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Uniformly Minimum Variance Unbiased Estimation for Distributions with Support Depending on Two Parameters †

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ABSTRACT

When a random sample is taken from a certain class of discrete and continuous distributions whose support depend on two parameters, we could find that there exists the complete and sufficient statistic for parameters which belong to a certain class, and formulate the uniformly minimum variance unbiased estimator(*UMVUE*) of any estimable function. Some *UMVUE*'s of parametric functions are illustrated for the class of the distribution. Especially, we find that the *UMVUE* of some estimable parametric function from the truncated normal distribution could be expressed by the version of the Mill's ratio.

KEYWORDS : Completeness, Estimable parametric function, Mill's ratio, Minimal sufficiency,

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1. INTRODUCTION

It is well known that the completeness is not a property of a statistic, but a property of the family of distribution of the statistic. Stigler(1972) explained this point clearly with the motivation behind the use of the word "complete". He illustrated a random variable X having a discrete uniform distribution with probability density function (p.d.f.) given by

$$f_\theta(x) = \begin{cases} 1/\theta, & \text{if } x = 1, \dots, \theta \\ 0, & \text{otherwise} , \end{cases}$$

where $\theta \in \Theta$. Suppose that the parameter space $\Theta = N - \{m\}$, N is the set of all positive integers and m is any specified positive integer. Then X is not even boundedly complete, even though X is a sufficient statistic. Thus, we can see immediately that the family of the distribution induced by X is not complete.

Bahadur(1957) asserted in general that if the minimal sufficient statistic is not complete, then there must exist some estimable parametric function that does not admit a uniformly minimum variance unbiased estimator (*UMVUE*). However, Stigler(1972) showed that even though the random variable X is not complete, there exists a complete sufficient statistic $T(X)$ with

$$T(x) = \begin{cases} x, & \text{if } x \neq m \\ m+1, & \text{if } x = m , \end{cases}$$

and the *UMVUE* of θ is obtained as being a function of $T(X)$ and essentially a function of X such that

$$U(x) = \begin{cases} 2x-1, & \text{if } x = 1, \dots, m-1, m+2, \dots \\ 2m, & \text{if } x = m, m+1 . \end{cases}$$

He proved this result by using the well-known *Lehmann-Scheffe* theorem (See Lehmann 1983, p 77).

Via Stigler's example, Ghosh and Datta(1989) showed that every parametric function $\gamma(\theta)$ is estimable and has a *UMVUE* of $\gamma(\theta)$. Ghosh and Datta emphasized the importance of the minimality in Bahadur's theory and found that the *UMVUE* of $\gamma(\theta)$ could be obtained as a function of T which itself is a function of X. They explored this result within the cases of discrete distributions with support depending on parameter.

When density functions are absolutely continuous on $(\theta,\ c)$ and $(c,\ \theta)$, where c is a constant, which are called a Type I or II truncation parameter density, respectively. Tate(1959) and Zacks(1971) obtained the $UMVUE$ of any estimable parameter function, which is the function of the minimal sufficient complete statistic $X_{(1)} = \min\{X_1, \cdots, X_n\}$ or $X_{(n)} = \max\{X_1, \cdots, X_n\}$.

In this paper, we extend the results of Ghosh and Datta(1989), Tate(1959) and Zacks(1971) not only to the cases of discrete and continuous distributions with support depending on two parameters but also to the cases of which $X_{(1)}$ and $X_{(n)}$ are not complete statistics any more. Under such circumstances, we find that there exists the complete and minimal sufficient statistic for parameters which belong to a certain parameter space. Those are discussed in section 2. Moreover we obtain the $UMVUE$'s of every parametric functions for discrete and continuous density functions in section 3. Also we can show that results of Ghosh and Datta, Tate and Zacks are the special cases of our results. In section 4 many examples for some parametric functions consisted with two parameters are illustrated.

2. MINIMAL SUFFICIENCY AND COMPLETENESS

2.1 Discrete Models

Consider parameter space $\Theta = \{\theta_1, \theta_2, \ldots\} \subset I$, a set of integers. For two parameters θ_i and θ_j $(\theta_i < \theta_j)$ which belong to Θ, let X be a discrete random variable with p.d.f. given by

$$P_{\theta_i,\theta_j}(x) = P_{\theta_i,\theta_j}(X = x) = \begin{cases} h(x)/S(\theta_i,\theta_j), & \text{for } x \in A(\theta_i,\theta_j) \\ 0, & \text{otherwise }, \end{cases} \tag{2.1}$$

where $A(\theta_i, \theta_j)$ is a non-empty subset of I and

$$S(\theta_i,\theta_j) = \sum_{x \in A(\theta_i,\theta_j)} h(x).$$

Note that $A(\theta_i, \theta_j) = \{x \mid x = \theta_i, \ldots, \theta_j\}$ may not be a subset of Θ (see examples 4.1 and 4.2).

Let $X_1, \ldots, X_n$ be a random sample of size n $(n \geq 2)$ from the distribution of X given by (2.1). Then for any $\theta_i, \theta_j \in \Theta$,

$$\begin{aligned} & P_{\theta_i,\theta_j}(X_1 = x_1, \ldots, X_n = x_n) \\ = & \prod_{k=1}^{n} h(x_k) I\{x_k \in A(\theta_i,\theta_j)\}/S(\theta_i,\theta_j)^n \\ = & \prod_{k=1}^{n} h(x_k) I\{x_{(1)} \in A(\theta_i,\theta_j)\} I\{x_{(n)} \in A(\theta_i,\theta_j)\}/S(\theta_i,\theta_j)^n, \end{aligned} \tag{2.3}$$

where $x_{(1)} = \min\{x_1, \ldots, x_n\}$ and $x_{(n)} = \max\{x_1, \ldots, x_n\}$. By the factorization theorem, $(X_{(1)}, X_{(n)})$ is jointly sufficient for (θ_i, θ_j) but may not be minimal. In order to find a minimal sufficient statistic for (θ_i, θ_j), we first show the existence of the sufficient and complete statistic. Here we might extend a mapping $T(\cdot)$ of Ghosh and Datta(1989) to the support depending on two parameters, so that define a function $T(\cdot)$ from $A(\theta_i, \theta_j)$ to $B(\theta_i, \theta_j) = \{t \mid t = \theta_i, \ldots, \theta_j\}$, which should be a subset of Θ. Since the support $A(\theta_i, \theta_j)$ of the random variable X may not belong to the parameter space Θ, $(X_{(1)}, X_{(n)})$ could not be complete and minimal as well. By considering a function $T(\cdot)$, the support $B(\theta_i, \theta_j)$ of the random variable T must belong to the parameter space Θ (see examples 4.1 and 4.2). Now (2.3) will become

$$\prod_{k=1}^{n} h(t_k) I\{t_{(1)} \in B(\theta_i,\theta_j)\} I\{t_{(n)} \in B(\theta_i,\theta_j)\}/S(\theta_i,\theta_j)^n. \tag{2.4}$$

Then $S(\theta_i, \theta_j)$ in (2.2) equals to

$$S(\theta_i,\theta_j) = \sum_{t_k \in B(\theta_i,\theta_j)} h(t_k). \tag{2.5}$$

Note that $(T_{(1)}, T_{(n)}) \equiv (T(X_{(1)}), T(X_{(n)}))$ is sufficient for (θ_i, θ_j).

Define $\mathcal{P}^T$ as a certain class of distributions $\{\mathcal{P}^T_{\theta_i,\theta_j}$ for all $\theta_i, \theta_j \in \Theta\}$, where $P^T_{\theta_i,\theta_j}(t_1, t_n) = P_{\theta_i,\theta_j}(T_{(1)} = t_1, T_{(n)} = t_n)$. Now we will show the family $\mathcal{P}^T$ induced by $T_{(1)}$ and $T_{(n)}$ is complete. For θ_k, and $\theta_l \in B(\theta_i, \theta_j)$ with $i \leq k \leq l \leq j$, $P^T_{\theta_i,\theta_j}(T_{(1)} \leq \theta_k, T_{(n)} \leq \theta_l) = [F^T(\theta_l)]^n - [F^T(\theta_l) - F^T(\theta_k)]^n$, where $F^T(t) = P_{\theta_i,\theta_j}(T \leq t)$, so that $T_{(1)}$ and $T_{(n)}$ has the jointly p.d.f.(see David (1981))

$$P^T_{\theta_i,\theta_j}(\theta_k, \theta_l) = P_{\theta_i,\theta_j}(T_{(1)} = \theta_k, T_{(n)} = \theta_l)$$

$$= -[F^T(\theta_l) - F^T(\theta_k)]^n - [F^T(\theta_{l-1}) - F^T(\theta_{k-1})]^n$$
$$+[F^T(\theta_l) - F^T(\theta_{k-1})]^n + [F^T(\theta_{l-1}) - F^T(\theta_k)]^n.$$

From (2.4) and (2.5), one gets

$$P_{\theta_i,\theta_j}(T_{(1)} = \theta_k, T_{(n)} = \theta_l) =$$
$$[S(\theta_k,\theta_l)^n + S(\theta_{k+1},\theta_{l-1})^n - S(\theta_{k+1},\theta_l)^n - S(\theta_k,\theta_{l-1})^n]/S(\theta_i,\theta_j)^n, (> 0)$$

where $S(\theta_k,\theta_l) = 0$ for any k and l such that $l < k$.

Consider any function $g(\cdot,\cdot)$ with

$$E_{\theta_i,\theta_j} g(T_{(1)}, T_{(n)}) = 0 \quad \text{for all} \quad \theta_i, \quad \theta_j \in \Theta.$$

For the sake of simplicity, henceforth we will define $S(\theta_i,\theta_j)$ as S_{ij} . Then this implies

$$\sum_{k=i}^{j}\sum_{l=k}^{j} g(\theta_k,\theta_l)[S^n_{k,l} + S^n_{k+1,l-1} - S^n_{k+1,1} - S^n_{k,l-1}]/S(\theta_i,\theta_j)^n = 0. (2.6)$$

For $i = j$, (2.6) implies that $g(\theta_i,\theta_i) = 0$. Hence we explore the case that $i < j$. Under θ_{i+1} and $\theta_j \in \Theta$,

$$E_{\theta_{i+1},\theta_j} g(T_{(1)}, T_{(n)}) = 0,$$

which implies that

$$\sum_{k=i+1}^{j}\sum_{l=k}^{j} g(\theta_k,\theta_l)[S^n_{k,l} + S^n_{k+1,l-1} - S^n_{k+1,1} - S^n_{k,l-1}]/S^n_{i,j} = 0. \quad (2.7)$$

With similar arguments under θ_{i+1} and/or θ_{j-1}, one obtains

$$\sum_{k=i}^{j-1}\sum_{l=k}^{j-1} g(\theta_k,\theta_l)[S^n_{k,l} + S^n_{k+1,l-1} - S^n_{k+1,1} - S^n_{k,l-1}]/S^n_{i,j} = 0, \quad (2.8)$$
$$\sum_{k=i+1}^{j-1}\sum_{l=k}^{j-1} g(\theta_k,\theta_l)[S^n_{k,l} + S^n_{k+1,l-1} - S^n_{k+1,1} - S^n_{k,l-1}]/S^n_{i,j} = 0. \quad (2.9)$$

Taking calculation for (2.6) − (2.7) − (2.8) + (2.9), we get for all $\theta_i, \theta_j \in \Theta$,

$$g(\theta_i,\theta_j)[S^n_{i,j} + S^n_{i+1,j-1} - S^n_{i+1,j} - S^n_{i,j-1}]/S^n_{i,j} = 0,$$

so that $g(\theta_i,\theta_j) = 0$ for all $\theta_i, \theta_j \in \Theta$. We conclude that for all $\theta_i, \theta_j \in \Theta$ $(i \leq j)$, $E_{\theta_i,\theta_j} g(T_{(1)}, T_{(n)}) = 0$ implies $g(T_{(1)}, T_{(n)}) = 0$ except on a set of points that has probabilty zero, i.e., the family $\mathcal{P}^T$ induced by $T_{(1)}$ and $T_{(n)}$ is complete. Therefore, one can conclude there exists $(T_{(1)}, T_{(n)})$ which is the joint minimal sufficient and complete statistic for (θ_i, θ_j).

2.2 Continuous Models

In order to distinguish parameters of discrete density function, we will use different subscriptions. For parameters θ_1 and $\theta_2 \in \Theta \subset R$, let X be a continuous random variable with p.d.f. given by

$$f_{\theta_1,\theta_2}(x) = \begin{cases} h(x)/S(\theta_1,\theta_2), & \text{if } x \in A(\theta_1,\theta_2) \\ 0, & \text{otherwise} \end{cases}, \tag{2.10}$$

where $A(\theta_1,\theta_2)$ is some non-empty subset of real line which is an interval between θ_1 and θ_2, and for all θ_1 and $\theta_2 \in \Theta$, $S(\theta_1,\theta_2)$ could be defined as

$$S(\theta_1,\theta_2) = \int_{x \in A(\theta_1,\theta_2)} h(x)\, dx.$$

Let $X_1, \ldots, X_n$ be a random sample of size $n (n \geq 2)$ from the distribution of X given by (2.10). Then

$$\begin{aligned} f_{\theta_1,\theta_2}(x_1,\ldots,x_n) &= \frac{\prod_{i=1}^n h(x_i)}{[S(\theta_1,\theta_2)]^n} I\{x_i \in A(\theta_1,\theta_2)\} \\ &= \frac{\prod_{i=1}^n h(x_i)}{[S(\theta_1,\theta_2)]^n} I\{x_{(1)} \in A(\theta_1,\theta_2)\} I\{x_{(n)} \in A(\theta_1,\theta_2)\}, \end{aligned} \tag{2.11}$$

where $x_{(1)} = \min\{x_1,\ldots,x_n\}$ and $x_{(n)} = \max\{x_1,\ldots,x_n\}$. By the factorization theorem, $(X_{(1)}, X_{(n)})$ is jointly sufficient for (θ_1,θ_2). But $(X_{(1)}, X_{(n)})$ may not necessarily be a minimal sufficient for (θ_1,θ_2). In order to find a minimal sufficient statistics, find a function $T(\cdot)$ from $A(\theta_1,\theta_2)$, which may not be a subset of Θ, to a counter domain $B(\theta_1,\theta_2) = \{$ an interval between θ_1 and $\theta_2\}$, which is necessarily a subset of Θ, and satisfies that for t_1 and $t_2 \in B(\theta_1,\theta_2)$ with $t_1 < t_2$,

$$\{T(x_{(1)}) \leq t_1,\ \ T(x_{(n)}) \leq t_2\} \text{ iff } \{x_{(1)} \leq t_1,\ \ x_{(n)} \leq t_2\}.$$

Note that the function $T(\cdot)$ is not always an identity function on a domain $A(\theta_1, \theta_2)$ (see example 4.4). Hence (2.11) turns out to be

$$f_{\theta_1,\theta_2}(x_1, \ldots, x_n) = \frac{\prod_{i=1}^{n} h(x_i)}{[S(\theta_1, \theta_2)]^n}\ \ I\{T(x_{(1)}) \in B(\theta_1, \theta_2)\} I\{T(x_{(n)}) \in B(\theta_1, \theta_2)\}.$$

Therefore $(T_1, T_2) \equiv (T(X_{(1)}), T(X_{(n)}))$ is jointly sufficient for (θ_1, θ_2) by the factorization theorem.

Now we will show the family of distribution induced by (T_1, T_2) is complete. The family of distributions would be denoted $\mathcal{P}^T = \{f^T_{\theta_1,\theta_2}\ ;\ \theta_1 \text{ and } \theta_2 \in \Theta\}$. For t_1 and $t_2 \in B(\theta_1, \theta_2)$ with $t_1 < t_2$, (T_1, T_2) has the joint c.d.f.

$$\begin{aligned} F^T_{\theta_1,\theta_2}(t_1, t_2) &= P[T_1 \leq t_1,\ T_2 \leq t_2] \\ &= P[\{X_{(1)} \leq t_1,\ X_{(n)} \leq t_2] \\ &= [F_{\theta_1,\theta_2}(t_2)]^n - [F_{\theta_1,\theta_2}(t_2) - F_{\theta_1,\theta_2}(t_1)]^n, \end{aligned} \tag{2.12}$$

where $F_{\theta_1,\theta_2}(\cdot)$ is the cumulative distribution function (c.d.f.) of the random variable defined in (2.10). If the joint c.d.f. is differentiable with respect to t_1 and $t_2 \in B(\theta_1, \theta_2)$, then one can get

$$F^T_{\theta_1,\theta_2}(t_2) - F^T_{\theta_1,\theta_2}(t_1) = \int_{t_1}^{t_2} f_{\theta_1,\theta_2}(x)\ dx = \frac{S(t_1, t_2)}{S(\theta_1, \theta_2)} \quad \text{for} \quad t_1 < t_2, \tag{2.13}$$

and from (2.12) and (2.13), we obtain

$$f^T_{\theta_1,\theta_2}(t_1, t_2) = n(n-1)[S(t_1, t_2)]^{n-2} h(t_1) h(t_2) / [S(\theta_1, \theta_2)]^n. \tag{2.14}$$

But if the joint c.d.f. ${F_{\theta_1,\theta_2}}^T(t_1, t_2)$ is not differntiable at some points t_1 and $t_2 \in B(\theta_1, \theta_2)$, the joint p.d.f. can be defined as

$${f_{\theta_1,\theta_2}}^T(t_1, t_2) = {F_{\theta_1,\theta_2}}^T(t_1, t_2) - {F_{\theta_1,\theta_2}}^T({t_1}^-, t_2) - {F_{\theta_1,\theta_2}}^T(t_1, {t_2}^-) + {F_{\theta_1,\theta_2}}^T({t_1}^-, {t_2}^-).$$

Hence, in order to show the completeness we consider any function $g(T_1, T_2)$ such that

$$E_{\theta_1,\theta_2}[g(T_1, T_2)] = 0 \quad \text{for} \quad \theta_1 \text{ and } \theta_2 \in \Theta.$$

The distribution function of T_1 and T_2 might be expressed as the linear combination of absolutely continuous and discrete distribution functions. For the discrete parts, we already showed the completeness for the family induced by $(T_{(1)}, T_{(n)})$ in section 2.1. So we here prove the case of absolutely continuous distribution.

The above expectation implies that for all θ_1 and $\theta_2 \in \Theta$

$$E_{\theta_1,\theta_2}[g(T_1, T_2)] = 0,$$

iff

$$\int_{\theta_1}^{\theta_2} \int_{\theta_1}^{t_2} g(t_1, t_2)[S(t_1, t_2)]^{n-2} h(t_1) h(t_2)\, dt_1 dt_2 = 0.$$

Taking the second derivative of the above equation with respect to both θ_1 and θ_2 turns out to be

$$\frac{\partial^2}{\partial\theta_1 \partial\theta_2} \int_{\theta_1}^{\theta_2} \int_{\theta_1}^{t_2} g(t_1, t_2)[S(t_1, t_2)]^{n-2} h(t_1) h(t_2)\, dt_1 dt_2 = 0,$$

iff

$$\frac{\partial}{\partial\theta_1} \int_{\theta_1}^{t_2} g(t_1, \theta_2)[S(t_1, \theta_2)]^{n-2} h(t_1) h(\theta_2) dt_1 = 0,$$

iff

$$g(\theta_1, \theta_2)[S(\theta_1, \theta_2)]^{n-2} h(\theta_1) = 0,$$

iff

$$g(\theta_1, \theta_2) = 0,$$

for all θ_1 and $\theta_2 \in \Theta$ except on a set of points that has probability zero.

Therefore the family of distributions $\mathcal{P}^T$ induced by $(T(X_{(1)}), T(X_{(n)}))$ is complete. Since the complete sufficient statistic is always minimal, $(T(X_{(1)}), T(X_{(n)}))$ is the complete and minimal sufficient statistic for (θ_1, θ_2). Hence after combining this completeness for continuous models with that of discrete models, we could state the following theorem.

Theorem 1. If $X_1, \ldots, X_n$ is a random sample of size $n (n \geq 2)$ from the distributions given (2.1) and (2.10), $(T(X_{(1)}), T(X_{(n)}))$ is the jointly complete and minimal sufficient statistic for (θ_1, θ_2).

3. UMVUE

3.1 Discrete Models

We consider every estimable parametric function $\gamma(\theta_i, \theta_j)$ for $\theta_i, \theta_j \in \Theta$. Now it will be shown that every parametric functions $\gamma(\theta_i, \theta_j)$ has an unbiased estimator based on $(T_{(1)}, T_{(n)})$. Then this estimator is the *UMVUE* of $\gamma(\theta_i, \theta_j)$ by the Rao-Blackwell-Lehmann-Scheffe Theorem(see Lehmann 1983, p.77).

In order to prove the existance of *UMVUE*, suppose $U(T_{(1)}, T_{(n)})$ is an unbiased estimator of $\gamma(\theta_i, \theta_j)$ for all $\theta_i, \theta_j \in \Theta$.

$$E_{\theta_i,\theta_j} U(T_{(1)}, T_{(n)}) = \gamma(\theta_i, \theta_j) \equiv \gamma_{i,j}(\text{say}),$$

iff

$$\gamma_{i,j} S_{i,j}^n = \sum_{k=i}^{j} \sum_{l=k}^{j} U(\theta_k, \theta_l)[S_{k,l}^n + S_{k+1,l-1}^n - S_{k+1,1}^n - S_{k,l-1}^n].$$

That is,

$$\gamma_{i,j} S_{i,j}^n = \gamma_{i+1,j} S_{i+1,j}^n + \gamma_{i,j-1} S_{i,j-1}^n - \gamma_{i+1,j-1} S_{i+1,j-1}^n$$
$$+ U(\theta_i, \theta_j)[S_{i,j}^n + S_{i+1,j-1}^n - S_{i+1,j}^n - S_{i,j-1}^n].$$

Therefore, the *UMVUE* of $\gamma(\theta_i, \theta_j)$ could be expressed as

$$U(\theta_i, \theta_j) = \frac{\gamma_{i,j} S_{i,j}^n + \gamma_{i+1,j-1} S_{i+1,j-1}^n - \gamma_{i+1,j} S_{i+1,j}^n - \gamma_{i,j-1} S_{i,j-1}^n}{S_{i,j}^n + S_{i+1,j-1}^n - S_{i+1,j}^n - S_{i,j-1}^n},$$

where θ_i and θ_j are the observed values of $T_{(1)}$ and $T_{(n)}$, respectively. So the following theorem could be stated.

Theorem 2. Let $(T_{(1)}, T_{(n)})$ be the jointly complete and minimal sufficient statistic for the distribution with the support $B(\theta_1, \theta_2)$ defined in (2.5). The *UMVUE* of any estimable parametric function $\gamma(\theta_i, \theta_j)$ could be expressed as

$$U(t_i, t_j) = \frac{\begin{bmatrix} \gamma(t_i, t_j) S(t_i, t_j)^n + \gamma(t_{i+1}, t_{j-1}) S(t_{i+1}, t_{j-1})^n \\ -\gamma(t_{i+1}, t_j) S(t_{i+1}, t_j)^n - \gamma(t_i, t_{j-1}) S(t_i, t_{j-1})^n \end{bmatrix}}{S(t_i, t_j)^n + S(t_{i+1}, t_{j-1})^n - S(t_{i+1}, t_j)^n - S(t_i, t_{j-1})^n}, \quad (3.1)$$

where t_i and $t_j \in B(\theta_i, \theta_j)$ with $t_i < t_j$.

If we consider a support of a random variable depends on one parameter, for example $\theta_j(\theta_i$ is a given constant), then it was the task of Ghosh and Datta(1989). Since there is no interest with the terms including t_{i+1} in (3.1), for the observed value t_j of $T_{(n)}$, the *UMVUE* of $\gamma(\theta_j)$ could be formulated as following :

Corollary 1. The *UMVUE* of $\gamma(\theta_j)$ is

$$U(t_j) = \frac{\gamma(t_j)S_1(t_j)^n - \gamma(t_{j-1})S_1(t_{j-1})^n}{S_1(t_j)^n - S_1(t_{j-1})^n}, \tag{3.2}$$

where $S_1(t_j) \equiv S(\theta_i, t_j | \theta_i \text{ is constant }) = \sum_{k=\theta_i}^{t_j} h(k)$.

This *UMVUE* of $\gamma(\theta_j)$ coincides with (2.8) expression in Ghosh and Datta(1989).

Now assume that θ_j is a given constant in $B(\theta_i, \theta_j)$, and a support depends on θ_i only, so that support $B(\theta_i, \theta_j)$ may be written $B_2(\theta_i) \equiv \{t = \theta_i, \ldots, \theta_j | \theta_j \text{is constant}\}$. And for $t_i \in B_2(\theta_i)$, $S(t_i, \theta_j)$ might be $S(t_i, \theta_j \mid \theta_j$ is constant) $\equiv S_2(t_i) = \sum_{k=t_i}^{\theta_j} h(k)$. With the similar arguments to obtain the formulation (3.2), the *UMVUE* of $\gamma(\theta_i)$ could be expressed as following :

Corollary 2

$$U(t_i) = \frac{\gamma(t_i)S_2(t_i)^n - \gamma(t_{i+1})S_2(t_{i+1})^n}{S_2(t_i)^n - S_2(t_{i+1})^n}, \tag{3.3}$$

where t_i is the observed value of $T_{(1)}$.

Note that for any $\theta_i \in \Theta$, the p.d.f. of the minimal sufficient and complete statistic $T_{(1)}$ is given by

$$\begin{aligned} P_{\theta_i}(T_{(1)} = t_k) &= P_{\theta_i}(T_{(1)} \geq t_k) - P_{\theta_i}(T_{(1)} \geq t_{k+1}) \\ &= [1 - F^T(t_{k-1})]^n - [1 - F^T(t_k)]^n \\ &= [S_2(t_k)^n - S_2(t_{k+1})^n]/S_2(\theta_i)^n. \end{aligned}$$

3.2 Continuous Models

Next, we show that every parametric function $r(\theta_1, \theta_2)$ has an unbiased estimator based on $(T_1, T_2) = (T(X_{(1)}), T(X_{(n)}))$, which is then the *UMVUE* of $\gamma(\theta_1, \theta_2)$ by *Rao-Blackwell-Lehmann-Sheffe* theorem.

Theorem 3. Let the random variable (T_1, T_2) follow an absolutely continuous distribution with support $B(\theta_1, \theta_2)$ depending on two parameters. Then the $UMVUE$ of any parametric function $\gamma(\theta_1, \theta_2)$ based on (T_1, T_2) is

$$U(t_1, t_2) = -\frac{\gamma_{12}''S^2 + nS(\gamma_1'S_2' + \gamma_2'S_1') + n(n-1)\gamma S_1'S_2' + n\gamma SS_{12}''}{n(n-1)h(t_1)h(t_2)}, \quad (3.4)$$

where γ, γ_1', γ_2', γ_{12}'', S, S_1', S_2' and S_{12}'' is defined in equation (3.6) with substituting θ with t.

Proof. Suppose that there exists $U(T_1, T_2)$ which is an unbiased estimator of $\gamma(\theta_1, \theta_2)$ such that, for all θ_1 and $\theta_2 \in \Theta$

$$E_{\theta_1,\theta_2}[U(T_1, T_2)] = \gamma(\theta_1, \theta_2).$$

One gets from (2.14),

$$\int_{\theta_1}^{\theta_2} \int_{\theta_1}^{t_2} U(t_1, t_2)n(n-1)[S(t_1, t_2)]^{n-2}h(t_1)h(t_2)dt_1dt_2 = \gamma(\theta_1, \theta_2)[S(\theta_1, \theta_2)]^n. \quad (3.5)$$

Taking derivative on both sides of (3.5) with respect to θ_1 and θ_2, the RHS becomes

$$\begin{aligned} &\frac{\partial^2}{\partial\theta_1\partial\theta_2}\gamma(\theta_1, \theta_2)[S(\theta_1, \theta_2)]^n \\ &\quad = \gamma_{12}''S^n + n\gamma_1'S_2'S^{n-1} + n\gamma_2'S^{n-1}S_1' \\ &\quad\quad + n(n-1)\gamma S^{n-2}S_1'S_2' + n\gamma S^{n-1}S_{12}'', \end{aligned} \quad (3.6)$$

where $\gamma = \gamma(\theta_1, \theta_2)$, $\gamma_1' = \partial\gamma/\partial\theta_1$, $\gamma_{12}'' = \partial^2\gamma/\partial\theta_1\partial\theta_2$, $S = S(\theta_1, \theta_2)$, $S_1' = \partial S/\partial\theta_1$ and $S_{12}'' = \partial^2 S/\partial\theta_1\partial\theta_2$.
The LHS of (3.5) becomes

$$\begin{aligned} &\frac{\partial^2}{\partial\theta_1\partial\theta_2}\int_{\theta_1}^{\theta_2} \int_{\theta_1}^{t_2} U(t_1, t_2)n(n-1)[S(t_1, t_2)]^{n-2}h(t_1)h(t_2)\,dt_1dt_2 \\ &\quad = \frac{\partial}{\partial\theta_1}\int_{\theta_1}^{\theta_2} U(t_1, \theta_2)n(n-1)[S(t_1, \theta_2)]^{n-2}h(t_1)h(\theta_2)dt_1 \end{aligned}$$

$$= -U(\theta_1, \theta_2) n(n-1)[S(\theta_1, \theta_2)]^{n-2} h(\theta_1) h(\theta_2). \tag{3.7}$$

Put (3.6) and (3.7) together, then

$$\begin{aligned} U(\theta_1, \theta_2) &= \frac{\gamma_{12}'' S^2 + n\gamma_1' S_2' S + n\gamma_2' S S_1' + n(n-1)\gamma S_1' S_2' + n\gamma S S_{12}''}{-n(n-1)h(\theta_1)h(\theta_2)} \\ &= -\frac{\gamma_{12}'' S^2 + nS(\gamma_1' S_2' + \gamma_2' S_1') + n(n-1)\gamma S_1' S_2' + n\gamma S S_{12}''}{n(n-1)h(\theta_1)h(\theta_2)}, \end{aligned}$$

if $h(\theta_1) \neq 0$ and $h(\theta_2) \neq 0$. Therefore, the uniformly minimum variance unbiased estimate of $\gamma(\theta_1, \theta_2)$ corresponding to the observed value (t_1, t_2) of (T_1, T_2) is given by (3.4). □

Next as restricted cases of Theorem 3, we consider the absolutely continuous distribution whose support depends on one parameter. The p.d.f. in (2.10) would be given by

$$f_\theta(x) = \begin{cases} h(x)/S(\theta), & \text{if } x \in A(\theta) \\ 0, & \text{otherwise} \end{cases}, \tag{3.8}$$

where $A(\theta)$ can be either $A(\theta_1 \; ; \; \theta_2 = \theta)$ with constant θ_1 or $A(\theta_1 = \theta \; ; \; \theta_2)$ with constant θ_2, which are denoted as $A_1(\theta)$ and $A_2(\theta)$, respectively. Each parameter space Θ_1 and Θ_2 would be subsets of $\{\theta \mid \theta > \theta_1\}$ and $\{\theta \mid \theta < \theta_2\}$, respectively. Also $S(\theta)$ in (3.8) is defined as, for all $\theta \in \Theta_1$ or Θ_2,

$$S(\theta) = \int_{x \in A(\theta)} h(x) dx.$$

In two cases of $A(\theta)$, there exists some function $T(\cdot)$ which could be defined so that $T_1 = T(X_{(1)})$ or $T_2 = T(X_{(n)})$ could be the minimal sufficient and complete statistics. When X is a discrete random variable, Ghosh and Datta(1989) already obtained the *UMVUE* of any parametric function $\gamma(\theta)$ with the support $A_1(\theta)$. We extend to get *UMVUE* with $A_2(\theta)$ as well as $A_1(\theta)$ in section 3.1. And when $h(x)$ is absolutely continuous over two kinds of $A(\theta)$, Tate(1959) and Zacks(1971) obtained the *UMVUE* of $\gamma(\theta)$ which is the function of the minimal sufficient statistic $X_{(1)}$ or $X_{(n)}$. Now we can state that every parametric function $\gamma(\theta)$ has an unbiased estimator based on T_1 and T_2, which is the *UMVUE*.

First, we deal with $A_2(\theta)$, where Zacks called this density function as a Type *I* truncation parameter density.

Corollary 3. The $UMVUE$ of $\gamma(\theta)$ based on T_1 is

$$U_2(t) = -\gamma(t) - \frac{\gamma'(t)S(\theta,t)}{nh(t)} \qquad \text{if } h(t) \neq 0\ , \tag{3.9}$$

where t is the observed value of T_1 .

Now consider the case of $A_1(\theta)$, where Zacks called this density function a Type II truncation parameter density.

Corollary 4. The $UMVUE$ of $\gamma(\theta)$ based on T_2 is

$$U_1(t) = \gamma(t) + \frac{\gamma'(t)S(\theta_1,t)}{nh(t)} \qquad \text{if } h(t) \neq 0\ , \tag{3.10}$$

where t is the observed value of T_2.

These two corollaries coincide with Tate(1959) and Zacks(1971)'s Theorems if T_1 and T_2 are $X_{(1)}$ and $X_{(n)}$, respectively.

4. SOME ILLUSTRATIVE EXAMPLES

4.1 Discrete Uniform Distributions.

Consider a parameter space $\Theta = \{$ even integers$\}$. For any $\theta_i, \theta_j \in \Theta$, let the random variable X have the following p.d.f.

$$\begin{aligned} P_{\theta_i,\theta_j}(x) &= 2(\theta_j - \theta_i + 2)^{-1}, && \text{for } x \in \{\theta_i, \theta_{i+2}, \ldots, \theta_{j-2}, \theta_j\} \subset \Theta \\ &= 0, && \text{otherwise.} \end{aligned}$$

Then $(T_{(1)}, T_{(n)}) = (X_{(1)}, X_{(n)})$ is a jointly minimal sufficient and complete statistic, and from (3.1) the $UMVUE$ of $\gamma(\theta_i, \theta_j) = \theta_i + \theta_j$ can be obtained

$$\begin{aligned} U(X_{(1)}, X_{(n)}) = [&(X_{(1)} + X_{(n)})(X_{(n)} - X_{(1)} + 2)^n \\ &+(X_{(1)} + X_{(n)})(X_{(n)} - X_{(1)} - 2)^n \\ &-(X_{(1)} + X_{(n)} + 2)(X_{(n)} - X_{(1)})^n \end{aligned}$$

$$-(X_{(1)} + X_{(n)} - 2)(X_{(n)} - X_{(1)})^n]$$

$$/[(X_{(n)} - X_{(1)} + 2)^n + (X_{(n)} - X_{(1)} - 2)^n$$

$$-2(X_{(n)} - X_{(1)})^n].$$

If $n = 2$, one can easily get and prove that the *UMVUE* is $X_{(1)} + X_{(2)}$.

For $\theta_i, \theta_j \in \Theta^* = \{$ integers $\} - \{m, m+1\}$, where m is any specfied integer, let X be a random variable whose p.d.f. is given by

$$\begin{aligned} P_{\theta_i,\theta_j}(x) = &\quad (\theta_j - \theta_i + 1)^{-1}, \quad \text{for } x \in \{\theta_i, \cdots, m-1, m, m+1, m+2, \cdots, \theta_j\} \\ = &\quad 0, \quad \text{otherwise.} \end{aligned}$$

In order to find a complete statistic, we may consider the function $T(\cdot)$ such that

$$T(x) = \begin{cases} m-1, & \text{if } x = m \\ m+2, & \text{if } x = m+1 \\ x, & \text{otherwise.} \end{cases}$$

Then it is trivial to show that $(T_{(1)}, T_{(n)})$ is a jointly minimal sufficient and complete statistic. Therefore the *UMVUE* of $\gamma(\theta_i, \theta_j)$ could be obtained by using (3.1) in Theorem 2.

4.2 Restricted Discrete Triangular Distribution.

For $\theta \in \Theta^* = \{$ negative integers $\} - \{m\}$, where m is any specfied negative integer, let X be a random variable whose p.d.f. is given by

$$\begin{aligned} P_\theta(x) = &\quad -2x/[\theta(\theta-1)], \quad \text{for } x \in A_2(\theta) = \{\theta, \theta+1, \ldots, -1\} \\ = &\quad 0, \quad \text{otherwise.} \end{aligned}$$

Note that $S_2(\theta) = \sum_{x=\theta}^{-1}(-2x) = \theta(\theta-1)$ and $X_{(1)}$ is no longer complete. In order to find a complete statistic, consider the function $T(\cdot)$ such that

$$T(x) = \begin{cases} m-1, & \text{if } x = m \\ x, & \text{otherwise.} \end{cases}$$

Then there exists a minimal sufficient and complete statistic $T_{(1)}$. And the *UMVUE* of $\gamma(\theta) = \theta \in \Theta^*$ can be found

$$
\begin{aligned}
U(T_{(1)}) &= \frac{T_{(1)}^{n+1}(T_{(1)}-1)^n-(T_{(1)}+1)^{n+1}T_{(1)}^n}{T_{(1)}^n(T_{(1)}-1)^n-(T_{(1)}+1)^nT_{(1)}^n}, \quad \text{if } t \neq m-1 \\
&= \frac{(m-1)^{n+1}(m-2)^n-(m+1)^{n+1}m^n}{(m-1)^n(m-2)^n-(m+1)^nm^n}, \quad \text{if } t = m-1.
\end{aligned}
$$

If $n = 1$, one can easily show and prove that the *UMVUE* of $\gamma(\theta) = \theta$ is

$$
U(T_{(1)}) = \begin{cases} \frac{3T_{(1)}+1}{2}, & \text{if } t \neq m-1 \\ \frac{3m^2-2m+1}{2m-1}, & \text{if } t = m-1. \end{cases}
$$

4.3 Truncated Triangular Distribution.

For all $\theta_i, \theta_j \in \Theta = \{$ natural numbers $\}$, let

$$
P_{\theta_i,\theta_j} = \begin{cases} (2x)/[\theta_j(\theta_j+1)-\theta_i(\theta_i-1)], & \text{if } x \in A(\theta_i,\theta_j) = \{\theta_i,\ldots,\theta_j\} \\ 0, & \text{otherwise.} \end{cases}
$$

Notes that $S(\theta_i,\theta_j) = \sum_{x=\theta_i}^{\theta_j} 2x$, and $(T_{(1)} = X_{(1)}, T_{(n)} = X_{(n)})$ is jointly minimal sufficient and complete. Taking $\gamma(\theta_i,\theta_j) = (\theta_i+\theta_j)^{-2}$, the *UMVUE* of $(\theta_i+\theta_j)^{-2}$ can be given by $U(T_{(1)}, T_{(n)})$ as the following :

$$
\begin{aligned}
U(X_{(1)}, X_{(n)}) &= [(X_{(1)}+X_{(n)})^{n-2}(X_{(n)}-X_{(1)}-1)^n \\
&\quad +(X_{(1)}+X_{(n)})^{n-2}(X_{(n)}-X_{(1)}+1)^n \\
&\quad -(X_{(1)}+X_{(n)}+1)^{n-2}(X_{(n)}-X_{(1)})^n \\
&\quad -(X_{(1)}+X_{(n)}-1)^{n-2}(X_{(n)}-X_{(1)})^n] \\
&\quad [(X_{(1)}+X_{(n)})^n(X_{(n)}-X_{(1)}-1)^n \\
&\quad +(X_{(1)}+X_{(n)})^n(X_{(n)}-X_{(1)}+1)^n \\
&\quad -(X_{(1)}+X_{(n)}+1)^n(X_{(n)}-X_{(1)})^n \\
&\quad -(X_{(1)}+X_{(n)}-1)^n(X_{(n)}-X_{(1)})^n]^{-1}, \quad \text{if } X_{(1)} < X_{(n)} \\
&= 1/(2X_{(1)})^n, \quad \text{if } X_{(1)} = X_{(n)}.
\end{aligned}
$$

If $n = 2$, the *UMVUE* of $(\theta_i + \theta_j)^{-2}$ is

$$
U(X_{(1)}, X_{(2)}) = 1/[4X_{(1)}X_{(2)}], \text{if } X_{(1)} \leq X_{(2)}
$$

Now we take the expectation of the above *UMVUE*.

$$E[U(X_{(1)}, X_{(2)})] = \sum_{t_1=\theta_i}^{\theta_j} \sum_{t_2=t_1}^{\theta_j} \frac{1}{[4t_1t_2]}$$
$$= \frac{1}{(\theta_i + \theta_j)^2} ,$$

which completes the unbiasedness.

4.4 Restricted Continuous Uniform Distribution.

For θ_1 and $\theta_2 \in \Theta^* \equiv \{\theta \mid 0 < \theta \leq \theta_0 \quad \text{or} \quad \theta > \theta_0 + 2r\}$, where θ_0 and r are known constants, let $X_1, X_2, \ldots, X_n$ be a continuous random sample of size n which follows a uniform distribution from θ_1 to θ_2. Then $h(x)$ and $S(\theta_1, \theta_2)$ in (2.1) turn out to be 1 and $(\theta_2 - \theta_1)$, respectively. For any $\theta \in \Theta^*$, the support $A_1(\theta)$ is $\{x \mid 0 \leq x \leq \theta\}$. If $\theta \in \Theta^*$ is less than θ_0 , then it is trivial. Hence we consider the case when $\theta > \theta_0 + 2r$. Then $A_1(\theta)$ could be written as $\{S_0 \cup S_1 \cup S_2 \cup S_3\}$, where $S_0 = \{x \mid 0 \leq x \leq \theta_0\}$, $S_1 = \{x \mid \theta_0 < x \leq \theta_0 + r\}$, $S_2 = \{x \mid \theta_0 + r < x \leq \theta_0 + 2r\}$ and $S_3 = \{x \mid \theta_0 + 2r < x \leq \theta\}$.

If we consider nontrivial function $g_0(\cdot)$ given by

$$g_0(x) = \begin{cases} 0, & \text{for } x \in S_0 \textit{ or } S_3 \\ a, & \text{for } x \in S_1 \\ -a, & \text{for } x \in S_2 \ , \end{cases}$$

where a is any non-zero constant, then it is easily seen that

$$E_\theta[g_0(X)] = 0 \quad \text{for} \quad \text{all } \theta \in \Theta^* ,$$

so that the non-trivial function g_0 is an unbiased estimator of zero. Thus, the random variable X is not complete even though it is sufficient.

It can be found also that there exists a complete sufficient statistic, $T(X)$, such that

$$T(x) = \begin{cases} x, & \text{for } x \in S_0 \textit{ or } S_3 \\ \theta_0 + 2r, & \text{for } x \in S_1 \textit{ or } S_2 \ . \end{cases}$$

Then for any $\theta \in \Theta^*$, one gets the p.d.f. of T given by

$$f_\theta^T(t) = \begin{cases} 1/\theta, & \text{for } t \in S_0 \text{ or } S_3 \\ 2r/\theta, & \text{for } t = \theta_0 + 2r \ . \end{cases}$$

It notes that support of the random variable T is a subset of the parameter space Θ^*, whereas support of the random variable X is not a subset of Θ^*.

Since this p.d.f. of T is a convex combination of a continuous and a discrete density function, we could use (3.10) for continuous parts and (3.2) in section 3.1 or (2.8) of Ghosh and Datta (1989) for a discrete point in order to obtain the $UMVUE$ of $\gamma(\theta) = \theta$ as

$$U(t) = \begin{cases} 2t, & \text{for } t \in S_0 \text{ } or \text{ } S_3 \\ 2(\theta_0 + r), & \text{for } t \in S_1 \text{ } or \text{ } S_2 \text{ .} \end{cases} \tag{4.1}$$

Also the $UMVUE$ above could be expressed as a function of the random variable X, via the function of $T(X)$,

$$U(x) = \begin{cases} 2x, & \text{for} x \in S_0 \text{ } or \text{ } S_3 \\ 2(\theta_0 + r), & \text{for} x \in S_1 \text{ } or \text{ } S_2. \end{cases} \tag{4.2}$$

This details are described in Appendix with some notes.

4.5 Continuous Triangular Distribution.

For any θ_1 and $\theta_2 \in \Theta = \{\theta | \theta > 0\}$, let $X_1, \ldots, X_n$ be a random sample of size n whose distribution is

$$f_{\theta_1,\theta_2}(x) = \begin{cases} 2x/(\theta_2^2 - \theta_1^2), & \text{for } x \in A(\theta_1, \theta_2) = \{\theta_1 \le x \le \theta_2\} \\ 0, & \text{otherwise.} \end{cases}$$

It is seen that $h(x) = x$ and $S(\theta_1, \theta_2) = (\theta_2^2 - \theta_1^2)/2$. From (3.4), the $UMVUE$ of $\gamma(\theta_1, \theta_2) = (\theta_1 + \theta_2)/2$ could be expressed.

$$U(X_{(1)}, X_{(n)}) = \frac{(X_{(1)} + X_{(n)})[2(n-1)X_{(1)}X_{(n)} - (X_{(1)} - X_{(n)})^2]}{4(n-1)X_{(1)}X_{(n)}}.$$

4.6 Truncated Normal Distribution.

For $\theta \in \Theta = \{\theta \mid \theta > 0\}$ define a truncated normal distribution as

$$f_\theta(x) = \begin{cases} \frac{\phi(x)}{\Phi(\theta) - 1/2}, & \text{for } x \in \{0 \le x \le \theta\} \\ 0, & \text{otherwise ,} \end{cases}$$

where $h(x) = \phi(x)$ and $S(\theta) = \Phi(\theta) - 1/2$, with $\phi(\cdot)$ and $\Phi(\cdot)$ are p.d.f. and c.d.f. of the standard normal random variable, respectively, and note that $T(X) = X_{(n)}$. Then from (3.10), we might express the *UMVUE* of any estimable function $\gamma(\theta)$ as the followings :

$$U(X_{(n)}) = \gamma(X_{(n)}) + \frac{\gamma'(X_{(n)})[\Phi(X_{(n)}) - 1/2]}{n\phi(X_{(n)})}$$

$$= \gamma(X_{(n)}) + \frac{\gamma'(X_{(n)})}{n}[\frac{1}{2\phi(X_{(n)})} - R(X_{(n)})] \tag{4.3}$$

$$= \gamma(X_{(n)}) + \frac{\gamma'(X_{(n)})}{n}\bar{R}(X_{(n)}), \tag{4.4}$$

where $R(x) = [1 - \Phi(x)]/\phi(x)$ and $\bar{R}(x) = [\Phi(x) - 1/2]/\phi(x)$ are well known as the *Mill's ratio* (see Kendall and Stuart, 1976). Therefore, the *UMVUE* in (4.3) and (4.4) of any parametric function $\gamma(\theta)$ for the truncated normal distribution could be well approximated by using the *Mill's ratio.*

APPENDIX

It is trivial when parameter θ belongs to $\{0 < \theta \leq \theta_0\}$, so we consider the case $\theta \in \{\theta \geq \theta_0 + 2r\}$. Then the *UMVUE* of $\gamma(\theta) = \theta$ is obtained as the followings :

1) Continuous parts (use (3.10));

$$\begin{aligned} U(t) &= \gamma(t) + S(t) \\ &= \begin{cases} t + t = 2t, & \text{. for } t \in S_0 \text{ ,} \\ t + [\int_0^{\theta_0} 1dt + 2rI\{t = \theta_0 + 2r\} + \int_{\theta_0+2r}^{t} 1dt] = 2t, & \text{for } t \in S_3 \text{ .} \end{cases} \end{aligned}$$

2) Discrete part (use (3.2) in section 3.1 or (2.8) of Ghosh and Datta(1989)) ;
Since $\gamma_i = \theta_0+2r$, $S_i = \int_0^{\theta_0} 1dt+2r = \theta_0+2r$, $\gamma_{i-1} = \theta_0$ and $S_{i-1} = \int_0^{\theta_0} 1dt = \theta_0$,

$$U(t) = \frac{(\theta_0 + 2r)^2 - {\theta_0}^2}{(\theta_0 + 2r) - \theta_0} = 2\theta_0 + 2r \text{ for } t = \theta_0 + 2r \text{ .}$$

For any $\theta \in \Theta^* = \{\theta \mid 0 < \theta \le \theta_o \text{ or } \theta > \theta_o + 2r\}$, the expectation of the estimator $U(T)$ in (4.1) is

$$E_\theta[U(T)] = \begin{cases} \int_0^\theta 2t/\theta dt = \theta, & \text{for } \theta \in (0, \theta_0) \\ \int_0^{\theta_0} 2t/\theta dt + (2\theta_0 + 2r)2r/\theta I\{t = \theta_0 + 2r\} & \\ + \int_{\theta_0+2r}^\theta 2t/\theta dt = \theta, & \text{for } \theta \in (\theta_0 + 2r, \infty). \end{cases}$$

Also the expectation of $U(X)$ in (4.2) rather than $U(T)$ can be obtained.

$$E_\theta[U(X)] = \begin{cases} \int_0^\theta \frac{2x}{\theta} dx = \theta, & \text{for } 0 < \theta \le \theta_o \\ \{\int_0^{\theta_o} + \int_{\theta_o+2r}^\theta\} \frac{2x}{\theta} dx + \int_{\theta_o}^{\theta_o+2r} \frac{2(\theta_o+r)}{\theta} dx = \theta, & \text{for } \theta > \theta_o + 2r\ . \end{cases}$$

$U(X)$, which is identical with $U(T)$, is unbiased for $\theta \in \Theta^*$. Since $T(X)$ is a complete and minimal sufficient statistic, this is the *UMVUE*.

If we consider $\Theta = \{\theta \mid \theta > 0\}$ rather than Θ^*, then the *UMVUE* of $\theta \in \Theta$ is well obtained as

$$U_1(X) = 2X \quad \text{for} \quad 0 \le x \le \theta.$$

Now the variance of the *UMVUE* $U(X)$ in (4.2) is given by

$$Var_\theta[U(X)] = \begin{cases} Var_\theta[U_1(X)], & \text{for } 0 < \theta \le \theta_o \\ Var_\theta[U_1(X)] - 8r^2/3\theta, & \text{for } \theta > \theta_o + 2r\ . \end{cases}$$

It is note that $U(X)$ in (4.2) has minimum variance among all unbiased estimator of $\theta \in \Theta^*$.

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