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A Note on Central Limit Theorem on $L^P(R)$ +

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Abstract

In this paper a central limit theorem on $L^P(R)$ for $1 \le p < \infty$ is obtained with an example when $\{X_n\}$ is a sequence of independent, identically distributed random variables on $L^P(R)$.

Keywords : L^P random variable, central limit theorem, kernel density estimation, $C_0(R)$ random variable.

Limit theorems on $L^P(R)$ have strong applications in statistical estimation. In paticular the L^1 consistency of kernel density estimators has been supported by several authors, most notably by Devroye and Györfi(1985) and central limit theorems for L^P norms of kernel density estimators were obtained by Csörgö and Horvåth(1988). Zinn(1977) reformulated the central limit theorem of Hoffmann-Jϕrgensen and Pisier(1976) and also obtained some central limit theorems on L^P [0,1], $1 \le p < \infty$. Central limit theorem on a separable Banach space can be defined as follows. Let E be a separable Banach space. A probability measure μ on E is said to be Gaussian if the finite dimensional distribution of μ are Gaussian, i.e., given any positive integer n and $f_1, f_2, \cdots, f_n \in E^*$, the dual space of E, the distribution induced in R^n by $(f_1, \cdots, f_n)$ is Gaussian. An E-valued random variable X is Gaussian if its distribution μ is. An E-valued random variable X with distribution μ is said to satisfy the central limit theorem if the sequence of measures $\{\mu_n\}$, induced by the sequence of random variables $\left\{n^{-\frac{1}{2}}(X_1 + \cdots + X_n)\right\}$, where $X_1, X_2, \cdots, X_n$ are independent copies of X, converges weakly to a Gaussian measure ν, i.e.,

$$\int f \, d\mu_n \to \int f \, d\nu, \qquad \text{for every } f \in E^*.$$

In this note we give a central limit theorem on $L^P(R)$, $1 \le p < \infty$, with applications to kernel

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density estimations, which is based on the following Zinn(1977)'s result.

Theorem 1. (Zinn(1977))

(i) A linear map $\nu : E \to F$ is of type 2 if and only if for every Radon probability μ on E satisfying $\int \|x\|^2 \mu(dx) < \infty$ and $\int x\mu(dx) = 0$, $\mu \circ \nu^{-1}$ satisfies the central limit theorem on E.

(ii) If μ is a Borel probability on $C[0,1]$ satisfying $\int \|x\|^2 \mu(dx) < \infty$ and $\int x\mu(dx) = 0$, then μ satisfies the central limit theorem on $L^P[0,1]$ for any $1 \le p < \infty$.

Note that $C(R)$, the space of all bounded continuous functions on R, is not a separable Banach space with sup norm. Hence the above central limit theorem on $L^P[0,1]$ can not be directly applied to $L^P(R)$. Now let,

$$C_0(R) = \{f : f \textit{ is continuous} \text{ and } \lim_{|t| \to \infty} f(t) = 0\}.$$

Then a central limit theorem on $L^P(R)$ can be obtained as follows.

Theorem 2. If μ is a Borel probability on $C_0(R)$ satisfying $\int \|x\|^2 \mu(dx) < \infty$ and $\int x\mu(dx) = 0$, then μ satisfies the central limit theorem on $L^P(R)$ for $1 \le p < \infty$.

Proof. Since $C_0(R)$ is a separable Banach space with sup norm, μ is a Radon probability. Thus, by Theorem1, we need only to show that a linear map $\upsilon : C_0(R) \to L^P(R)$ is of type 2. Since any continuous map from $\mathcal{L}^\infty$-space to $\mathcal{L}^P$-space is of type 2 (see Zinn(1977)) and $C_0(R) \subset \mathcal{L}^\infty$, υ is of type 2.

Remark. Statistical data analysis using the criterion of the least absolute value methods necessitates limit theorems on $L^1(R)$ space. When $p=1$, $L^1(R)$ is of cotype 2 and the central limit theorem for X on $L^1(R)$ holds if the X is pre-Gaussian. However, Theorem 2 is still useful in functional estimation as the following example indicates(cf. Taylor and Hu(1987)).

Example. Let $X_1, X_2, \cdots, X_n$ be a random sample having the same density $f(t)$ belonging to $C_0(R)$. Let $K(t)$ be an even, bounded, compactly supported probability density function which is strictly decreasing in its support as $|t|$ increases and satisfy $|K(x) - K(y)| \le H|x-y|^\alpha$, for all $x, y \in R$ and $H, \alpha > 0$. Then

$$X_{nk}(t) = K(\frac{t-X_k}{h_n}) - EK(\frac{t-X_k}{h_n}), \quad h_n \to 0,$$

is a random variable in $C_0(R)$ and hence $\{\mu_n\}$ induced by $\{n^{-1/2}(X_{n1}+\cdots+X_{nn})\}$, converges weakly to a Gaussian measure ν.

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