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Estimable Functions in Row-column Designs[1]

Dong Kwon Park[2]

Abstract

A method is presented for finding estimable functions in a row-column design. It can easily be applied because the method consists of solving equations derived from the design without using the design matrix. It determines not only the estimability of treatment effects but also between row(or column) and treatment effects.

1. Introduction

Consider an experiment with υ treatments in the presence of two crossed blocking factors having r and c levels respectively. We restrict attention to experiments in which exactly one treatment is observed at each combination of levels of the two blocking factors. A design for such an experiment is called a row-column design. Assume that the following linear additive model holds :

$$\begin{aligned} \mathbf{y} &= X_1\boldsymbol{\alpha} + X_2\boldsymbol{\beta} + X_\tau\boldsymbol{\tau} + E , \\ &= X\boldsymbol{\pi} + E , \end{aligned} \qquad (1.1)$$

where $\mathbf{y}$ is an $n(=rc) \times 1$ vector of observations; $\boldsymbol{\alpha}=[\alpha_1,\alpha_2,\ldots,\alpha_r]^t$ is a vector of effects of the r levels block parameters corresponding to the rows of the design; $\boldsymbol{\beta}=[\beta_1,\beta_2,\ldots,\beta_c]^t$ is a vector of effects of the c levels block parameters corresponding to the columns of the design; $\boldsymbol{\tau}=[\tau_1,\tau_2,\ldots,\tau_\upsilon]^t$ is a $\upsilon \times 1$ vector of treatment parameters; $X=[X_1|X_2|X_\tau]$ is a design matrix; $\boldsymbol{\pi}=[\boldsymbol{\alpha}|\boldsymbol{\beta}|\boldsymbol{\tau}]^t$; E is a vector of random errors, where A^t denotes transpose of matrix (or vector) A.

It is well known that every contrast of treatment effects in a connected block design is estimable. If a design is disconnected, we need to classify equivalent classes of treatment parameters so that any contrast among the parameters in an equivalent class is estimable. It is easy to determine the equivalent classes in a block design by drawing a treatment concurrence graph (Bose, 1947) or electrical network (Tjur, 1991). It was a difficult task for row-column designs. Park and Shah (1995) gave a simple procedure which enables one to determine which treatment contrasts are estimable in a row-column design. The procedure can

1) This paper was supported by Maeji Institute of Academic Research, Yonsei University, 1994-1995.
2) Department of Statistics, Yonsei University, Wonju, Kangwon-Do, 222-701, KOREA.

be done by looking at the basis formed by the $y_{ij} - y_{i'j} + y_{ij'} + y_{i'j'}$, which is called tetra-differences. Further, it also determines the estimability of row effects and of column effects.

A simple procedure presented here enables the results of Park and Shah to be used for finding estimable functions between row(or column) and treatment effects. It can easily be applied because the method consists of solving equations derived from the design without using the design matrix.

2. Problem Formulation

To deal with our problem easily, we transform the design matrix into a matrix having a simple form. Without loss of generality, assume that rows of the design matrix X are arranged in lexicographical order with respect to the rows and columns of the design. For the first step, we eliminate the parameter α from the model (1.1). Let X^c be the design matrix X pre-multiplied by the $n \times n$ matrix T, where

$$T = I_r \otimes L_c ,$$

where I_r is an $r \times r$ identity matrix, $\otimes$ denotes Kronecker product, and the $c \times c$ matrix L_c is

$$L_c = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 1 & -1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & -1 & 0 & 0 & \cdots & 0 \\ \cdot & & & \cdot & & & \\ \cdot & & & & \cdot & & \\ \cdot & & & & & \cdot & \\ 1 & & & & & & -1 \end{bmatrix}$$

As a simple example, take a 4×3 design with v=5 treatment labels. The design is shown below :

		Column 1	Column 2	Column 3
Row	1	1	2	3
	2	2	3	1
	3	3	1	2
	4	1	4	5

The design matrix X for model (1.1) and the matrix X^c are

$$
X = \left.\begin{array}{cccc|ccc|ccccc}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
\hline
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1
\end{array}\right.
\qquad
X^c = \left.\begin{array}{cccc|ccc|ccccc}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 & -1 & 0 & 0 \\
\hline
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & 1 & 0 & 0 & 0 \\
\hline
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & -1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & -1 & 1 & 0 & 0 \\
\hline
0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 & 0 & 0 & -1
\end{array}\right.
$$

Let the vector u_i be i-th unit vector of length c, and for a next step, we arrange the rows of matrix X^c in the order $(j,\ c+j,\ 2c+j,\ldots,(r-1)c+j)$, $j=1,2,\ldots,r$, to give

$$
X^* = \left[\begin{array}{c|cc}
I_r & U_1 & F_1 \\
\hline
 & U_2 & F_2 \\
\cline{2-3}
0 & U_3 & F_3 \\
\cline{2-3}
 & \cdot & \cdot \\
\cline{2-3}
 & U_c & F_c
\end{array}\right]
$$

where $U_1 = 1_r \otimes u_1^t$, $U_i(i=2,\ldots,c) = 1_r \otimes (u_1 - u_i)^t$, 1_r be an $r \times 1$ vector with all 1 and where F_1 is an $r \times v$ matrix whose rows consist of r unit vectors of length c, and where F_i $(i=2,\ldots,c)$ is an $r \times v$ matrix $F_i^t = [\ f_{i1}, f_{i2},\ldots,f_{ir}\ ]$, where f_{ij}^t is the difference of two unit row vectors of length v.

For the above 4 × 3 design,

$$
X^{*} = \left[\begin{array}{cccc|ccc|ccccc}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & -1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 & 0 & 0 & -1 & 0 \\
\hline
0 & 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 & 0 & 0 & -1
\end{array}\right]
$$

Before we state and prove our theorems, we state the following lemma whose proof is given in Butz (1982).

Lemma 2.1. (Butz, 1982) $\zeta(\boldsymbol{\pi}) = \boldsymbol{p}^t\boldsymbol{\pi}$ is estimable in the model (1.1) if and only if ζ vanishes for any point $\boldsymbol{\pi}$ in the kernel of X.

Since the matrix T is non-singular and X^* is just an arrangement of X^c,

$$
\begin{aligned}
\text{kernel of } X &= \text{kernel of } X^c \\
&= \text{kernel of } X^* \qquad (2.1)
\end{aligned}
$$

Thus, to apply the lemma 2.1, we will consider the kernel of X^* which satisfy the equations $X^*\boldsymbol{\pi} = \mathbf{0}_n$, where $\mathbf{0}_n$ be an $n \times 1$ vector with all 0. The equations become

$$\alpha_j + u_1{}^t\boldsymbol{\beta} + f_{ij}{}^t\boldsymbol{\tau} = 0, \quad i=1 \ ; \ j=1,\dots, r \qquad (2.2)$$

$$(u_1 - u_i)^t\boldsymbol{\beta} + f_{ij}{}^t\boldsymbol{\tau} = 0, \quad i=2,\dots, c \ ; \ j=1,\dots, r \qquad (2.3)$$

3. General form of estimable functions

From the representation of X^*, kernel of X^* becomes as follows.

$$\text{kernel of } X^* = \{\boldsymbol{\pi} | (\boldsymbol{\beta},\boldsymbol{\tau}) \in \text{kernel of} \begin{bmatrix} U_2 & F_2 \\ U_3 & F_3 \\ \cdot & \cdot \\ U_c & F_c \end{bmatrix},\ \alpha_j = -(u_1{}^t\beta + f_{1j}{}^t\tau)\}$$

$$= \{\boldsymbol{\pi} | \boldsymbol{\pi} \text{ which satisfy the equations (2.2) and (2.3) }\}$$

$$= [\boldsymbol{\pi} |\ \{(\boldsymbol{\beta},\boldsymbol{\tau}) |\ (u_1 - u_i)^t\boldsymbol{\beta} = -f_{ij}{}^t\boldsymbol{\tau},\ i=2,\dots,c\},$$
$$\{\boldsymbol{\alpha} |\ \alpha_j = -(u_1{}^t\boldsymbol{\beta} + f_{ij}{}^t\boldsymbol{\tau})\}\ ;\ j=1,\dots,r\ ] \qquad (3.1)$$

In the equations (3.1), for fixed $i=2,\dots,c$, $f_{ij}{}^t\boldsymbol{\tau} = \tau_k - \tau_{k'}$ for some $k, k'=1,\dots,\upsilon$.

Theorem 3.1. A parametric function $\zeta(\boldsymbol{\pi}) = \boldsymbol{p}^t\boldsymbol{\pi} = (\beta_1 - \beta_i) + (\tau_k - \tau_{k'})$ is estimable.

Proof. $\zeta(\boldsymbol{\pi}) = (\beta_1 - \beta_i) + (\tau_k - \tau_{k'})$ vanishes for $\boldsymbol{\beta},\boldsymbol{\tau}$ which satisfy the equations (3.1) for any $\boldsymbol{\alpha}$, and the proof follows from the lemma 2.1.

Obviously, columns can be treated as rows by exchanging rows and columns. If a design is connected, we will see in theorem 3.3 that every contrast of row and column effects is also estimable. Thus, if a row-column design is connected, then we do not proceed further and conclude that every contrast among row, column and treatment effects is estimable.

Now, we suggest a simple procedure for obtaining equivalent classes of treatments. For fixed $i>1$, look at the differences of pairs of rows in the equations (2.2) and (2.3). They are

$$f_{ij}{}^t\boldsymbol{\tau} - f_{ij'}{}^t\boldsymbol{\tau} = 0$$

or

$$f_{ij}{}^t\boldsymbol{\tau} = f_{ij'}{}^t\boldsymbol{\tau}, \qquad j \neq j' = 1,\dots,r \qquad (3.2)$$

The equations (3.2) in the above example are shown below :

$$i = 2: \quad \begin{aligned} \tau_1 - \tau_2 &= \tau_2 - \tau_3 \\ \tau_2 - \tau_3 &= \tau_3 - \tau_1 \\ \tau_3 - \tau_1 &= \tau_1 - \tau_4 \\ \tau_1 - \tau_2 &= \tau_1 - \tau_4 \end{aligned} \qquad\qquad i = 3: \quad \begin{aligned} \tau_1 - \tau_3 &= \tau_2 - \tau_1 \\ \tau_2 - \tau_1 &= \tau_3 - \tau_2 \\ \tau_3 - \tau_2 &= \tau_1 - \tau_5 \\ \tau_1 - \tau_3 &= \tau_1 - \tau_5 \end{aligned}$$

Each of these sets can be written as a single equation as follows :

$$EQ(2): \quad \tau_1-\tau_2 = \tau_2-\tau_3 = \tau_3-\tau_1 = \tau_1-\tau_4 \tag{3.3}$$

$$EQ(3): \quad \tau_1-\tau_3 = \tau_2-\tau_1 = \tau_3-\tau_2 = \tau_1-\tau_5 \tag{3.4}$$

Theorem 3.2. If the relation between two treatment parameters τ_k and $\tau_{k'}$ is derived as $\tau_k = \tau_{k'}$ from EQ(2), EQ(3), . . . , EQ(c), they are in the same equivalent class. Otherwise, they are in different classes. If only one class exists, the design is connected.

Proof. Since our concern is connectedness between two treatment parameters τ_k and $\tau_{k'}$, our parametric function is then

$$\xi(\boldsymbol{\pi}) = \boldsymbol{p}^t\boldsymbol{\pi} = \tau_k - \tau_{k'} \tag{3.5}$$

, where $\boldsymbol{p}^t = [0,...,0 \mid 0,...,0 \mid 0,...,1,...,-1,0,...,0]$ corresponding to $\boldsymbol{\pi} = [\boldsymbol{\alpha} \mid \boldsymbol{\beta} \mid \boldsymbol{\tau}]^t$. Without regarding any $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, replace $\boldsymbol{\pi}$ in the equations (3.1) by

$$[\boldsymbol{\pi} \mid \{\boldsymbol{\tau} \mid f_{ij}^t\boldsymbol{\tau} = \delta_i\}, \{\boldsymbol{\beta} \mid (u_1 - u_j)^t\boldsymbol{\beta} = -\delta_i\},$$

$$\{\boldsymbol{\alpha} \mid \alpha_j = -(u_1^t\boldsymbol{\beta} + f_{1j}^t\boldsymbol{\tau})\}] \tag{3.6}$$

,where δ_i's are some constants.

$$= [\boldsymbol{\pi} \mid \{\boldsymbol{\tau} \mid f_{ij}^t\boldsymbol{\tau} = \delta_i\} \text{ and for any } \boldsymbol{\alpha},\boldsymbol{\beta}\,]$$

$$= [\boldsymbol{\pi} \mid \{\boldsymbol{\tau} \mid \boldsymbol{\tau} \text{ which satisfy EQ}(i)\text{'s}\} \text{ and for any } \boldsymbol{\alpha},\boldsymbol{\beta}] \tag{3.7}$$

Then $\xi(\boldsymbol{\pi})$ of the equation (3.5) vanishes only if τ_k and $\tau_{k'}$ in the equation (3.7) have a relation with $\tau_k = \tau_{k'}$. Since $\boldsymbol{\pi}$ of the kernel of X^* (or X) is a subset of $\boldsymbol{\pi}$ in the equation (3.7), the proof follows from the lemma 2.1.

Now, we sketch the procedure for determining the equivalent classes. For each i, write the reduced equation EQ(i). Look for two occurrences of τ_k. Suppose these are $\tau_k - \tau_{k'} = \tau_k - \tau_{k''}$ (i.e., $\tau_{k'} = \tau_{k''}$) then $\tau_{k'}$ and $\tau_{k''}$ are in the same equivalent class. Replace $\tau_{k'}$ by $\tau_{k''}$ in all of EQ(i)'s. At any stage, combine 2 equivalent classes containing any label in common. We show how to do this, step by step, with the 4 × 3 design given in section 2. In the equation (3.3), EQ(2), $\tau_1-\tau_2 = \tau_2-\tau_3$ implies that $\tau_2 = (\tau_1+\tau_3)/2$. By inserting $\tau_2 = (\tau_1+\tau_3)/2$ into the equation $\tau_2-\tau_3 = \tau_3-\tau_1$, we have $(\tau_1-\tau_3)/2 = \tau_3-\tau_1$ or $\tau_1 = \tau_3$. From the theorem 3.2, the treatment parameters τ_1 and τ_3 are in the same equivalent class. Moreover, the equations (3.3) and (3.4) become zero and $\tau_1=\tau_2=\tau_3=\tau_4=\tau_5$, so that all the treatment parameters are in the same equivalent class. We conclude that the 4×3 design is connected.

The equation EQ(i)'s can be derived directly from the design without writing down the design matrix X or the matrix X^*. Now, a simple procedure for obtaining the EQ(i)'s is presented as follows :

A row-column design is given by r rows and c columns. Without loss of generality, we assume $r \geq c$. Take c-1 pairs of two columns (1,2), (1,3), ... , (1,c). For each fixed pair, subtract the second column from the first column so that the pair of columns has r treatment differences. Then an equation EQ(i) is deduced by equating the r treatment differences in the pair of columns.

The procedure follows from theorem 3.2, by looking at each i in turn so that choice of column blocks is the choice given by $(u_1 - u_i)^t$.

As far as the estimability of treatment effects is concerned, the results of theorem 3.2 are the same as those of Park and Shah, although their proof uses a different approach. However, a procedure given in here gives general form of estimable functions between row(or column) and treatment effects.

Theorem 3.3. If two treatments τ_k and $\tau_{k'}$ are in an equivalent class, the corresponding β_1 and β_i in the equations (3.1) are also in the same equivalent class of column effects.

Proof. If two treatments τ_k and $\tau_{k'}$ are in an equivalent class, $\delta_i=0$ in the equations (3.6). It implies that $\beta_1=\beta_i$. From the lemma 2.1, the proof is established.

Theorem 3.3 implies that if a design is connected, then every contrast of row and of column effects is also estimable. This results is also contained in Raghavarao and Federer (1975).

Two examples will show that how to obtain the EQ(i)'s and find the equivalent classes from the EQ(i)'s. During the procedure, we could find estimable functions $\zeta(\boldsymbol{\pi})=(\beta_1-\beta_i)+(\tau_k-\tau_{k'})$.

Example 3.1. Consider the following row-column design which is given in Butz(1982).

1	3	2
6	2	8
9	8	11
7	5	11
4	5	3
10	9	7

The column block pairs (1,2),(1,3) and the corresponding EQ(i)'s, i=2,3, are as shown below :

Columns ; (1,2) (1,3)

(1,2)		(1,3)	
1	3	1	2
6	2	6	8
9	8	9	11
7	5	7	11
4	5	4	3
10	9	10	7

$$EQ(2) : \tau_1-\tau_3 = \tau_6-\tau_2 = \tau_9-\tau_8 = \tau_7-\tau_5 = \tau_4-\tau_5 = \tau_{10}-\tau_9$$

$$EQ(3) : \tau_1-\tau_2 = \tau_6-\tau_8 = \tau_9-\tau_{11} = \tau_7-\tau_{11} = \tau_4-\tau_3 = \tau_{10}-\tau_7$$

In EQ(3), $\tau_9-\tau_{11} = \tau_7-\tau_{11}$ implies that τ_7 and τ_9 are in the same equivalent class. Since $\tau_7 = \tau_9$, $\tau_{10}-\tau_9$ and $\tau_{10}-\tau_7$ which are the final terms of the EQ(2) and EQ(3) are the same. Thus, we can equate EQ(2) and EQ(3), and we have $\tau_1-\tau_3 = \tau_1-\tau_2$ so that τ_2 and τ_3 are in the same equivalent class. By continuing this way and at any stage, combining 2 equivalent classes containing any label in common, we can verify that $\{\tau_j |\ j=1,4,6,7,9\}$, $\{\tau_j |\ j=2,3,5,8,11\}$ and $\{\tau_{10}\}$ form equivalent classes of treatment parameters. Obviously, the design is disconnected.

Replacing each treatment by the lowest numbered treatment in the corresponding class we get the following design

$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 1 & 2 & 2 \\ 2 & 1 & 2 & 2 \\ 3 & 1 & 2 & 2 \\ 4 & 1 & 2 & 2 \\ 5 & 1 & 2 & 2 \\ 6 & 10 & 1 & 1 \end{array} \tag{3.8}$$

From theorem 3.3, If two treatment parameters τ_k and $\tau_{k'}$ are in an equivalent class, we see that β_1 and β_i are in an equivalent class of column effects. In this way, we verify that $\{\beta_1\}$, $\{\beta_2, \beta_3\}$ and $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$, $\{\alpha_6\}$ form equivalent classes of column and row effects respectively. It is also clear from inspection of (3.8). Any contrast among the parameters in an equivalent class is estimable.

In theorem 3.1, we prove that for fixed $i=2,\ldots,c$, $\zeta(\boldsymbol{\pi})=(\beta_1-\beta_i)+(\tau_k-\tau_{k'})$ is estimable. Thus, estimable functions between row(or column) and treatment effects are

$$(\beta_1-\beta_2)+(\tau_1-\tau_2), \quad (\beta_1-\beta_2)+(\tau_{10}-\tau_1),$$

$$(\alpha_1-\alpha_6)+(\tau_2-\tau_1), \quad (\alpha_1-\alpha_6)+(\tau_1-\tau_{10}).$$

In a block design, once we know the equivalent classes we know that only *within* classes contrasts are estimable. In a row-column design this is no longer true. It could now happen, as an example, treatment 1,2 and 3 are in different classes, but that $\tau_1-2\tau_2+\tau_3$ is an estimable function, in other words some *between* classes contrasts are estimable. Working with the tetra-differences involving y_{11} we find that only estimable between classes contrasts is $2\tau_1-\tau_2-\tau_{10}$.

Of course sum of estimable functions, such as $(\beta_2-\beta_3)$ + $(2\tau_1-\tau_2-\tau_{10})$ or $(\beta_1-\beta_2)+(\tau_1-\tau_2)$ + $(\beta_1-\beta_2)+(\tau_{10}-\tau_1)$ = $2(\beta_1-\beta_2)+(\tau_{10}-\tau_2)$, is estimable.

Example 3.2. When we exchange the treatment labels 6 and 2 in second row of the design in Example 3.1, the EQ(i)'s are changed as follows :

$$\text{EQ}(2) : \tau_1-\tau_3 = \tau_2-\tau_6 = \tau_9-\tau_8 = \tau_7-\tau_5 = \tau_4-\tau_5 = \tau_{10}-\tau_9$$

$$\text{EQ}(3) : \tau_1-\tau_2 = \tau_2-\tau_8 = \tau_9-\tau_{11} = \tau_7-\tau_{11} = \tau_4-\tau_3 = \tau_{10}-\tau_7$$

With the same procedure as above, we can easily verify that the changed design is connected. And so every contrast is estimable.

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