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Smoothing Parameter Selection in Nonparametric Spectral Density Estimation[1)]

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Abstract

In this paper we consider kernel type estimator of the spectral density at a point in the analysis of stationary time series data. The kernel entails choice of smoothing parameter called bandwidth. A data-based bandwidth choice is proposed, and it is obtained by solving an equation similar to Sheather(1986) which relates to the probability density estimation. A Monte Carlo study is done. It reveals that the spectral density estimates using the data-based bandwidths show comparatively good performance.

1. Introduction

Investigating the structure of underlying spectral density is crucial in the analysis of stationary time series. See Brillinger(1981) and Priestly(1981) for interpretation of spectral density and its relation to the autocorrelation function. Fitting an autoregression, with an appropriate choice of order for the model, is most popular as a method of estimating spectral density. But, this method is based on fitting a parametric model and so is defective when the fitted model is inappropriate. Alternatively, a kernel estimate may be used. It is obtained by smoothing periodogram with a spectral window called kernel. Like all other methods of estimating spectral density nonparametrically, this method entails choice of smoothing parameter, called bandwidth. An unsuitable choice can produce poor estimates of spectral densities. An arbitrary choice is almost the same as an arbitrary choice of order for some approximating parametric model.

However, past years have seen a few literature on the bandwidth selection problem in kernel spectrum estimation. Robinson(1991) discussed for asymptotic theory of some kernel spectrum estimates in the presence of data-dependent bandwidths, Hurvich(1985), Beltrão and Bloomfield(1987) introduced a bandwidth selection method based on a cross-validation criterion. Park, Cho and Kang(1994) considered a plug-in bandwidth which is obtained by plugging some suitable estimates into the unknown parts, integrated squared density

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derivatives, of the theoretically optimum choice. The theoretical performance of the estimate of integrated squared density derivatives was investigated by Lee, Cho, Kim and Park(1995). All these are related to global bandwidth selection.

In this paper, we propose a data-driven bandwidth selection method which is suitable for estimating the spectral density at a point. It has similar flavor to the data-based method of Sheather(1986) in probability density function setting. We investigated the empirical performance of the proposed method by a simulation, where we compared two spectral density estimates, one with the data-based bandwidth selector and the other with a theoretically optimal bandwidth (the latter is not practicable and is considered only as a benchmark). The result is that the data-based method performs quite well and does not deteriorate, in comparison with the theoretically optimal bandwidth, the mean squared error of the resulting estimator.

In the next section, we describe the data-based bandwidth. And Section 3 contains the results of a simulation study for small sample properties of the estimate.

2. The data-based bandwidth

Let $X(t)$, $t=0, \pm 1, \cdots$, be a stationary time series with zero mean and the spectral density

$$f(\lambda) = (2\pi)^{-1} \sum_{s=-\infty}^{\infty} \exp(-i\lambda s) E\{X(0)X(s)\}, \quad -\infty < \lambda < \infty .$$

A kernel spectrum estimate is given by

$$\hat{f}_h(\lambda) = K_h * I_n(\lambda), \quad -\infty < \lambda < \infty . \tag{2.1}$$

Here and below $I_n(\lambda)$ is the periodogram of the data, defined by

$$I_n(\lambda) = (2\pi n)^{-1} \left| \sum_{s=0}^{n-1} \exp(-i\lambda s) X(s) \right|^2 , \quad -\infty < \lambda < \infty,$$

$K_h(\cdot) = K(\cdot/h)/h$, K is called kernel, $h = h_n$ is called the bandwidth, and $*$ denotes convolution.

An appealing approach to the problem of selecting h is to consider an error criterion such as Mean Squared Error,

$$\begin{aligned} MSE(h) &= E\{\hat{f}_h(\lambda) - f(\lambda)\}^2 \\ &= Bias^2\{\hat{f}_h(\lambda)\} + Var\{\hat{f}_h(\lambda)\}. \end{aligned}$$

The two terms on the right hand side of the above equation admit the asymptotic representations

$$Bias^2\{\hat{f}_h(\lambda)\} = \frac{1}{4}h^4\{f''(\lambda)\}^2\mu_2^2(K) + o(h^4)$$

and

$$Var\{\hat{f}_h(\lambda)\} = \begin{cases} 2\pi n^{-1}h^{-1}f^2(\lambda)R(K) + o(n^{-1}h^{-1}), & \lambda \neq 0, \pm\pi \\ 4\pi n^{-1}h^{-1}f^2(\lambda)R(K) + o(n^{-1}h^{-1}), & \lambda = 0, \pm\pi \end{cases}$$

where $R(K) = \int K^2(x)dx$, $\mu_j(K) = \int x^jK(x)dx.$

Then the value of h which minimizes the asymptotic mean squared error of $\hat{f}_h(\lambda)$ is given by

$$h_{opt}(\lambda) = \alpha(K)\beta(f(\lambda), f''(\lambda)) \cdot n^{-1/5}, \tag{2.2}$$

$$\alpha(K) = \begin{cases} [8\pi R(K)\mu_2^{-2}(K)]^{1/5}, & \lambda \neq 0, \pm\pi \\ [16\pi R(K)\mu_2^{-2}(K)]^{1/5}, & \lambda = 0, \pm\pi \end{cases}$$

and

$$\beta(f(\lambda), f''(\lambda)) = (f(\lambda)/f''(\lambda))^{2/5}.$$

In practice, however, the optimal bandwidth $h_{opt}(\lambda)$ can not be realized since it involves the unknown $f(\lambda)$ and $f''(\lambda)$. Thus, $f(\lambda)$ and $f''(\lambda)$ should be replaced by their corresponding estimators, $\hat{f}(\lambda)$ and $\hat{f}''(\lambda)$, and so (2.2) would become

$$\hat{h}_{opt}(\lambda) = \alpha(K)\beta(\hat{f}(\lambda), \hat{f}''(\lambda)) \cdot n^{-1/5}. \tag{2.3}$$

If the second derivative of K, K'' exists, then it is possible to estimate $f''(\lambda)$ by

$$\hat{f}_a''(\lambda) = a^{-2}(K'')_a * I_n(\lambda) \tag{2.4}$$

where a is a bandwidth for estimating $f''(\lambda)$.

The next problem is how to choose the bandwidth a. As in the case of $\hat{f}_h(\lambda)$, the same type of asymptotic representations are valid here. They are

$$Bias^2\{ \hat{f}_a''(\lambda)\} = \frac{1}{4} a^4 \{ f^{(iv)}(\lambda)\}^2 \mu_2^2(K) + o(a^4)$$

and

$$Var(\hat{f}_a''(\lambda)) = \begin{cases} 2\pi n^{-1} a^{-5} f^2(\lambda) R(K'') + o(n^{-1}a^{-5}), & \lambda \neq 0, \pm\pi \\ 4\pi n^{-1} a^{-5} f^2(\lambda) R(K'') + o(n^{-1}a^{-5}), & \lambda = 0, \pm\pi . \end{cases}$$

Hence an asymptotically optimal choice of a is given by

$$a_{opt}(\lambda) = \delta(K)\gamma(f(\lambda), f^{(iv)}(\lambda)) n^{-1/9}, \tag{2.5}$$

where

$$\delta(K) = \begin{cases} [8\pi R(K'')\mu_2^{-2}(K)]^{1/9}, & \lambda \neq 0, \pm\pi \\ [16\pi R(K'')\mu_2^{-2}(K)]^{1/9}, & \lambda = 0, \pm\pi \end{cases}$$

and

$$\gamma(f(\lambda), f^{(iv)}(\lambda)) = (f(\lambda)/f^{(iv)}(\lambda))^{2/9} .$$

Comparing (2.2) with (2.5) and defining

$$C(K,f) = \{R(K'')/R(K)\}^{1/9} \{f''(\lambda)/f^{(iv)}(\lambda)\}^{2/9},$$

we find

$$a_{opt}(\lambda) = h_{opt}(\lambda)^{5/9} C(K,f) .$$

In this case, the asymptotically optimal bandwidth again require to estimate unknown quantities $f''(\lambda)$, $f^{(iv)}(\lambda)$. Since the dependency of $a_{opt}(\lambda)$ on f at this stage seems to be less crucial than the other stage, one may use some reference spectral density instead of the unknowns. For example, one may replace f by AR(1) or MA(1) spectral density with the parameter estimated by maximum likelihood method. Thus if we let $\hat{h}_{opt}(\lambda)$ be the estimator of h in (2.1) and f_{ref} be a reference density, then

$$\hat{a}_{opt}(\lambda) = \hat{h}_{opt}(\lambda)^{5/9} \cdot C(K, f_{ref}) . \tag{2.6}$$

Define

$$\hat{a}(h) = \hat{a}(h,\lambda) = h^{5/9} C(K, f_{ref}) .$$

Then (2.3) and (2.6) lead us to solve

$$0=\alpha(K)\beta(\hat{f}_h(\lambda),\hat{f}''_{\hat{a}(h)}(\lambda))\cdot n^{-1/5}-h \tag{2.7}$$

to find $\hat{h}_{opt}(\lambda)$ using numerical methods. Detailed root finding algorithm is described in appendix.

For practical use, a discretized version of (2.1) may be used for fast computation. Let $\delta=h/M$ and $B_j=[(j-1/2)\delta,(j+1/2)\delta)$ where M is a positive integer which determines the amount of discretization errors. Let $r(x)$ denote the 'rounded point of x/δ', defined by $r(x)=j$ if and only if $x\in B_j$. Then the estimator can be approximated by

$$h^{-1}\int K\left(\frac{r(\lambda)-r(x)}{M}\right)I_n(r(x)\delta)dx .$$

If we take a nonnegative kernel K with bounded support $[-1,1]$, it is further approximated by

$$\hat{f}_h(\lambda)=\frac{1}{M}\sum_{l=1-M}^{M-1}K(\frac{l}{M})I_n((r(\lambda)+l)\delta) . \tag{2.8}$$

Similarly, (2.4) can be approximated by

$$\hat{f}_a''(\lambda)=a^{-2}\frac{1}{M}\sum_{l=1-M}^{M-1}K''(\frac{l}{M})I_n((r(\lambda)+l)\delta) , \tag{2.9}$$

where δ is defined by replacing h with a. Thus, (2.7) can be replaced by

$$0=\alpha(K)\beta(\hat{f}_h(\lambda),\hat{f}''_{\hat{a}(h)}(\lambda))\cdot n^{-1/5} - h . \tag{2.10}$$

3. A simulation results

A Monte Carlo study was carried out to evaluate the finite-sample performance of the data-based bandwidth when estimating f at a point λ using the discritized versions (2.8) and (2.9). The stationary time series models chosen for this study were

$$\text{MA}(1) \quad : X(t)=\varepsilon(t)-\beta\varepsilon(t-1)$$

$$\text{AR}(1) \quad : X(t)=\phi X(t-1)+\varepsilon(t) .$$

The values β and ϕ selected were 0.3 and 0.6. The distribution of $\varepsilon(t)$ was taken the standard normal, and $M=100$ was used. And we used the quartic kernel,

$K(x) = (15/16)(1-x^2)^2 I_{[-1,1]}(x)$. The three points where we estimate the spectral densities are 0, 1.5 and π (center, shoulder, and boundary). Figure 1 shows the spectral densities which we considered here. We found a root of (2.10) in the range of bandwidths $[h_{opt}(\lambda)/3, \; 3h_{opt}(\lambda)]$. For the definiteness, we let $\hat{h}_{opt}(\lambda) = 3h_{opt}(\lambda)$ when (2.10) has no root.

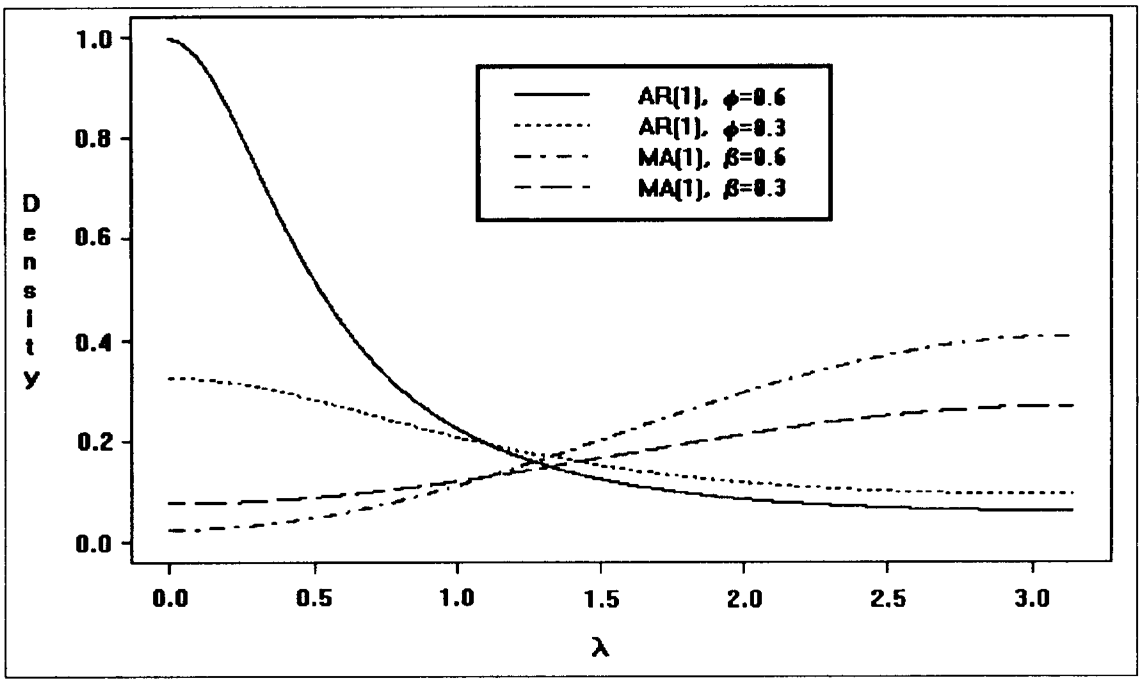


Figure 1. Spectral densities of AR(1) and MA(1) processes with range $[0, \pi]$

In this study we compare two estimates $\hat{f}_0(\lambda)$ and $\hat{f}_1(\lambda)$, where $\hat{f}_0(\lambda)$ is obtained by using the asymptotically optimal bandwidth $h_{opt}(\lambda)$, whereas $\hat{f}_1(\lambda)$ uses data-based bandwidth $\hat{h}_{opt}(\lambda)$. Tables 1 and 2 contain the Monte Carlo estimates of the bias and the root mean squared error of the two estimates based on 500 pseudo time series data of size 100 and 400 where ϕ and β are 0.6. The reference density used here was AR(1) spectral density with the parameter estimated by maximum likelihood method. The Monte Carlo estimate of the bias (Bias) and that of the root mean squared error (RMSE) are given by

$$\text{Bias} = \bar{f}(\lambda) - f(\lambda)$$

$$\text{RMSE} = \sqrt{\sum_{i=1}^{500} (\hat{f}_{(i)}(\lambda) - f(\lambda))^2 / 500} \;,$$

where $\bar{f}(\lambda) = \sum_{i=1}^{500} \hat{f}_{(i)}(\lambda)/500$ and $\hat{f}_{(i)}(\lambda)$ is the estimate computed from the i-th pseudo data set. For taking into account the Monte Carlo variability, we computed the standard errors of the Monte Carlo estimates of the bias and the root mean squared error. They are defined by

$$\text{SE(Bias)} = \hat{\sigma}(\text{ Bias})/\sqrt{500}$$

and

$$\text{SE(RMSE)} = (\sqrt{\text{RMSE}^2 + \hat{\sigma}(\text{ RMSE})/\sqrt{500}} - \sqrt{\text{RMSE}^2 - \hat{\sigma}(\text{ RMSE})/\sqrt{500}}\;)/2 \;,$$

where

$$\hat{\sigma}^2(\text{ Bias}) = \sum_{i=1}^{500} (\hat{f}_{(i)}(\lambda) - \bar{f}(\lambda))^2 \Big/ 500$$

$$\hat{\sigma}^2(\text{ RMSE}) = \sum_{i=1}^{500} ((\hat{f}_{(i)}(\lambda) - f(\lambda))^2 - \text{RMSE}^2)^2 \Big/ 500 \;.$$

Comparing $\hat{f}_1(\lambda)$ with $\hat{f}_0(\lambda)$, $\hat{f}_1(\lambda)$ shows comparable performance with $\hat{f}_0(\lambda)$ in almost all cases. Since $\hat{f}_0(\lambda)$ is not practicable, it is recommendable to use $\hat{f}_1(\lambda)$. Also we can find that, as n increases 100 to 400, the bias and the root mean squared error of the estimates decrease. One thing apparent from Table 2 is that the bias and the root mean squared error of the estimates are somewhat large when $\lambda = 0$ in AR(1) process. But, it is not so surprising since AR(1) process when $\phi = 0.6$ has a peak at $\lambda = 0$ as shown in Figure 1.

Tables 3 and 4 show the results when MA(1) reference density is used. We get the same lessons as in Tables 1 and 2. Table 4 and Table 5 show the results when ϕ and β are 0.3. We omit $n = 400$ cases here because we obtained virtually the same results as when ϕ and β are 0.6. Figure 2 show the density estimates of the population distributions of $\hat{f}_0(\lambda)$ and $\hat{f}_1(\lambda)$ when $\lambda = 1.5$. They are based on $\hat{f}_{(i)}$, $i = 1, \cdots, 500$. Many other cases show similar shapes or better. The impression we get from Figure 2 is that our procedure works quite well.

Table 1. Bias and root mean squared error of data-based bandwidths and the resulting spectral density estimators

Model	λ	n	$h_{opt}(\lambda)$	$f(\lambda)$		$\hat{h}_{opt}(\lambda)$	$\hat{f}_0(\lambda)$	$\hat{f}_1(\lambda)$
MA(1) ($\beta=0.6$)	π				Bias	-0.39535	-0.06752	-0.03885
		100	2.4027	0.40743		(.04041)	(.00311)	(.00517)
					RMSE	0.98634	0.09703	0.12199
						(.02554)	(.00331)	(.00469)
					Bias	-0.17929	-0.04359	-0.02847
		400	1.8209	0.40743		(.02569)	(.00216)	(.00287)
					RMSE	0.60179	0.06510	0.07013
						(.01639)	(.00236)	(.00221)
	1.5				Bias	-1.61157	0.01129	0.01235
		100	4.5661	0.20294		(.07906)	(.00157)	(.00165)
					RMSE	2.39221	0.03639	0.03701
						(.08664)	(.00142)	(.00121)
					Bias	-1.12306	0.00586	0.00390
		400	3.4605	0.20294		(.05198)	(.00096)	(.00104)
					RMSE	1.61619	0.02235	0.02351
						(.05236)	(.00077)	(.00082)
	0				Bias	0.35275	0.00975	0.01868
		100	0.7926	0.02546		(.01760)	(.00047)	(.00084)
					RMSE	0.52849	0.01430	0.02654
						(.02672)	(.00072)	(.00138)
					Bias	0.12353	0.00476	0.00710
		400	0.6007	0.02546		(.00708)	(.00023)	(.00032)
					RMSE	0.20084	0.00701	0.01007
						(.00805)	(.00033)	(.00045)

Note : AR(1) spectral density was used as a reference density. Bias and RMSE are the Monte Carlo estimates of the bias and root mean squared error, and the standard errors defined in the text are given in parentheses.

Table 2. Bias and root mean squared error of data-based bandwidths and the resulting spectral density estimator

Model	λ	n	$h_{opt}(\lambda)$	$f(\lambda)$		$\hat{h}_{opt}(\lambda)$	$\hat{f}_0(\lambda)$	$\hat{f}_1(\lambda)$
AR(1) ($\phi=0.6$)	π				Bias	-0.24484	0.01804	0.01308
		100	2.4027	0.06217		(.02288)	(.00064)	(.00092)
					RMSE	0.56735	0.02309	0.02439
						(.01717)	(.00118)	(.00109)
					Bias	-0.14672	0.00825	0.00628
		400	1.8209	0.06217		(.01429)	(.00037)	(.00046)
					RMSE	0.35165	0.01174	0.01209
						(.01187)	(.00051)	(.00046)
	1.5				Bias	0.01229	0.03141	0.03771
		100	1.2506	0.12482		(.03045)	(.00125)	(.00247)
					RMSE	0.68115	0.04200	0.06681
						(.04419)	(.00201)	(.00335)
					Bias	-0.00799	0.01683	0.02161
		400	0.9478	0.12482		(.04040)	(.00120)	(.00297)
					RMSE	0.46774	0.02186	0.04057
						(.01878)	(.00099)	(.00219)
	0				Bias	0.29023	-0.21953	-0.24469
		100	0.7926	0.99472		(.01873)	(.01281)	(.01559)
					RMSE	0.50954	0.36092	0.42598
						(.02659)	(.01065)	(.01173)
					Bias	0.16088	-0.14636	-0.16167
		400	0.6007	0.99472		(.01039)	(.00753)	(.00924)
					RMSE	0.28251	0.22316	0.26251
						(.01430)	(.00739)	(.00804)

Note : AR(1) spectral density was used as a reference density.

Table 3. Bias and root mean squared error of data-based bandwidths and the resulting spectral density estimators

Model	λ	n	$h_{opt}(\lambda)$	$f(\lambda)$		$\hat{h}_{opt}(\lambda)$	$\hat{f}_0(\lambda)$	$\hat{f}_1(\lambda)$
MA(1) (β=0.6)	π	100	2.4027	0.40744	Bias	0.28314	-0.06752	-0.07780
						(.00896)	(.00312)	(.00321)
					RMSE	0.34691	0.09703	0.10591
						(.01825)	(.00330)	(.00376)
		400	1.8209	0.40744	Bias	0.16758	-0.04247	-0.04786
						(.00594)	(.00191)	(.00199)
					RMSE	0.21394	0.06020	0.06543
						(.01088)	(.00230)	(.00253)
	1.5	100	4.5661	0.20294	Bias	1.98442	0.01036	0.01066
						(.09740)	(.00150)	(.00155)
					RMSE	2.94644	0.03516	0.03620
						(.15330)	(.00127)	(.00130)
		400	3.4605	0.20294	Bias	0.91730	0.00586	0.00847
						(.07464)	(.00096)	(.00118)
					RMSE	1.90442	0.02235	0.02788
						(.12798)	(.00077)	(.00101)
	0	100	0.7926	0.02546	Bias	0.32120	0.00939	0.01820
						(.01696)	(.00048)	(.00087)
					RMSE	0.49615	0.01432	0.02655
						(.02522)	(.00069)	(.00138)
		400	0.6007	0.02546	Bias	0.10841	0.00514	0.00713
						(.00635)	(.00026)	(.00034)
					RMSE	0.17865	0.00768	0.01035
						(.00805)	(.00045)	(.00033)

Note : MA(1) spectral density was used as a reference density.

Table 4. Bias and root mean squared error of data-based bandwidths and the resulting spectral density estimators

Model	λ	n	$h_{opt}(\lambda)$	$f(\lambda)$		$\hat{h}_{opt}(\lambda)$	$\hat{f}_0(\lambda)$	$\hat{f}_1(\lambda)$
AR(1) (ϕ=0.6)	π	100	2.4027	0.06217	Bias	-0.41379	0.01804	0.01041
						(.02458)	(.00064)	(.00099)
					RMSE	0.68789	0.02309	0.02446
						(.12564)	(.00118)	(.00215)
		400	1.8209	0.06217	Bias	-0.22042	0.00827	0.00639
						(.01974)	(.00038)	(.00070)
					RMSE	0.49338	0.01182	0.01699
						(.06861)	(.00054)	(.00143)
	1.5	100	1.2506	0.12482	Bias	0.71017	0.03318	0.04371
						(.07037)	(.00127)	(.00226)
					RMSE	1.72630	0.04367	0.06682
						(.19793)	(.00222)	(.00321)
		400	0.9478	0.12482	Bias	-0.12043	0.01626	0.01210
						(.01188)	(.00069)	(.00090)
					RMSE	0.29175	0.02241	0.02352
						(.05252)	(.00099)	(.00141)
	0	100	0.7926	0.99472	Bias	1.20820	-0.22171	-0.52030
						(.00648)	(.01317)	(.00639)
					RMSE	1.21686	0.36853	0.53954
						(.12365)	(.01229)	(.05055)
		400	0.6007	0.99472	Bias	0.91696	-0.13908	-0.38997
						(.00398)	(.00744)	(.00412)
					RMSE	0.92127	0.21684	0.40070
						(.07142)	(.00793)	(.04768)

Note : MA(1) spectral density was used as a reference density.

Table 5. Bias and root mean squared error of data-based bandwidths and the resulting spectral density estimators

Model	λ	n	$h_{opt}(\lambda)$	$f(\lambda)$		$\hat{h}_{opt}(\lambda)$	$\hat{f}_0(\lambda)$	$\hat{f}_1(\lambda)$
MA(1) (β = 0.3)	π	100	2.6851	0.26897	Bias	-0.13730	-0.04069	-0.03049
						(.04003)	(.00186)	(.00262)
					RMSE	0.90554	0.05821	0.06598
						(.03744)	(.00212)	(.00198)
	1.5	100	5.5695	0.16672	Bias	-0.49132	0.00609	0.00509
						(.13112)	(.00114)	(.00128)
					RMSE	2.97281	0.02628	0.02902
						(.12402)	(.00091)	(.00101)
	0	100	1.6364	0.07799	Bias	0.86307	0.01591	0.03064
						(.04582)	(.00091)	(.00148)
					RMSE	1.33970	0.02588	0.04507
						(.08985)	(.00110)	(.00208)
AR(1) (ϕ=0.3)	π	100	2.6851	0.09417	Bias	0.20761	0.02286	0.02043
						(.06212)	(.00084)	(.00131)
					RMSE	1.40447	0.02959	0.03574
						(.09559)	(.00142)	(.00149)
	1.5	100	1.8796	0.15193	Bias	-0.41121	0.01923	0.01274
						(.06912)	(.00119)	(.00166)
					RMSE	0.89959	0.03282	0.03932
						(.07347)	(.00140)	(.00204)
	0	100	1.6364	0.32481	Bias	0.56424	-0.05167	-0.05645
						(.03485)	(.00314)	(.00394)
					RMSE	0.96207	0.08709	0.10469
						(.04678)	(.00284)	(.00403)

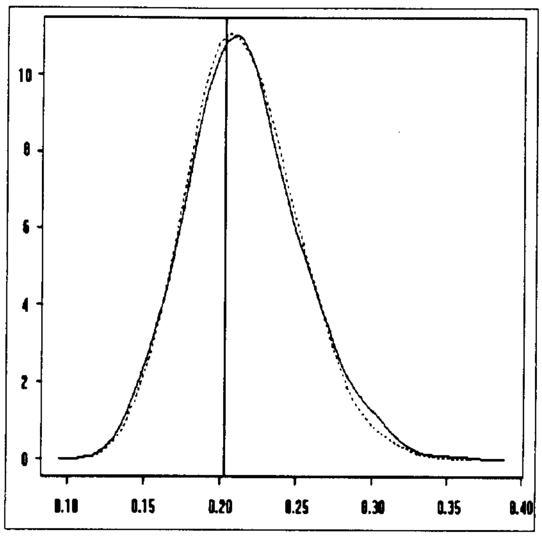
Note : AR(1) spectral density was used as a reference density.

Table 6. Bias and root mean squared error of data-based bandwidths and the resulting spectral density estimators

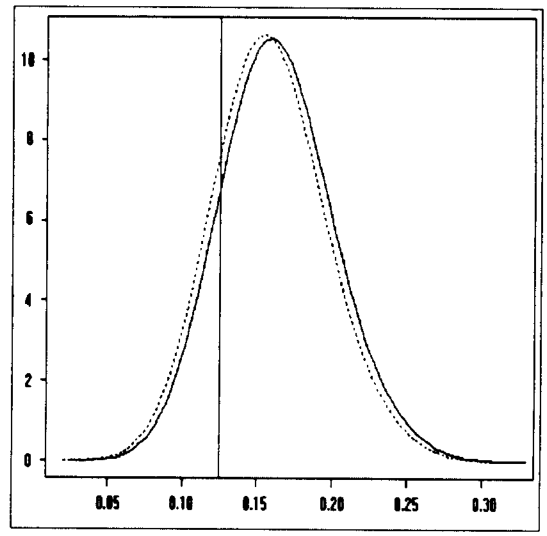
Model	λ	n	$h_{opt}(\lambda)$	$f(\lambda)$		$\hat{h}_{opt}(\lambda)$	$\hat{f}_0(\lambda)$	$\hat{f}_1(\lambda)$
MA(1) (β = 0.3)	π	100	2.6851	0.26897	Bias	0.50494	-0.03852	-0.04587
						(.02554)	(.00206)	(.00221)
					RMSE	0.76227	0.06013	0.06744
						(.06644)	(.00191)	(.00219)
	1.5	100	5.5695	0.16672	Bias	2.87756	0.00726	0.00718
						(.16522)	(.00121)	(.00129)
					RMSE	4.68282	0.02806	0.02971
						(.19736)	(.00726)	(.00718)
	0	100	1.6364	0.07799	Bias	0.76937	0.01662	0.02827
						(.04589)	(.00094)	(.00136)
					RMSE	1.28258	0.02675	0.04152
						(.12013)	(.00116)	(.00183)
AR(1) (ϕ=0.3)	π	100	2.6851	0.09417	Bias	-0.10602	0.02129	0.01541
						(.03889)	(.00088)	(.00121)
					RMSE	0.87622	0.02897	0.03105
						(.08153)	(.00131)	(.00113)
	1.5	100	1.8796	0.15193	Bias	2.51368	0.01761	0.02051
						(.04301)	(.00119)	(.00133)
					RMSE	2.69134	0.03206	0.03619
						(.23706)	(.00143)	(.00153)
	0	100	1.6364	0.32481	Bias	1.36585	-0.03798	-0.08501
						(.02231)	(.00323)	(.00258)
					RMSE	1.45413	0.08166	0.10281
						(.12037)	(.00255)	(.00426)

Note : MA(1) spectral density was used as a reference density.

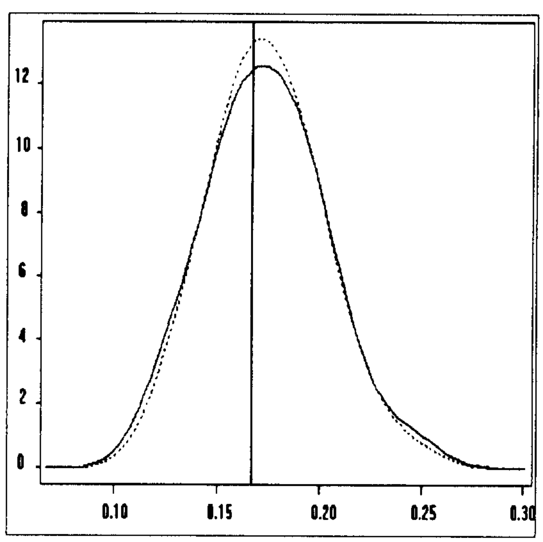
Figure 2. Density estimates of the population distributions of $\hat{f}_0(\lambda)$ and $\hat{f}_1(\lambda)$. Solid curve corresponds to $\hat{f}_1(\lambda)$, dotted one corresponds to $\hat{f}_0(\lambda)$ and vertical line indicates the location of the true value of $f(\lambda)$.



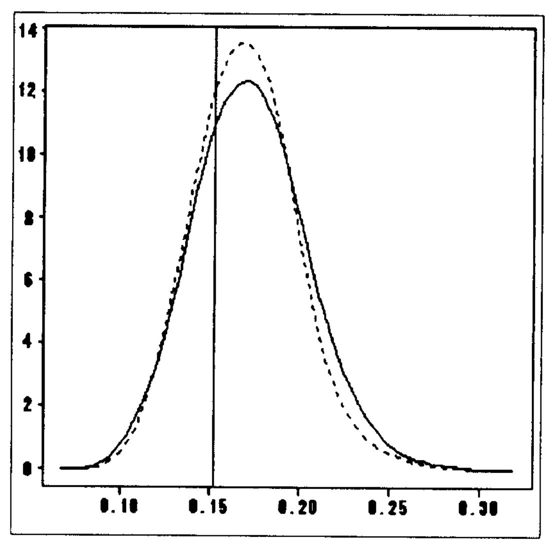
(a) MA(1) with $\beta = 0.6, \lambda = 1.5, n = 100$, AR(1) reference



(b) AR(1) with $\phi = 0.6, \lambda = 1.5, n = 100$, AR(1) reference



(c) MA(1) with $\beta = 0.3, \lambda = 1.5, n = 100$, MA(1) reference



(d) AR(1) with $\phi = 0.3, \lambda = 1.5, n = 100$, MA(1) reference

Appendix

We describe an algorithm for finding a root in (2.7). Let (2.7) be a form of $F(h)=0$.

Step 1 : Give two initial values h_0, h_1 .

Step 2 : For $n=1,2,\cdots$, until $|h_{n+1}-h_n|$ or $|F(h_n)|$ sufficiently small, do :

Calculate $h_{n+1}=h_n-F(h_n)(h_n-h_{n-1})/(F(h_n)-F(h_{n-1}))$.

If $h_m<0$ for some m , then add some increment to the initial values and do Step 2 again.

If $F(h_m)F(h_{m+1}) < 0$ for some m , then apply the bisection method at $(h_m,F(h_m))$ and $(h_{m+1},F(h_{m+1}))$.

If h obtained in Step 2 is out of the range $[h_{opt}/3, 3h_{opt}]$, then add some increment to the initial values and do Step 2 again.

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