\right]^2} d\theta = \infty$$
$$= \int_{\underline{\theta}}^{a} \frac{|h'(\theta)|\, g[h(\theta)]\, c^n(\theta)}{\left[\int_{\max\{h(b^{-1}(a(\theta))),h(\bar{\theta})\}}^{\min\{h(a^{-1}(b(\theta))),h(\underline{\theta})\}} u\, g(u)\, du\right]^2} d\theta \tag{2.12}$$

for $\underline{\theta} < a, b < \bar{\theta}$.

Remark 2.1 The conditions (2.2) and (2.11) of above theorems are necessary for the existence of generalized Bayes estimators in (2.1) and (2.10), respectively. Also, the conditions (2.3) and (2.12) involve the divergence of a certain integral in the neighborhood of both endpoints $\underline{\theta}$ and $\bar{\theta}$ of the parameter space which guarantees the admissibility of generalized Bayes estimators in (2.1) and (2.10), respectively.

3. Examples

In order to apply Theorem 2.1 and Theorem 2.2 to some examples, we need to choose appropriate nonnegative functions g which satisfy the condition (2.2) of Theorem 2.1 and the condition (2.11) of Theorem 2.2. In the following examples, we consider $g(u) = u^{\alpha-1}$, $u > 0$,

$-\infty < \alpha < \infty$ in Theorem 2.1 and $g(u) = u^{-\alpha-2}$, $u > 0$, $-\infty < \alpha < \infty$ in Theorem 2.2. Note that these g's satisfy conditions (2.2) and (2.11) of Theorems 2.1 and 2.2, respectively.

Example 3.1 Let $X_1, X_2, \cdots, X_n$ be a random sample from the uniform distribution with a density

$$f(x;\theta) = \begin{cases} 1 & \text{for } -\theta - \frac{1}{2} < x < -\theta + \frac{1}{2}, \ -\infty < \theta < \infty \\ 0 & \text{otherwise}. \end{cases}$$

In this case, $r(x) \equiv 1$, $c(\theta) \equiv 1$, $a(\theta) = -\theta - \frac{1}{2}$, $b(\theta) = -\theta + \frac{1}{2}$ and $\underline{\theta} = -\infty$ and $\overline{\theta} = \infty$. First, note that $b(\overline{\theta}) < a(\underline{\theta})$. Suppose it is desired to estimate $h(\theta) = e^{\theta}$ under squared error loss which is monotone increasing. With $g(u) = u^{\alpha-1}$, $u > 0$, $-\infty < \alpha < \infty$, the prior (1.2) becomes $\pi_\alpha(\theta) = \pi_g(\theta) = e^{\alpha\theta}$, $-\infty < \theta < \infty$, and the generalized Bayes estimator (2.1) turns out to be

$$\delta_\alpha(Y_1, Y_n) = \delta_g(Y_1, Y_n) = \begin{cases} \dfrac{\alpha}{\alpha+1}\left[\dfrac{e^{(-Y_n+\frac{1}{2})(\alpha+1)} - e^{(-Y_1-\frac{1}{2})(\alpha+1)}}{e^{(-Y_n+\frac{1}{2})\alpha} - e^{(-Y_1-\frac{1}{2})\alpha}}\right], & \alpha \neq -1, 0 \\[2ex] \dfrac{Y_1 - Y_n + 1}{e^{Y_1+\frac{1}{2}} - e^{Y_n-\frac{1}{2}}}, & \alpha = -1 \\[2ex] \dfrac{e^{-Y_n+\frac{1}{2}} - e^{-Y_1-\frac{1}{2}}}{Y_1 - Y_n + 1}, & \alpha = 0. \end{cases}$$

Now, it can be easily shown that the condition (2.3) of Theorem 2.1 is satisfied if $\alpha = -2$. Hence

$$\delta_{-2}(Y_1, Y_n) = 2\left[\frac{e^{Y_n-\frac{1}{2}} - e^{Y_1+\frac{1}{2}}}{e^{2Y_n-1} - e^{2Y_1+1}}\right]$$

is admissible for estimating e^{θ}.

Example 3.2 For the model governed by the density of Example 3.1 we want to estimate $h(\theta) = e^{-\theta}$ under squared error loss which is monotone decreasing. With $g(u) = u^{-\alpha-2}$, $u > 0$, $-\infty < \alpha < \infty$, the prior (1.2) becomes $\pi_\alpha(\theta) = \pi_g(\theta) = e^{(\alpha+2)\theta}$, $-\infty < \theta < \infty$, and the generalized Bayes estimator (2.10) turns out to be

$$\delta_\alpha(Y_1,Y_n)=\delta_g(Y_1,Y_n)=\begin{cases}\dfrac{\alpha+1}{\alpha}\left[\dfrac{e^{-\alpha(Y_1+\frac{1}{2})}-e^{-\alpha(Y_n-\frac{1}{2})}}{e^{(-\alpha-1)(Y_1+\frac{1}{2})}-e^{(-\alpha-1)(Y_n-\frac{1}{2})}}\right], & \alpha\neq-1,0\\[2ex] \dfrac{e^{Y_1+\frac{1}{2}}-e^{Y_n-\frac{1}{2}}}{Y_1-Y_n+1}, & \alpha=-1\\[2ex] \dfrac{Y_1-Y_n+1}{e^{-Y_n+\frac{1}{2}}-e^{-Y_1-\frac{1}{2}}}, & \alpha=0\,.\end{cases}$$

A simple calculation shows that the condition (2.12) of Theorem 2.2 is satisfied if $\alpha=1$. Hence $\delta_1(Y_1,Y_n)=2\left[\dfrac{e^{-Y_1-\frac{1}{2}}-e^{-Y_n+\frac{1}{2}}}{e^{-2Y_1-1}-e^{-2Y_n+1}}\right]$ is admissible for estimating $e^{-\theta}$. Note that $\delta_1(Y_1,Y_n)$ is the best scale invariant estimator of $e^{-\theta}$ under the loss $L(\theta,d)=e^{2\theta}(e^{-\theta}-d)^2$. Therefore, $\delta_1(Y_1,Y_n)$ is also admissible for estimating $e^{-\theta}$ under the loss $L(\theta,d)$ since $\delta_1(Y_1,Y_n)$ is admissible for estimating $e^{-\theta}$ under squared error loss and the weighting factor $e^{2\theta}$ does not affect admissibility of an estimator.

Example 3.3 Let $X_1, X_2, \cdots, X_n$ be a random sample from the distribution with a density

$$f(x;\theta)=\begin{cases}\dfrac{\theta^2}{\theta-1} & \text{for } \dfrac{1}{\theta^2}<x<\dfrac{1}{\theta},\ 1<\theta<\infty\\ 0 & \text{otherwise}\,.\end{cases}$$

Here, $r(x)\equiv1$, $c(\theta)\equiv\dfrac{\theta^2}{\theta-1}$, $a(\theta)=\dfrac{1}{\theta^2}$, $b(\theta)=\dfrac{1}{\theta}$, $\underline{\theta}=1$, and $\overline{\theta}=\infty$. Note that $b(\overline{\theta})=0<1=a(\underline{\theta})$. We want to estimate $h(\theta)=\theta^k$, $k>\theta$, under squared error loss which is monotone increasing. With $g(u)=u^{\alpha-1}$, $u>0$, $-\infty<\alpha<\infty$, the prior (1.2) becomes $\pi_\alpha(\theta)=\pi_g(\theta)=k\theta^{k\alpha-2n-1}(\theta-1)^n$, $1<\theta<\infty$, and the generalized Bayes estimator (2.1) is given by

$$\delta_\alpha(Y_1,Y_n)=\delta_g(Y_1,Y_n)=\begin{cases}\dfrac{\alpha}{\alpha+1}\left[\dfrac{Y_n^{-k(\alpha+1)}-Y_1^{-\frac{k}{2}(\alpha+1)}}{Y_n^{-k\alpha}-Y_1^{-\frac{k}{2}\alpha}}\right], & \alpha\neq-1,0\\[2ex] \dfrac{\frac{k}{2}\ln Y_1-k\ln Y_n}{Y_1^{\frac{k}{2}}-Y_n^{k}}, & \alpha=-1\\[2ex] \dfrac{Y_n^{-k}-Y_1^{-\frac{k}{2}}}{\frac{k}{2}\ln Y_1-k\ln Y_n}, & \alpha=0\,.\end{cases}$$

It can be shown after some calculations that the condition (2.3) of Theorem 2.1 is satisfied if $\alpha\le\frac{n-4k}{3k}$. Hence, $\delta_\alpha(Y_1,Y_n)$ is admissible for estimating θ^k, $k>0$, if $\alpha\le\frac{n-4k}{3k}$.

Example 3.4 Let $X_1,X_2,\cdots,X_n$ be as in Example 3.3. But , it is desired to estimate $h(\theta)=\theta^k$, $k<0$, under squared error loss which is monotone decreasing. With $g(u)=u^{-\alpha-2}$, $u>0$, $-\infty<\alpha<\infty$, the prior (2.1) becomes $\pi_\alpha(\theta)=\pi_g(\theta)$ $=|k|\theta^{-k\alpha-k-1-2n}(\theta-1)^n$, $1<\theta<\infty$, and the generalized Bayes estimator (2.10) is given by

$$\delta_\alpha(Y_1,Y_n)=\delta_g(Y_1,Y_n)=\begin{cases}\dfrac{\alpha+1}{\alpha}\left[\dfrac{Y_1^{\frac{k\alpha}{2}}-Y_n^{k\alpha}}{Y_1^{\frac{k}{2}(\alpha+1)}-Y_n^{k(\alpha+1)}}\right], & \alpha\neq-1,0\\[2ex] \dfrac{Y_1^{-\frac{k}{2}}-Y_n^{-k}}{k\ln Y_n-\frac{k}{2}\ln Y_1}, & \alpha=-1\\[2ex] \dfrac{k\ln Y_n-\frac{k}{2}\ln Y_1}{Y_n^{k}-Y_1^{\frac{k}{2}}}, & \alpha=0\,.\end{cases}$$

After lengthy calculations it can be shown that the condition (2.12) of Theorem 2.2 is satisfied if $\alpha\le\frac{k-n}{3k}$. Hence, $\delta_\alpha(Y_1,Y_n)$ is admissible for estimating θ^k, $k<0$, if $\alpha\le\frac{k-n}{3k}$.

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