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A Central Limit Theorem for Linearly Positive Quadrant Dependent Random Fields

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Abstract

In this note, we obtain the central limit theorem for linearly positive quadrant dependent random fields satisfying some assumptions on the covariances and the moment condition $\sup E|X_j|^3 < \infty$. The proofs are similar to those of a central limit theorem for associated random field of Cox and Grimmett.

1. Introduction

A random field is a collection of nondegenerate random variables indexed by Z^d and is denoted by $\{X_j: j\in Z^d\}$. In the last years there has been growing interest in concepts of positive dependence for random fields. Such concepts are of considerable use in deriving inequalities in probability and statistics.

Lehmann[6] introduced a simple and natural definition of positive dependence :

A random field $\{X_j: j\in Z^d\}$ is said to be pairwise positive quadrant dependent if for any real r_i, r_j and $i\neq j$,

$$P\{X_i>r_i, X_j>r_j\} \geq P\{X_i>r_i\}P\{X_j>r_j\}.$$

A much stronger concept than positive quadrant dependence was considered by Esary, Proschan and Walkup[4]:

A random field $\{X_j: j\in Z^d\}$ is said to be associated if for any subset $A\subset Z^d$ and for any pair of coordinatewise increasing functions f, g on $R^{\#A}$

$$\mathrm{Cov}\,(f(X_j: j\in A),\ g(X_j: j\in A)) \geq 0,$$

whenever the covariance is defined. Here $\#A$ is the cardinality of A.

Newman[8] was the first who showed that for positive dependent random fields approximate uncorrelatedness implies approximate independence, such that useful limit theorems can be obtained.

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In the following years several extensions and generalizations of these results were given(for examples [3, 4, 7, 9]). Most of these results, however, cannot be applied to weaker concepts of positive dependent random fields.

Newman[7] has shown that a stationary associated random field $\{X_{\underline{j}}: \underline{j} \in Z^d\}$, having the property that

$$0 < \sigma^2 = \sum_{\underline{j} \in Z^d} \mathrm{Cov}(X_{\underline{0}}, X_{\underline{j}}) < \infty, \tag{1.1}$$

satisfies the central limit theorem.

Cox and Grimmett[3] have shown that the assumption of stationarity may be relaxed and replaced by certain conditions on the moments of the X's, that is, instead of (1.1) Cox and Grimmett[3] proved the following central limit for associated random fields using conditions on the coefficient of maximal covariance

$$u(r) = \sup_{\underline{1} \le \underline{k} \le \underline{n1}} \sum_{\underline{j}: |\underline{j}-\underline{k}| \ge r} \mathrm{Cov}(X_{\underline{j}}, X_{\underline{k}})$$

where $|\underline{j}-\underline{k}| = \sup\{|j_i - k_i| : i = 1, \cdots, d\}$.

Theorem A (Cox and Grimmett(1984)). Let $\{X_{\underline{j}}: \underline{j} \in Z^d\}$ be an associated random field with $EX_{\underline{j}} = 0$. Assume

(i) $\inf_{\underline{1} \le \underline{j} \le \underline{n1}} \mathrm{Var}(X_{\underline{j}}) > 0$,

(ii) $\sup_{\underline{1} \le \underline{j} \le \underline{n1}} E(|X_{\underline{j}}|^3) \le \infty$,

(iii) $u(0) < \infty, \quad u(r) \to 0$ as $r \to \infty$.

Then $\dfrac{S_{\underline{n1}} - ES_{\underline{n1}}}{\sqrt{\mathrm{Var}\, S_{\underline{n1}}}}$ is asymptotically normally distributed as $n \to \infty$.

The purpose of this note is to extend Theorem A to a linearly positive quadrant dependent random fields(see Definition 2.1) using the similar methods to those of Cox and Grimmett(1984).

The preliminaries and results are stated in Section 2. The proofs of our theorems as well as some lemmas are given in Section 3.

2. Preliminaries and Results

Newman(1984) first introduced the concept of linearly positive quadrant dependence. We extend this concept to the random field.

Definition 2.1. A random field $\{X_{\mathbf{j}}: \mathbf{j}\in Z^d\}$ is said to be linearly positive quadrant dependent if for any disjoint subsets A, $B\subset Z^d$ and positive $r_{\mathbf{j}}$'s,

$$\sum_{\mathbf{i}\in A} r_{\mathbf{i}}X_{\mathbf{i}} \quad \text{and} \quad \sum_{\mathbf{j}\in B} r_{\mathbf{j}}X_{\mathbf{j}} \text{ are positive quadrant dependent.}$$

For any fixed $l=1, 2, 3, \cdots$, we let $m=[n/l]$. Put

$$\begin{aligned}
S_{n\mathbf{1}} &= \sum_{\mathbf{1}\le \mathbf{j}\le n\mathbf{1}} X_{\mathbf{j}}, \\
Y_{\mathbf{j}}(l) &= \sum_{(\mathbf{j}-\mathbf{1})l<\mathbf{i}\le \mathbf{j}l} X_{\mathbf{i}} \quad \text{for} \quad \mathbf{1}\le \mathbf{j}\le m\mathbf{1}, \\
S_{ml\mathbf{1}} &= \sum_{\mathbf{1}\le \mathbf{j}\le m\mathbf{1}} Y_{\mathbf{j}}(l), \qquad Z(n) = S_{n\mathbf{1}}-S_{ml\mathbf{1}}, \\
\sigma^2_{ml\mathbf{1}} &= \mathrm{Var}(S_{ml\mathbf{1}}), \qquad s^2(n,l) = \sum_{\mathbf{1}\le \mathbf{j}\le ml\mathbf{1}} \mathrm{Var}(Y_{\mathbf{j}}(l)), \\
\sigma^2_{n\mathbf{1}} &= \mathrm{Var}(S^2_{n\mathbf{1}}).
\end{aligned}$$

Note that $Y_{\mathbf{j}}(l)$ and $Z(n)$ depend upon the choice of l and that $m=\left[\frac{n}{l}\right]\to\infty$, as $n\to\infty$.

Lemma 2.2. Let $\{X_{\mathbf{j}}: \mathbf{j}\in Z^d\}$ be a linearly positive quadrant dependent random field with $EX_{\mathbf{j}}=0$, $EX_{\mathbf{j}}^2<\infty$. Then we obtain

$$\limsup_{n\to\infty} \frac{\sigma^2_{n\mathbf{1}}}{s^2(n,l)} \le 1+\frac{2d}{lc_1}\sum_{r=1}^{l} u(r) \tag{2.1}$$

where $c_1=\inf_{\mathbf{1}\le \mathbf{j}\le n\mathbf{1}}\mathrm{Var}(X_{\mathbf{j}})>0$.

Put

$$\begin{aligned}
\varphi_{n\mathbf{1}}(t) &= E(\exp(itS_{n\mathbf{1}})), \\
\varphi_{n,l}(t) &= E(\exp(itS_{ml\mathbf{1}})), \\
\varphi_{n\mathbf{j}}(t) &= E(\exp(itY_{\mathbf{j}}(l))).
\end{aligned}$$

Lemma 2.3. Let $\{X_j : j \in Z^d\}$ be a linearly positive quadrant dependent random field with $EX_j = 0$, $\sup_{1 \le j \le n1} E(|X_j|^2) < \infty$. Then we have the following statements :

$$\limsup_{n \to \infty} |\varphi_{n1}(\frac{t}{\sigma_{n1}}) - \varphi_{n,l}(\frac{t}{s(n,l)})| \le |t| \frac{2d}{lc_1} \sum_{r=1}^{l} u(r), \tag{2.2}$$

$$\begin{aligned} \limsup_{n \to \infty} & |\varphi_{n,l}(\frac{t}{s(n,l)}) - \Pi_{1 \le j \le m1} \varphi_{nj}(\frac{t}{s(n,l)})| \\ & \le \frac{dt^2}{lc_1} \sum_{r=1}^{l} u(r), \end{aligned} \tag{2.3}$$

$$\limsup_{n \to \infty} \left| \Pi_{1 \le j \le m1} \varphi_{nj}(\frac{t}{s(n,l)}) - \exp(\frac{-t^2}{2}) \right| = 0 \ . \tag{2.4}$$

Theorem 2.4. Let $\{X_j : j \in Z^d\}$ be a linearly positive quadrant dependent random field with $EX_j = 0$. Assume

(i) $\inf_{1 \le j \le n1} \mathrm{Var}(X_j) > 0$,

(ii) $\sup_{1 \le j \le n1} E(|X_j|^3) < \infty$

(iii) $u(0) < \infty$, $u(r) \to 0$ as $r \to \infty$

where $u(r) = \sup_{1 \le j \le n1} \sum_{j : |j-k| \ge r} \mathrm{Cov}(X_j, X_k)$.

Then $\dfrac{S_{n1} - ES_{n1}}{(\mathrm{Var}(S_{n1}))^{1/2}}$ is asymptotically normally distributed as $n \to \infty$.

3. Proof

Proof of Lemma 2.2 : First we show that

$$c_1 n^d \le \sigma^2_{n1} \le c_3 n^d \tag{3.1}$$

where $c_3 = u(0)$. The left hand side of (3.1) follows from

$$\sigma^2_{n1} = \sum_{1 \le j \le n1} \mathrm{Var}(X_j) + \sum_{1 \le i \ne j \le n1} \mathrm{Cov}(X_i, X_j) \ge c_1 n^d$$

since the X_j are linearly positive quadrant dependent. That is, the X_j are nonnegatively correlated and the right hand side of (3.1) follows

$$\sum_{\underline{1}\le \underline{i},\underline{j}\le n\underline{1}} \mathrm{Cov}(X_{\underline{i}},X_{\underline{j}}) \le \sum_{\underline{1}\le \underline{i}\le n\underline{1}} u(0) = u(0)n^d \le c_3 n^d.$$

By the similar arguments we have

$$(ml)^d c_1 \le s^2(n,l) \le \sigma^2_{ml\underline{1}} \le (m\,l)^d c_3 \tag{3.2}$$

and

$$\sigma^2_{ml\underline{1}} \le \sigma^2_{n\underline{1}}. \tag{3.3}$$

By expanding $S_{n\underline{1}}$ in terms of the $Y_{\underline{j}}(l)$'s we find that

$$\sigma^2_{n\underline{1}} = \mathrm{Var}(Z(n)) + 2\mathrm{Cov}(Z(n),\ S_{ml\underline{1}}) + \mathrm{Var}(S_{ml\underline{1}})\ . \tag{3.4}$$

Since $Z(n) = \sum_{ml\underline{1}\le \underline{j}\le n\underline{1}} X_{\underline{j}}$

$$\mathrm{Var}(Z(n)) \le c_3 l^d((m+1)^d - m^d), \tag{3.5}$$

$$\mathrm{Cov}(Z(n),S(n,l)) \le c_3 l^d((m+1)^d - m^d)\ . \tag{3.6}$$

Thus (3.2), (3.5) and (3.6) yield that

$$\frac{\mathrm{Var}(Z(n)) + 2\mathrm{Cov}(Z(n),\ S_{ml\underline{1}})}{s^2(n,\ l)} \le \frac{3c_3 l^d\{(m+1)^d - m^d\}}{c_1(ml)^d} \to 0 \quad \text{as} \quad n\to\infty. \tag{3.7}$$

For the final term in (3.4)

$$\mathrm{Var}(S_{ml\underline{1}})) = s^2(n,l) + \Sigma(n\underline{1})$$

where

$$\begin{aligned} \Sigma(n\underline{1}) &= \sum_{\underline{1}\le \underline{i}\le m\underline{1}} \sum_{\underline{1}\le \underline{i}\ne \underline{j}\le m\underline{1}} \mathrm{Cov}(Y_{\underline{i}}(l), Y_{\underline{j}}(l)) \\ &\le \sum_{\underline{1}\le \underline{i}\le m\underline{1}} \sum_{\alpha\in\Lambda_{\underline{i}}} \sum_{\beta\notin\Lambda_{\underline{i}}} \mathrm{Cov}(X_{\underline{\alpha}}, X_{\underline{\beta}}) \end{aligned} \tag{3.8}$$

and $\Lambda_{\underline{i}} = \{\underline{\alpha} \in Z^d : (\underline{i} - \underline{1})l < \underline{\alpha} \le \underline{i}l\}$. But

$$\sum_{\underline{\alpha} \in \Lambda_{\underline{i}}} \sum_{\underline{\beta} \notin \Lambda_{\underline{i}}} \mathrm{Cov}(X_{\underline{\alpha}}, X_{\underline{\beta}}) \le 2dl^{d-1} \sum_{r=1}^{l} u(r) \tag{3.9}$$

since there are at most $2dl^{d-1}$ points of $\Lambda_{\underline{i}}$ which are within distance r of some point outside $\Lambda_{\underline{i}}$.

Combining (3.2), (3.4) and (3.7)-(3.9) we find that

$$\limsup_{n\to\infty} \frac{\sigma^2_{n\underline{1}}}{s^2(n,l)} \le 1 + \frac{2d}{lc_1} \sum_{r=1}^{l} u(r)$$

as required.

Proof of Lemma 2.3 : To prove(2.2) by a standard inequality, we have that

$$\begin{aligned}
&\left|\varphi_{n\underline{1}}\left(\frac{t}{\sigma_{n\underline{1}}}\right) - \varphi_{n,l}\left(\frac{t}{s(n,l)}\right)\right| \\
&\le |t| \left(\mathrm{Var}\left(\frac{S_{n\underline{1}}}{\sigma_{n\underline{1}}} - \frac{S_{ml\underline{1}}}{s(n,l)}\right)\right)^{1/2} \\
&= |\underline{t}| \left(\mathrm{Var}\left(\frac{Z_n}{\sigma_{n\underline{1}}} - S_{ml\underline{1}}\left(\frac{1}{s(n,l)} - \frac{1}{\sigma_{n\underline{1}}}\right)\right)\right)^{\frac{1}{2}} \\
&\le |t|\left(\sigma_{ml\underline{1}}\left(\frac{1}{s(n,l)} - \frac{1}{\sigma_{n\underline{1}}}\right) + \left(\frac{1}{\sigma_{n\underline{1}}}\right)\{\mathrm{Var}(Z(n))\}^{1/2}\right) \\
&\le |t|\left(\frac{\sigma_{n\underline{1}}}{s(n,l)} - 1 + \left(\frac{c_3\{(m+1)^d - m^d\}}{c_1 m^d}\right)^{1/2}\right)
\end{aligned}$$

by (3.2), (3.3), and (3.5). Thus (2.2) follows from Lemma 2.2.

Proof of (2.3) : We use Theorem 1 of [9] to find that

$$\begin{aligned}
&\left|\varphi_{n,l}\left(\frac{t}{s(n,l)}\right) - \Pi_{\underline{1}\le \underline{j}\le m\underline{1}}\varphi_{n\underline{j}}\left(\frac{t}{s(n,l)}\right)\right| \\
&\le \left(\frac{t^2}{2s^2(n,l)}\right) \sum_{\underline{i}\ne \underline{j}} \mathrm{Cov}(Y_{\underline{i}}(l), Y_{\underline{j}}(l)) \\
&= \frac{t^2}{2s^2(n,l)}(\sigma^2(n,l) - s^2(n,l)) \\
&\le \frac{t^2}{2}\left(\frac{\sigma^2_{n\underline{1}}}{s^2(n,l)} - 1\right)
\end{aligned}$$

Now use Lemma 2.2.

Proof of (2.4) : This follows from Lyapunov's Theorem(see Theorem 7.12 of [2], for example). Just note that

$$\begin{aligned} E(|Y_{\underline{j}}(l)|^3) &\le \sum_{\underline{\alpha},\underline{\beta},\underline{\gamma}\in\Lambda_{\underline{j}}} E|X_{\underline{\alpha}}X_{\underline{\beta}}X_{\underline{\gamma}}| \\ &\le l^{3d}c_2 \end{aligned}$$

by Holder's inequality , and so by (3.2),

$$\frac{1}{s^3(n,l)}\sum_{\underline{1}\le\underline{j}\le m\underline{1}} E(|Y_{\underline{j}}(l)|^3) \le \frac{m^d l^{3d} c_2}{(m^d l^d c_1)^{3/2}} \to 0 \quad \text{as} \quad n\to\infty,$$

where $\Lambda_{\underline{j}} = \{\underline{\alpha}\in Z^d : (\underline{j}-\underline{1})l < \underline{\alpha} \le \underline{j}l\}$, $c_2 = \sup_{\underline{1}\le\underline{j}\le n\underline{1}} E|X_{\underline{j}}|^3$.

Proof of Theorem 2.4. The theorem follows immediately from Lemma 2.3 since

$$\frac{1}{l}\sum_{r=1}^{l} u(r) \to 0 \quad \text{as} \quad l\to\infty.$$

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