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On the Almost Certain Rate of Convergence of Series of Independent Random Variables [†]

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ABSTRACT

The rate of convergence to a random variable S for an almost certainly convergent series $S_n = \sum_{j=1}^{n} X_j$ of independent random variables is studied in this paper. More specifically, when S_n converges to S almost certainly, the tail series $T_n = \sum_{j=n}^{\infty} X_j$ is a well-defined sequence of random variables with $T_n \to 0$ a.c. Various sets of conditions are provided so that for a given numerical sequence $0 < b_n = o(1)$, the tail series strong law of large numbers $b_n^{-1} T_n \to 0$ a.c. holds. Moreover, these results are specialized to the case of the weighted i.i.d. random variables. Finally, examples are provided and an open problem is posed.

KEYWORDS: Rate of convergence, Convergence of series of random variables, Tail series, Strong law of large numbers, Weighted i.i.d. random variables

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1. INTRODUCTION

Let $\{X_n,\ n \geq 1\}$ be independent random variables defined on a probability space $(\Omega,\ \mathcal{F},\ \mathrm{P})$ and their partial sums will be denoted by $S_n = \sum_{j=1}^{n} X_j,\ n \geq 1$. If S_n converges almost certainly (a.c.) to a random variable S, then (set $S_0 = 0$)

$$T_n = S - S_{n-1} = \sum_{j=n}^{\infty} X_j,\ n \geq 1$$

is a well-defined sequence of random variables (referred to as the *tail series*) with

$$T_n \to 0 \text{ a.c.}$$

To study the rate in which the partial sum S_n converges to S is equivalent to studying the rate in which the tail series T_n converges to 0.

Pioneering work on the limiting behavior of the tail series $\{T_n,\ n \geq 1\}$ was conducted by Chow and Teicher (1973) wherein they obtained for the tail series of suitably bounded summands a counterpart to Kolmogorov's (1929) celebrated law of the iterated logarithm (LIL) (see, e.g., Chow and Teicher (1988, p.355)). Barbour (1974) established a tail series analogue of the Lindeberg-Feller version of the central limit theorem (CLT). Numerous other investigations on the tail series LIL problem have been conducted; see Heyde (1977), Wellner (1978), Kesten (1979), Budianu (1981), Chow, Teicher, Wei, and Yu (1981), Klesov (1983), and Rosalsky (1983) for work in this direction. The tail series LIL for Banach space valued random elements was investigated by Dianliang (1988, 1991). Nam and Rosalsky (1995) provided various sets of conditions so that for a given numerical sequence $0 < b_n = o(1)$, the limit law $\sup_{k \geq n} |T_k|/b_n \xrightarrow{\mathrm{P}} 0$ holds.

Random variables $\{X_n,\ n \geq 1\}$ are said to obey the tail series *strong law of large numbers* (SLLN) with the norming constants $0 < b_n = o(1)$ if the tail series T_n is well defined and

$$b_n^{-1} T_n \to 0 \text{ a.c.}$$

Klesov (1983, 1984) studied the tail series SLLNs which are tail series analogues of the SLLNs of Petrov (1975, p.272) for partial sums of independent random variables.

In this paper, we will establish tail series SLLNs for independent summands which are tail series counterparts to the SLLNs for partial sums of Teicher (1979). As special cases of these tail series SLLNs, we will investigate the tail series SLLN problem for weighted *independent and identically distributed* (i.i.d.) random variables. Moreover, examples are provided which illustrate the current work and an open problem is posed.

Finally, a remark about notation is in order. Throughout, it proves convenient to define for $x > 0$

$$\log x = \log_1 x = \log e \vee x, \ \log_r x = \log_1 \log_{r-1} x, \ r \geq 2$$

where $\log x$ (when $x \geq e$) denotes the natural logarithm.

2. PRELIMINARY LEMMAS

Several lemmas are needed to establish the main result, Theorem 1. Lemma 1 is a tail series analogue of the Kronecker lemma. This lemma is initially due to Heyde (1977), but Rosalsky (1983) reproved it in an alternative manner because Heyde's original proof was not quite clear to him. Independently from Rosalsky's paper, Klesov (1984) also proved the lemma in a manner similar to that of Rosalsky.

Lemma 1. (Heyde (1977), Rosalsky (1983), Klesov (1984)). Let $\{x_n, \ n \geq 1\}$ be a sequence of constants and let $\{b_n, \ n \geq 1\}$ be a sequence of positive constants with $b_n \downarrow 0$. If the series $\sum_{n=1}^{\infty} b_n^{-1} x_n$ converges, then $b_n^{-1} \sum_{j=n}^{\infty} x_j \to 0$.

In Lemma 2, there are no assumptions concerning the integrability of the random variables $\exp\{tS\}$, $\exp\{tS_n\}$, $n \geq 1$. Moreover, Lemma 2 cannot be proved by invoking the continuity theorem for moment generating functions unless S_n, $n \geq 1$, and S are all defined on a common interval of the t-axis containing 0 as an interior point.

Lemma 2. Let $S_n = \sum_{j=1}^{n} X_j$, $n \geq 1$, where $\{X_n, \ n \geq 1\}$ are independent random variables with $\mathrm{E}(X_n) = 0$ and $\sum_{n=1}^{\infty} \mathrm{E}(X_n^2) < \infty$. Then there exists a random variable S with $\mathrm{E}(S) = 0$, $\mathrm{Var}(S) = \sum_{n=1}^{\infty} \mathrm{E}(X_n^2)$ and $S_n \to S$ a.c.

and such that

$$\lim_{n\to\infty} \mathrm{E}\Big(\exp\{tS_n\}\Big) = \mathrm{E}\Big(\exp\{tS\}\Big), \ -\infty < t < \infty.$$

Proof. The existence of a random variable S with $\mathrm{E}(S) = 0$, $\mathrm{Var}(S) = \sum_{n=1}^{\infty} \mathrm{E}(X_n^2)$ and $S_n \to S$ a.c. follows directly from the Khintchine-Kolmogorov convergence theorem (see, e.g., Chow and Teicher (1988, p.113)).

Next, let $T_n = \sum_{j=n}^{\infty} X_j$, $n \geq 1$. For all $n \geq 1$, Jensen's inequality ensures that

$$\mathrm{E}\Big(\exp\{tT_{n+1}\}\Big) \geq \exp\{t\mathrm{E}(T_{n+1})\} = e^0 = 1$$

and so

$$\begin{aligned}\mathrm{E}\Big(\exp\{tS\}\Big) &= \mathrm{E}\Big(\exp\{tT_{n+1}\}\exp\{tS_n\}\Big)\\ &= \mathrm{E}\Big(\exp\{tT_{n+1}\}\Big)\,\mathrm{E}\Big(\exp\{tS_n\}\Big) \text{ (by independence)}\\ &\geq \mathrm{E}\Big(\exp\{tS_n\}\Big).\end{aligned}$$

Thus,

$$\limsup_{n\to\infty} \mathrm{E}\Big(\exp\{tS_n\}\Big) \leq \mathrm{E}\Big(\exp\{tS\}\Big).$$

Moreover,

$$\begin{aligned}\mathrm{E}\Big(\exp\{tS\}\Big) &= \mathrm{E}\Big(\lim_{n\to\infty}\exp\{tS_n\}\Big)\\ &\leq \liminf_{n\to\infty} \mathrm{E}\Big(\exp\{tS_n\}\Big) \text{ (by Fatou's lemma)}\end{aligned}$$

which when combined with above inequality yields the conclusion. □

Lemma 3 is a tail series analogue of the exponential bounds lemma of Teicher (1979, Lemma 1). The proof of Lemma 3 employs the function $2^{-1}(1 + 2^{-1}x)$ playing a similar role as the function $g(x) = x^{-2}(e^x - 1 - x)$ of Lemma 1 of Teicher (1979).

Lemma 3. Let $\{X_n,\ n \geq 1\}$ be independent random variables with $\mathrm{E}(X_n) = 0$, $\mathrm{E}(X_n^2) = \sigma_n^2$, and $|X_n| \leq M_n$, where $\{M_n,\ n \geq 1\}$ is a bounded sequence of positive constants. Suppose that $\sum_{n=1}^{\infty} \sigma_n^2 < \infty$, and set $t_n^2 = \sum_{j=n}^{\infty} \sigma_j^2$, $n \geq 1$. Let $C_n = \sup_{j \geq n} M_j / t_n$, $n \geq 1$.

(i) The inequalities

$$\mathrm{E}\left(\exp\left\{\frac{t\,T_n}{t_n}\right\}\right) < \exp\left\{\frac{t^2}{2}\left(1 + \frac{t\,C_n}{2}\right)\right\},\ n \geq 1$$

hold for all $t \in (0, C_n^{-1}]$.

(ii) Let $\{x_n,\ n \geq 1\}$ be a sequence of positive real numbers satisfying

$$0 < C_n\,x_n \leq u,\ n \geq 1 \tag{1}$$

for some constant $u < \infty$. Then the inequalities

$$\mathrm{P}\left\{\sup_{j \geq n} T_j > \lambda\,x_n\,t_n\right\} \leq \exp\left\{-x_n^2\left(v\,\lambda - \frac{v^2}{2}\left(1 + \frac{u\,v}{2}\right)\right)\right\},\ n \geq 1$$

hold for all $\lambda > 0$ and all $v \in (0, u^{-1}]$.

Proof. (i) The argument is contained in the *proof* of Theorem 2 of Chow and Teicher (1973).

(ii) Fix $n \geq 1$ and for $M > n$ set

$$S_{j,M} = \sum_{i=j}^{M} X_i,\ n \leq j \leq M$$

$$S_j^{(M)} = \sum_{i=1}^{j} X_{M+1-i},\ \mathcal{F}_j^{(M)} = \sigma(X_M,\ \ldots,\ X_{M+1-j}),\ 1 \leq j \leq M+1-n.$$

Note at the outset that the independence and mean 0 hypotheses ensure that $\left\{S_j^{(M)},\ \mathcal{F}_j^{(M)},\ 1 \leq j \leq M+1-n\right\}$ is a martingale in j for fixed $M > n$ and so for $t > 0$ and fixed $M > n$, $\left\{\exp\left\{\frac{t}{t_n}\,S_j^{(M)}\right\},\ \mathcal{F}_j^{(M)},\ 1 \leq j \leq M+1-n\right\}$ is a submartingale in j (see, e.g., Chow and Teicher (1988, p.236)) since the function $\varphi(s) = \exp\left\{\frac{t}{t_n}\,s\right\}$ is convex. Next, observe that

$$\{S_{j,M},\ j = n,\ \ldots,\ M\} = \{S_j^{(M)},\ j = M+1-n,\ \ldots,\ 1\}. \tag{2}$$

Then for $N \geq n$, $v \in (0, u^{-1}]$, and $t = v\, x_n$ we have

$$\begin{aligned}
\mathrm{P}\Big\{\max_{n \leq j \leq N} T_j > \lambda\, x_n\, t_n\Big\} &= \mathrm{P}\Big\{\lim_{M\to\infty} \max_{n \leq j \leq N} S_{j,M} > \lambda\, x_n\, t_n\Big\} \\
&\leq \liminf_{M\to\infty} \mathrm{P}\Big\{\max_{n \leq j \leq M} S_{j,M} > \lambda\, x_n\, t_n\Big\} \\
&\quad \text{(by Chow and Teicher (1988, p.260))} \\
&= \liminf_{M\to\infty} \mathrm{P}\Big\{\max_{1 \leq j \leq M+1-n} S_j^{(M)} > \lambda\, x_n\, t_n\Big\} \text{ (by (2))} \\
&= \liminf_{M\to\infty} \mathrm{P}\Big\{\max_{1 \leq j \leq M+1-n} \exp\{\frac{t}{t_n} S_j^{(M)}\} > \exp\{\frac{t}{t_n} \lambda\, x_n\, t_n\}\Big\} \\
&\leq \liminf_{M\to\infty} \frac{\mathrm{E}\Big(\exp\Big\{\frac{t}{t_n} S_{M+1-n}^{(M)}\Big\}\Big)}{\exp\{t\, \lambda\, x_n\}} \\
&\quad \text{(by Doob's submartingale maximal inequality, see, e.g.,} \\
&\quad \text{Doob (1953, p.314) or Billingsley (1986, p.487))} \\
&= \liminf_{M\to\infty} \frac{\mathrm{E}\Big(\exp\Big\{\frac{t}{t_n} S_{n,M}\Big\}\Big)}{\exp\{t\, \lambda\, x_n\}} \\
&= \frac{\mathrm{E}\Big(\exp\Big\{\frac{t}{t_n} T_n\Big\}\Big)}{\exp\{t\, \lambda\, x_n\}} \text{ (by Lemma 2)} \\
&\leq \exp\Big\{-t\, \lambda\, x_n + \frac{t^2}{2}\Big(1 + \frac{t\, C_n}{2}\Big)\Big\} \\
&\quad \text{(by part } (i) \text{ noting that } t \in (0, C_n{}^{-1}]) \\
&= \exp\Big\{-v\, \lambda\, x_n^2 + \frac{1}{2} v^2\, x_n^2 \Big(1 + \frac{1}{2} v\, x_n\, C_n\Big)\Big\} \\
&\leq \exp\Big\{-x_n^2 \Big(v\, \lambda - \frac{v^2}{2}\Big(1 + \frac{u\, v}{2}\Big)\Big)\Big\} \text{ (by (1)).}
\end{aligned}$$

Letting $N \to \infty$ yields

$$\begin{aligned}
\mathrm{P}\Big\{\sup_{j \geq n} T_j > \lambda\, x_n\, t_n\Big\} &= \lim_{N\to\infty} \mathrm{P}\Big\{\max_{n \leq j \leq N} T_j > \lambda\, x_n\, t_n\Big\} \\
&\leq \exp\Big\{-x_n^2 \Big(v\, \lambda - \frac{v^2}{2}\Big(1 + \frac{u\, v}{2}\Big)\Big)\Big\},
\end{aligned}$$

thereby proving part (ii). □

3. TAIL SERIES STRONG LAWS OF LARGE NUMBERS

To establish almost certain convergence of the series S_n, the Khintchine-Kolmogorov convergence theorem and the Kolmogorov three-series criterion (see, e.g., Chow and Teicher (1988, p.117)) are very useful devices. It will be shown in the proof of Theorem 1 that the hypotheses guarantee that $\{T_n, n \geq 1\}$ is a well-defined sequence of random variables. As will be apparent, Theorem 1 owes much to the work of Teicher (1979).

Theorem 1. Let $1 \leq p \leq 2$ and let $\{X_n,\ n \geq 1\}$ be independent random variables with $\mathrm{E}(X_n) = 0$, $\mathrm{E}(|X_n|^p) \leq e_n$ where $\{e_n,\ n \geq 1\}$ are positive constants with $\sum_{n=1}^{\infty} e_n < \infty$. Assume that

$$B_n^p = \mathrm{O}(B_{n+1}^p) \tag{3}$$

where $B_n^p = \sum_{j=n}^{\infty} e_j,\ n \geq 1$. If for some $\alpha \in (-\infty, \frac{1}{p})$

$$\sum_{n=1}^{\infty} \mathrm{P}\Big\{|X_n| > \delta\, B_n\, (\log_2 B_n^{-p})^{1-\alpha}\Big\} < \infty \text{ for some } \delta > 0 \tag{4}$$

and for *all* $\varepsilon > 0$

$$\sum_{n=1}^{\infty} \frac{\mathrm{E}\Big(X_n^2\, \mathrm{I}_{[\varepsilon B_n (\log_2 B_n^{-p})^{-\alpha} < |X_n| < \delta B_n (\log_2 B_n^{-p})^{1-\alpha}]}\Big)}{\Big(B_n\, (\log_2 B_n^{-p})^{1-\alpha}\Big)^2} < \infty, \tag{5}$$

then the tail series SLLN

$$\frac{T_n}{B_n\, (\log_2 B_n^{-p})^{1-\alpha}} \to 0 \text{ a.c.} \tag{6}$$

holds.

Proof. Observe at the outset that the tail series $\{T_n, n \geq 1\}$ is well defined by taking $g(x) = |x|^p$ in Assertion 4 of Klesov (1984). (Alternatively, with the choice of $g_n(x) \equiv |x|^p,\ n \geq 1$, Loève's (1977, p.252) generalization of the Khintchine-Kolmogorov convergence theorem ensures that $\{T_n, n \geq 1\}$ is a well-defined sequence of random variables.)

Let $\varepsilon > 0$ be arbitrary and let $0 \le \alpha < \frac{1}{p}$. For each $n \ge 1$, set $U_n = X_n \, \mathrm{I}_{[|X_n| \le \varepsilon B_n (\log_2 B_n^{-p})^{-\alpha}]}$, $V_n = X_n \, \mathrm{I}_{[|X_n| > \delta B_n (\log_2 B_n^{-p})^{1-\alpha}]}$, and $W_n = X_n - U_n - V_n$. Now, for each $j \ge n \ge 1$,

$$\begin{aligned} \mathrm{E}(|V_j|) &\le \mathrm{E}\Big(|X_j| \, \mathrm{I}_{[\delta B_j (\log_2 B_j^{-p})^{1-\alpha} < |X_j| \le B_n (\log_2 B_n^{-p})^{-\alpha}]}\Big) \\ &\quad + \mathrm{E}\Big(|X_j| \, \mathrm{I}_{[|X_j| > B_n (\log_2 B_n^{-p})^{-\alpha}]}\Big) \\ &\le B_n (\log_2 B_n^{-p})^{-\alpha} \, \mathrm{P}\Big\{|X_j| > \delta B_j (\log_2 B_j^{-p})^{1-\alpha}\Big\} \\ &\quad + B_n^{1-p} (\log_2 B_n^{-p})^{\alpha(p-1)} \, \mathrm{E}\Big(|X_j|^p \, \mathrm{I}_{[|X_j| > B_n (\log_2 B_n^{-p})^{-\alpha}]}\Big) \end{aligned}$$

and so

$$\begin{aligned} \Big|\sum_{j=n}^{\infty} \mathrm{E}(V_j)\Big| &\le B_n (\log_2 B_n^{-p})^{-\alpha} \sum_{j=n}^{\infty} \mathrm{P}\Big\{|X_j| > \delta B_j (\log_2 B_j^{-p})^{1-\alpha}\Big\} \\ &\quad + B_n^{1-p} (\log_2 B_n^{-p})^{\alpha(p-1)} \sum_{j=n}^{\infty} \mathrm{E}\left(|X_j|^p \, \mathrm{I}_{\left[|X_j| > B_n (\log_2 B_n^{-p})^{-\alpha}\right]}\right) \\ &= \mathrm{o}\Big(B_n (\log_2 B_n^{-p})^{1-\alpha}\Big) \text{ (since } \alpha p < 1), \end{aligned}$$

using (4) and the fact that

$$\sum_{j=n}^{\infty} \mathrm{E}\Big(|X_j|^p \, \mathrm{I}_{[|X_j| > B_n (\log_2 B_n^{-p})^{-\alpha}]}\Big) \le B_n^p.$$

Note that (4) ensures via the Borel-Cantelli lemma that with probability 1, V_n is eventually 0 and consequently so is $\sum_{j=n}^{\infty} V_j$. Thus

$$\frac{\sum_{j=n}^{\infty} \{V_j - \mathrm{E}(V_j)\}}{B_n (\log_2 B_n^{-p})^{1-\alpha}} \to 0 \text{ a.c.} \tag{7}$$

In view of (5) and the Khintchine-Kolmogorov convergence theorem, Lemma 1 yields

$$\frac{\sum_{j=n}^{\infty} \{W_j - \mathrm{E}(W_j)\}}{B_n (\log_2 B_n^{-p})^{1-\alpha}} \to 0 \text{ a.c.} \tag{8}$$

Now, observe that $\mathrm{E}(X_n) = \mathrm{E}(U_n) + \mathrm{E}(V_n) + \mathrm{E}(W_n) = 0$. Then, in view of (7) and (8), in order to show that (6) holds, it suffices to show (since ε is arbitrary) that $R_n = \sum_{j=n}^{\infty} \{U_j - \mathrm{E}(U_j)\}$, $n \geq 1$, is a well-defined sequence of random variables satisfying

$$\limsup_{n\to\infty} \frac{|R_n|}{B_n\,(\log_2 B_n^{-p})^{1-\alpha}} \leq \frac{6\,\varepsilon}{\gamma^2} \text{ a.c.} \tag{9}$$

for some $0 < \gamma < 1$ not depending on ε.

To this end, firstly observe for each $n \geq 1$ that

$$\begin{aligned}\mathrm{E}(|U_n - \mathrm{E}(U_n)|^p) &\leq 2^p\{\mathrm{E}(|U_n|^p + |\mathrm{E}(U_n)|^p)\} \\ &\leq 2^{p+1}\,\mathrm{E}(|U_n|^p) \text{ (by Jensen's inequality)}\end{aligned}$$

and so

$$\sum_{n=1}^{\infty} \mathrm{E}(|U_n - \mathrm{E}(U_n)|^p) \leq 2^{p+1} \sum_{n=1}^{\infty} e_n < \infty.$$

Thus, by taking $g(x) \equiv |x|^p$, via Assertion 4 of Klesov (1984), $\{R_n,\ n \geq 1\}$ is a well-defined sequence of random variables.

Next, note that (3) ensures that $B_{n+1}/B_n \geq \gamma$ for some $0 < \gamma < 1$ and all $n \geq 1$, and define $n_k = \inf\left\{n \geq 1 : B_n \leq \gamma^k\right\}$, $k \geq 1$. Then, for all k such that $n_k \geq 2$, $B_{n_k} \geq \gamma\, B_{n_k-1} > \gamma^{k+1} \geq B_{n_{k+1}}$. Hence $\{n_k,\ k \geq 1\}$ is a strictly increasing sequence of integers. Moreover, for all $k \geq 2$ such that $n_k \geq 2$, since $B_{n_k} > \gamma^{k+1}$ and $B_{n_{k-1}} \leq \gamma^{k-1}$, it follows that

$$\frac{B_{n_k}}{B_{n_{k-1}}} > \gamma^2. \tag{10}$$

For each $n \geq 1$,

$$\begin{aligned}&\mathrm{P}\left\{R_n > \frac{6\,\varepsilon}{\gamma^2}\, B_n\,(\log_2 B_n^{-p})^{1-\alpha} \text{i.o.}\,(n)\right\} \\ &\quad\leq \mathrm{P}\left\{\max_{n_{k-1}\leq n<n_k} R_n > \frac{6\,\varepsilon}{\gamma^2}\, B_{n_k}\left(\log_2 B_{n_k}^{-p}\right)^{1-\alpha} \text{ i.o.}\,(k)\right\} \\ &\quad\leq \mathrm{P}\left\{\sup_{n\geq n_{k-1}} R_n > \frac{6\,\varepsilon}{\gamma^2}\, B_{n_k}\left(\log_2 B_{n_k}^{-p}\right)^{1-\alpha} \text{ i.o.}\,(k)\right\}\end{aligned}$$

$$\leq \mathrm{P}\left\{\sup_{n\geq n_{k-1}} R_n > 6\,\varepsilon\, B_{n_{k-1}}\left(\log_2 B_{n_{k-1}}^{-p}\right)^{1-\alpha} \text{ i.o.}\,(k)\right\}$$

(by (10) and the fact that $B_{n_k}^{-p} \uparrow$ as $k \uparrow$)

$$= \mathrm{P}\left\{\sup_{n\geq n_k} R_n > 6\,\varepsilon\, B_{n_k}\left(\log_2 B_{n_k}^{-p}\right)^{1-\alpha} \text{ i.o.}\,(k)\right\}. \tag{11}$$

Now, for each $n \geq 1$, let $r_n^2 = \mathrm{E}(R_n^2)$. Then

$$\begin{aligned} r_n^2 &\leq \sum_{j=n}^{\infty} \mathrm{E}\left(X_j^2\, \mathrm{I}_{[|X_j|\leq \varepsilon B_j(\log_2 B_j^{-p})^{-\alpha}]}\right) \\ &\leq \frac{\varepsilon^{2-p}\, {B_n}^{2-p}}{(\log_2 B_n^{-p})^{\alpha(2-p)}} \sum_{j=n}^{\infty} \mathrm{E}(|X_j|^p) \\ &= \frac{\varepsilon^{2-p}\, {B_n}^2}{(\log_2 B_n^{-p})^{\alpha(2-p)}}. \end{aligned} \tag{12}$$

For each $n \geq 1$, note that

$$|U_n - \mathrm{E}(U_n)| \leq 2\,\varepsilon\, B_n\, (\log_2 B_n^{-p})^{-\alpha} \equiv M_n.$$

Then, setting

$$C_n = \frac{M_n}{r_n},\quad \lambda_n = \frac{6\varepsilon\, B_n^2\, (\log_2 B_n^{-p})^{1-2\alpha}}{r_n^2},\quad x_n = \frac{r_n\, (\log_2 B_n^{-p})^{\alpha}}{B_n}$$

it follows that

$$C_{n_k} x_{n_k} = 2\,\varepsilon \tag{13}$$

$$\lambda_{n_k} x_{n_k} r_{n_k} = 6\varepsilon\, B_{n_k}\left(\log_2 B_{n_k}^{-p}\right)^{1-\alpha} \tag{14}$$

$$\lambda_{n_k} x_{n_k}^2 = 6\varepsilon\left(\log_2 B_{n_k}^{-p}\right) \to \infty \tag{15}$$

$$x_{n_k}^2 \leq \varepsilon^{2-p}\, (\log_2 B_n^{-p})^{\alpha p} = \mathrm{o}(\log_2 B_n^{-p}) \tag{16}$$

by (12) and the fact that $\alpha p < 1$.

Now by (11) and (14) we obtain

$$\mathrm{P}\left\{R_n > \frac{6\varepsilon}{\gamma^2} B_n (\log_2 B_n^{-p})^{1-\alpha} \text{ i.o.} (n)\right\} \leq \mathrm{P}\left\{\sup_{n \geq n_k} R_n > \lambda_{n_k} x_{n_k} r_{n_k} \text{ i.o.} (k)\right\}. \quad (17)$$

But for all $k \geq 2$ such that $n_k \geq 2$, it follows from (13) and part (ii) of Lemma 3 by taking $u = 2\varepsilon$ and $v = (2\varepsilon)^{-1}$ that

$$\begin{aligned}
\mathrm{P}\left\{\sup_{n \geq n_k} R_n > \lambda_{n_k} x_{n_k} r_{n_k}\right\} &\leq \exp\left\{-x_{n_k}^2 \left(\frac{\lambda_{n_k}}{2\varepsilon} - \frac{3}{16\varepsilon^2}\right)\right\} \\
&= \exp\left\{-3 \left(\log_2 B_{n_k}^{-p}\right) + K\, x_{n_k}^2\right\} \\
&\quad \text{(by (15) where } K = \tfrac{3}{16\varepsilon^2} \text{)} \\
&\leq \exp\left\{-2 \left(\log_2 B_{n_k}^{-p}\right)\right\} \text{ (for all large } k \text{ by (16))} \\
&\leq (-p\, k \log \gamma)^{-2} \text{ (since } B_{n_k}^{-p} \geq \gamma^{-pk})
\end{aligned}$$

and so

$$\sum_{k=1}^{\infty} \mathrm{P}\left\{\sup_{n \geq n_k} R_n > \lambda_{n_k} x_{n_k} r_{n_k}\right\} < \infty.$$

Hence, by the Borel-Cantelli lemma and (17),

$$\mathrm{P}\left\{R_n > \frac{6\,\varepsilon}{\gamma^2} B_n (\log_2 B_n^{-p})^{1-\alpha} \text{ i.o.}(n)\right\} = 0$$

implying that

$$\limsup_{n \to \infty} \frac{R_n}{B_n (\log_2 B_n^{-p})^{1-\alpha}} \leq \frac{6\,\varepsilon}{\gamma^2} \text{ a.c.} \quad (18)$$

Now by replacing $\{X_n,\ n \geq 1\}$ by $\{-X_n,\ n \geq 1\}$, it follows from (18) that (18) likewise holds with R_n replaced by $-R_n$ thereby proving (9) and the theorem. □

By taking $p = 2$ in Theorem 1, we obtain the following corollary which is an analogue of a corollary of Teicher (1979).

Corollary 1. Let $\{X_n,\ n \geq 1\}$ be independent random variables with $\mathrm{E}(X_n) = 0$, $\mathrm{E}(X_n^2) = \sigma_n^2$, $t_n^2 = \sum_{j=n}^{\infty} \sigma_j^2 = \mathrm{o}(1)$, and $t_n^2 = \mathrm{O}(t_{n+1}^2)$. If for some $-\infty < \alpha < \frac{1}{2}$

$$\sum_{n=1}^{\infty} \mathrm{P}\Big\{|X_n| > \delta\, t_n(\log_2 t_n^{-2})^{1-\alpha}\Big\} < \infty \text{ for some } \delta > 0 \tag{19}$$

and for all $\varepsilon > 0$

$$\sum_{n=1}^{\infty} \frac{\mathrm{E}\Big(X_n^2\, \mathrm{I}_{[\varepsilon t_n(\log_2 t_n^{-2})^{-\alpha} < |X_n| < \delta t_n(\log_2 t_n^{-2})^{1-\alpha}]}\Big)}{\Big(t_n\,(\log_2 t_n^{-2})^{1-\alpha}\Big)^2} < \infty, \tag{20}$$

then the tail series SLLN

$$\frac{T_n}{t_n\,(\log_2 t_n^{-2})^{1-\alpha}} \to 0 \text{ a.c.}$$

holds.

Remark. Observe that this corollary precludes $\alpha = \frac{1}{2}$. In fact, the conditions (19) and (20) when $\alpha = \frac{1}{2}$ comprise two of the three conditions for the tail series LIL of Rosalsky (1983, Theorem 1).

The two conditions (4) and (5) of Theorem 1 will now be combined into a single one in the ensuing Corollary 2 which is comparable with the tail series LIL of Rosalsky (1983, Corollary 1). That is, a condition which ensures that the conditions (4) and (5) are simultaneously satisfied will be presented in the following corollary.

Corollary 2. Let $1 \le p \le 2$ and let $\{X_n,\ n \ge 1\}$ be independent random variables with $\mathrm{E}(X_n) = 0$, $\mathrm{E}(|X_n|^p) \le e_n$ where $\{e_n,\ n \ge 1\}$ are positive constants with $\sum_{n=1}^{\infty} e_n < \infty$. Assume that (3) holds where $B_n^p = \sum_{j=n}^{\infty} e_j,\ n \ge 1$. Let $-\infty < \alpha < \frac{1}{p}$ and $0 \le \beta \le 1$. If for all $\varepsilon > 0$

$$\sum_{n=1}^{\infty} \frac{\mathrm{E}\Big(|X_n|^{2\beta}\, \mathrm{I}_{[|X_n| > \varepsilon B_n(\log_2 B_n^{-p})^{-\alpha}]}\Big)}{\Big(B_n\,(\log_2 B_n^{-p})^{1-\alpha}\Big)^{2\beta}} < \infty, \tag{21}$$

then the tail series SLLN (6) holds.

Remark. Observe that for $\beta = 0$, the condition (21) reduces to

$$\sum_{n=1}^{\infty} \mathrm{P}\Big\{|X_n| > \varepsilon\, B_n\, (\log_2 B_n^{-p})^{-\alpha}\Big\} < \infty \text{ for all } \varepsilon > 0$$

and for $\beta = 1$, it becomes

$$\sum_{n=1}^{\infty} \frac{\mathrm{E}\Big(X_n^2\, \mathrm{I}_{[|X_n| > \varepsilon B_n (\log_2 B_n^{-p})^{-\alpha}]}\Big)}{\Big(B_n\, (\log_2 B_n^{-p})^{1-\alpha}\Big)^2} < \infty.$$

Proof of Corollary 2. Let $\delta = 1$. For an arbitrary $\varepsilon > 0$, the series of (21) majorizes the two series obtained from (21) by restricting the range of integration to $[|X_n| > B_n(\log_2 B_n^{-p})^{1-\alpha}]$ and $[\varepsilon B_n(\log_2 B_n^{-p})^{-\alpha} < |X_n| \le B_n(\log_2 B_n^{-p})^{1-\alpha}]$, and these series, in turn, majorize the series of (4) and (5), respectively. □

4. THE WEIGHTED I.I.D. CASE

For i.i.d. random variables $\{Y_n,\ n \ge 1\}$ with $\mathrm{E}(Y_1) = 0$ and $\mathrm{E}(Y_1^2) = 1$ and nonzero contants $\{a_n,\ n \ge 1\}$, consider the sequence of weighted i.i.d. random variables $\{a_n Y_n,\ n \ge 1\}$. Then there exists a random variable S with $\sum_{j=1}^{n} a_j Y_j \to S$ a.c. iff $\sum_{n=1}^{\infty} a_n^2 < \infty$. (Sufficiency follows directly from the Khintchine-Kolmogorov convergence theorem whereas necessity results from the work of Kac and Steinhaus (1936) or Marcinkiewicz and Zygmund (1937) or Abbott and Chow (1973).) In such a case, $\mathrm{E}(S) = 0$, $\mathrm{E}(S^2) = \sum_{n=1}^{\infty} a_n^2$.

Corollaries 1 and 2 reduce to Corollaries 3 and 4 below, respectively, in the weighted i.i.d. case.

Corollary 3. Let $\{Y_n,\ n \ge 1\}$ be i.i.d. random variables with $\mathrm{E}(Y_1) = 0$, $\mathrm{E}(Y_1^2) = 1$, and let $\{a_n,\ n \ge 1\}$ be nonzero constants satisfying $t_n^2 = \sum_{j=n}^{\infty} a_j^2 = \mathrm{o}(1)$ and $t_n^2 = \mathrm{O}(t_{n+1}^2)$. If for some $-\infty < \alpha < \frac{1}{2}$

$$\sum_{n=1}^{\infty} \mathrm{P}\Big\{|Y_1| > \delta\, |a_n|^{-1}\, t_n\, (\log_2 t_n^{-2})^{1-\alpha}\Big\} < \infty \text{ for some } \delta > 0$$

and for all $\varepsilon > 0$

$$\sum_{n=1}^{\infty} \frac{a_n^2\, \mathrm{E}\Big(Y_1^2\, \mathrm{I}_{[\varepsilon |a_n|^{-1} t_n (\log_2 t_n^{-2})^{-\alpha} < |Y_1| < \delta |a_n|^{-1} t_n (\log_2 t_n^{-2})^{1-\alpha}]}\Big)}{\Big(t_n\, (\log_2 t_n^{-2})^{1-\alpha}\Big)^2} < \infty,$$

then the tail series SLLN

$$\frac{\sum_{j=n}^{\infty} a_j Y_j}{t_n (\log_2 t_n^{-2})^{1-\alpha}} \to 0 \text{ a.c.} \tag{22}$$

holds.

Corollary 4. Let $\{Y_n, n \geq 1\}$ be i.i.d. random variables with $E(Y_1) = 0$, $E(Y_1^2) = 1$, and let $\{a_n, n \geq 1\}$ be nonzero constants satisfying $t_n^2 = \sum_{j=n}^{\infty} a_j^2 = o(1)$ and $t_n^2 = O(t_{n+1}^2)$. Let $-\infty < \alpha < \frac{1}{2}$ and $0 \leq \beta \leq 1$. If for all $\varepsilon > 0$

$$\sum_{n=1}^{\infty} \frac{|a_n|^{2\beta} E\Big(|Y_1|^{2\beta} I_{[|Y_1| > \varepsilon |a_n|^{-1} t_n (\log_2 t_n^{-2})^{-\alpha}]}\Big)}{\Big(t_n (\log_2 t_n^{-2})^{1-\alpha}\Big)^{2\beta}} < \infty,$$

then the tail series SLLN (22) holds.

Theorem 2, which is an analogue of the tail series LIL of Rosalsky (1983, Theorem 2), may now be stated.

Theorem 2. Let $\{Y_n, n \geq 1\}$ be i.i.d. random variables with $E(Y_1) = 0$, $E(Y_1^2) = 1$, and let $\{a_n, n \geq 1\}$ be nonzero constants satisfying $t_n^2 = \sum_{j=n}^{\infty} a_j^2 = o(1)$ and $t_n^2 = O(t_{n+1}^2)$. If

$$\frac{n a_n^2}{t_n^2} = O\Big((\log_2 t_n^{-2})^{\tau}\Big) \text{ for some } -\infty < \tau < \infty, \tag{23}$$

then the tail series SLLN

$$\frac{\sum_{j=n}^{\infty} a_j Y_j}{t_n (\log_2 t_n^{-2})^{1-\alpha}} \to 0 \text{ a.c.}$$

holds for every $-\infty < \alpha < \frac{1}{2}$ provided in the case $\tau > 2(1-\alpha)$ that

$$E\Big(Y_1^2 (\log_2 |Y_1|)^{\tau - 2(1-\alpha)}\Big) < \infty. \tag{24}$$

Remark. Actually, under the assumption (23) when $\tau < 1$, the result follows directly from the tail series LIL of Rosalsky (1983, Theorem 2). In the case $1 \leq \tau \leq 2(1-\alpha)$, the additional assumption is not needed in Theorem 2, although an alternative additional assumption in the same spirit as (24) is required for the tail series LIL of Rosalsky when (23) holds with $\tau \geq 1$. And for $\tau > 2(1-\alpha)$, we assumed the moment condition (24) which is weaker than the additional moment condition in the tail series LIL of Rosalsky since $\tau - 2(1-\alpha) < \tau - 1$.

Proof of Theorem 2. Without loss of generality, it may be assumed that $\tau \geq 0$ and $\alpha \geq 0$. For each $n \geq 1$, let

$$q_n^2 = \frac{\varepsilon^2 n}{K(\log_2 t_n^{-2})^{\tau+2\alpha}}$$

where $\varepsilon > 0$ is fixed but arbitrary. Then the result follows from Corollary 4 by suitably modifying the *argument* in the proof of Theorem 2 of Rosalsky (1983), *mutatis mutandis*. The details are left to the reader. □

5. EXAMPLES

Two examples are provided which illustrate the current work. In the first example, we will consider the almost certain rate of convergence of the *harmonic series* with a random choice of signs.

Example 1. Let $\{X_n,\ n \geq 1\}$ be independent random variables such that

$$P\left\{X_n = n^{-1}\right\} = P\left\{X_n = -n^{-1}\right\} = \frac{1}{2},\ n \geq 1.$$

The series of partial sums $S_n = \sum_{j=1}^{n} X_j$ can be interpreted as the harmonic series with a random choice of signs. Consider the weighted i.i.d. random variables $X_n = n^{-1} Y_n$, $n \geq 1$, where $\{Y_n,\ n \geq 1\}$ is a sequence of i.i.d. random variables with $P\{Y_1 = 1\} = P\{Y_1 = -1\} = \frac{1}{2}$. Then the tail series SLLN

$$\frac{n^{\frac{1}{2}} T_n}{(\log_2 n)^{\frac{1}{2}+\varepsilon}} \to 0 \text{ a.c. } (\varepsilon > 0)$$

follows from Theorem 2 with $\tau = 0$ and $\alpha = \frac{1}{2} - \varepsilon$.

The second example provides an application of the tail series SLLN to the field of *time series analysis.*

Example 2. Let $\{S_t,\ t = 0,\ \pm 1,\ \pm 2,\ ...\}$ be the *moving average process* of infinite order given by

$$S_t = \sum_{j=0}^{\infty} a_j X_{t-j} \tag{25}$$

where $\{X_t,\ t = 0,\ \pm 1,\ \pm 2,\ ...\}$ are i.i.d. normal random variables with mean 0 and variance 1 and $\{a_j,\ j \geq 0\}$ is a square summable sequence of constants. As a specific example, consider a *long memory process*, which is represented by (25) with $a_0 = 1$ and

$$a_j = \frac{\Gamma(j+d)}{\Gamma(j+1)\,\Gamma(d)} = \prod_{0<k\leq j} \frac{k-1+d}{k},\ j \geq 1 \tag{26}$$

where

$$|d| < \frac{1}{2} \text{ and } \Gamma(x) = \begin{cases} \int_0^{\infty} t^{x-1} e^{-t} dt, & \text{if } x > 0 \\ \infty, & \text{if } x = 0 \\ x^{-1}\,\Gamma(1+x), & \text{if } x < 0. \end{cases}$$

By applying Stirling's formula to (26), we obtain (see, e.g., Brockwell and Davis (1987, p.466)) for $d \neq 0$

$$a_j \sim \frac{j^{d-1}}{\Gamma(d)} \text{ as } j \to \infty.$$

Then for every integer t and $\varepsilon > 0$, the tail series SLLN

$$\frac{n^{\frac{1}{2}-d}}{(\log_2 n)^{\frac{1}{2}+\varepsilon}} \sum_{j=n}^{\infty} a_j X_{t-j} \to 0 \text{ a.c.}$$

follows from Theorem 2 (with $\tau = 0$ and $\alpha = \frac{1}{2} - \varepsilon$). We have thus determined for every integer t an order bound on the almost certain rate in which $\sum_{j=0}^{n} a_j X_{t-j}$ converges to S_t. Observe that this order bound is independent of the time t. Of course, $\sum_{j=0}^{n} a_j X_{t-j}$ is structurally far simpler than S_t.

6. AN OPEN PROBLEM

In Section 3, we established the counterpart to the SLLN for partial sums of Teicher (1979). But Theorem 1 is indeed an incomplete analogue of the SLLN of Teicher (1979) because we assumed that the condition (5) holds for *all* $\varepsilon > 0$ rather than for *some* $\varepsilon > 0$ which was the case in a partial sum version of condition (5) which was used by Teicher (1979) to prove a SLLN. The reason for this is that our tail series exponential bound (part (*ii*) of Lemma 3), which was employed to prove Theorem 1, was proved only for all $v \in (0, u^{-1}]$ rather than for all $v \in (0, \infty)$. Thus, by establishing an extension of this exponential bound lemma without the restriction on v (which is the case in an exponential bound for partial sums), the assumption (5) for all $\varepsilon > 0$ might be able to be weakened to (5) for some $\varepsilon > 0$. Conceivably, under no additional conditions or under mild conditions, the convergence of the series in (5) for some $\varepsilon > 0$ guarantees convergence for all $\varepsilon > 0$ but this would require further investigation.

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