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A Diffusion Model for a System Subject to Random Shocks †

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ABSTRACT

A diffusion model for a system subject to random shocks is introduced. It is assumed that the state of system is modeled by a Brownian motion with negative drift and an absorbing barrier at the origin. It is also assumed that the shocks coming to the system according to a Poisson process decrease the state of the system by a random amount. It is further assumed that a repairman arrives according to another Poisson process and repairs or replaces the system if the system, when he arrives, is in state zero. A forward differential equation is obtained for the distribution function of $X(t)$, the state of the system at time t, some boundary conditions are discussed, and several interesting characteristics are derived, such as the first passage time to state zero, $F(0,t)$, the probability of the system being in state zero at time t, and $F(0)$, the limit of $F(0,t)$ as t tends to infinity.

KEYWORDS : Diffusion model, Random shocks, Poisson process, First passage time

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1. INTRODUCTION

Baxter and Lee (1987) introduced a diffusion model for a system subject to continuous wear. The state of the system was modeled by a Brownian motion with negative drift and an absorbing barrier at the origin. Recently, Lee and Lee (1993) introduced a pure jump model for a system subject to random shocks coming to the system according to a Poisson process. In this paper, we generalize the previous two analyses by introducing a mixed model for a system whose state changes both continuously and jumpwise with time. It is assumed that the state of the system is initially $\beta > 0$ and thereafter follows a Brownian motion with negative drift and an absorbing barrier at the origin, unless a shock arrives at the system. We assume that the shocks come to the system according to a Poisson process of rate $\nu > 0$ and, instantaneously, decreases the state of the system by a random amount Z with distribution function H. If the amount of a shock is larger than the current state of the system, it is assumed that the state of the system becomes zero after the shock, that is, the state zero is considered as the complete breakdown of the system. It is further assumed that the system is repaired or replaced by a repairman who arrives according to another Poisson process of rate $\lambda > 0$; when he arrives, if the system is in positive states, no action is taken, otherwise, he instantaneously increases the state of the system up to β. Let $X(t)$ denote the state of the system at time t and let $F(x,t) = P\{X(t) \leq x\}$ be the distribution function of $X(t)$. In section 2, we derive a forward differential equation for $F(x,t)$ of Kolmogorov's type and discuss some boundary conditions. By making use of the martingale argument, in section 3, we obtain the Laplace transform of the distribution function of the first passage time to state zero, and use this result and the renewal argument to derive an expression for $F(0,t) = P\{X(t) = 0\}$ in section 4. An explicit formula for $F(0) = \lim_{t\to\infty} F(0,t)$ is also calculated by applying the key renewal theorem.

2. FORWARD DIFFERENTIAL EQUTATION FOR $F(x,t)$

If $\Delta(\delta t) = A(t+\delta t) - A(t)$, where $\{A(t), t \geq 0\}$ is an ordinary Brownian motion with parameters $\mu < 0$ and $\sigma^2 > 0$, then one of the following four

mutually exclusive events will occur during the small interval $(t, t+\delta t)$:
(a) Neither the repairman nor the shock comes, then

$$X(t+\delta t) = \begin{cases} X(t)+\Delta(\delta t), & \text{almost surely if } X(t)+\Delta(\delta t) > 0, \\ 0, & \text{almost surely if } X(t)+\Delta(\delta t) \le 0. \end{cases}$$

(b) The repairman does not come but the shock comes, then

$$X(t+\delta t) = \begin{cases} X(t)+\Delta(\delta t)-Z, & \text{almost surely if } X(t)+\Delta(\delta t)-Z > 0, \\ 0, & \text{almost surely if } X(t)+\Delta(\delta t)-Z \le 0, \end{cases}$$

(c) The repairman comes but does nothing since $X(t) > 0$, and the shock does not come, then

$$X(t+\delta t) = X(t)+\Delta(\delta t) \text{ and } X(t) > 0, \text{ almost surely.}$$

(d) The repairman comes and repairs the system since $X(t) = 0$, and the shock does not come, then

$$X(t+\delta t) = \beta+\Delta(\delta t) \text{ and } X(t) = 0, \text{ almost surely.}$$

Notice that the probability of the event that both the repairman and the shock come during the interval $(t, t+\delta t)$ is $o(\delta t)$. Thus, for $x \ge 0$,

$$\begin{aligned} F(x, t+\delta t) &= (1-\nu\delta t)(1-\lambda\delta t)P\{X(t)+\Delta(\delta t) \le x\} \\ &\quad + \nu\delta t(1-\lambda\delta t)P\{X(t)+\Delta(\delta t)-Z \le x\} \\ &\quad + (1-\nu\delta t)\lambda\delta t P\{X(t)+\Delta(\delta t) \le x,\, X(t) > 0\} \\ &\quad + (1-\nu\delta t)\lambda\delta t P\{\beta+\Delta(\delta t) \le x,\, X(t) = 0\} + o(\delta t). \end{aligned}$$

Now,

$$\begin{aligned} P\{X(t)+\Delta(\delta t) \le x\} &= \int_{-\infty}^{\infty} F(x-y, t)dP\{\Delta(\delta t) \le y\} \\ &= F(x,t) - E[\Delta(\delta t)]\frac{\partial}{\partial x}F(x,t) \\ &\quad + \frac{1}{2}E[\{\Delta(\delta t)\}^2]\frac{\partial^2}{\partial x^2}F(x,t) + o(\delta t), \end{aligned}$$

on performing a Taylor series expansion, assuming that $\frac{\partial}{\partial x}F(x,t)$ and $\frac{\partial^2}{\partial x^2}F(x,t)$ exist. Substituting this expression into the above formula for $F(x,t+\delta t)$, rearranging and letting $\delta t \to 0$, we obtain the following forward differential equation for $F(x,t)$:

$$\frac{\partial}{\partial t}F(x,t) = \begin{cases} \frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2}F(x,t) - \mu\frac{\partial}{\partial x}F(x,t) + \nu\int_0^\infty F(x+y,t)dH(t) \\ -\nu F(x,t) - \lambda F(0,t), & \text{for } x < \beta, \\ \frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2}F(x,t) - \mu\frac{\partial}{\partial x}F(x,t) + \nu\int_0^\infty F(x+y,t)dH(t) \\ -\nu F(x,t), & \text{for } x \geq \beta. \end{cases} \tag{2.1}$$

Since the origin is an absorbing state, we can prove by using an argument similar to that of Cox and Miller (1965, p. 219–220) that $f(0,t) = \frac{\partial}{\partial x}F(x,t)|_{x=0} = 0$, for all $t > 0$, which is a boundary condition. In section 4, we derive an expression for $F(0,t)$, for all $t > 0$ by a purely probabilistic argument and this result will serve as another boundary condition.

3. THE FIRST PASSAGE TIME TO ZERO

Notice that until the state of the system reaches zero, $X(t)$ can be expressed as

$$X(t) = B(t) - C(t),$$

where $\{B(t), t \geq 0\}$ is a Brownian motion starting at β with parameters $\mu < 0$ and $\sigma^2 > 0$, and $\{C(t), t \geq 0\}$, $C(t) = \sum_{i=0}^{N(t)} Z_i$, is a compound Poisson process with $\{N(t), t \geq 0\}$ being a Poisson process of rate ν and Z_i's are *i.i.d.* random variables having the distribution function H.

For the convenience of the calculation, let $\{B'(t), t \geq 0\}$ be a Brownian motion starting at 0 with parameters $-\mu > 0$ and $\sigma^2 > 0$, and let's define a new process

$$X'(t) = B'(t) + C(t).$$

Then, by symmetry, it can be easily seen that the first passage time of $X(t)$ reaching state 0 is equal in distribution to that of $X'(t)$ reaching state β, say

T_β:

$$T_\beta = \begin{cases} \inf\{t : X'(t) \geq \beta\}, & \text{if } X'(t) \geq \beta, \text{ for some } t \geq 0, \\ \infty, & \text{if } X'(t) < \beta, \text{ for all } t \geq 0. \end{cases}$$

It can be shown that the moment generating function of $X'(t)$ is given by

$$E[e^{uX'(t)}] = exp\{(-u\mu + \frac{1}{2}u^2\sigma^2 - \nu + \nu m_H(u))t\}, \tag{3.1}$$

where $m_H(u) = E[e^{uZ}]$, the moment generating function of Z. By noting that both Brownian motion and compound Poisson process are Markovian and possess the stationary increments, it can be easily seen that $D(t) = \exp\{uX'(t) - \eta t\}$ is a martingale, where $\eta = -u\mu + \frac{1}{2}u^2\sigma^2 - \nu + \nu m_H(u)$. Since T_β is a Markov time, an argument similar to that of Karlin and Taylor (1975, p. 361–362) shows that the Laplace transform of T_β is given by

$$E[e^{-\eta T_\beta}] = e^{-u\beta}, \tag{3.2}$$

where u is related to η by equation

$$\eta = -u\mu + \frac{1}{2}u^2\sigma^2 - \nu + \nu m_H(u). \tag{3.3}$$

For example, in the case that H is an exponential distribution of rate θ, u is the solution of

$$\frac{1}{2}\sigma^2 u^3 - (\mu + \frac{1}{2}\sigma^2\theta)u^2 + (\mu\theta - \nu - \eta)u + \eta\theta = 0. \tag{3.4}$$

4. FORMULA FOR $F(0,t)$

Consider the points where the actual repairs occur. Notice that the sequence of these points forms an embedded renewal process. Let T^* be the generic random variable denoting the time between successive renewals, then we see that

$$T^* \stackrel{D}{=} T_\beta + E^\lambda, \tag{4.1}$$

where $\stackrel{D}{=}$ denotes equality in distribution and E^{λ} is an exponential random variable of rate λ.

By inverting the Laplace transform of T_{β} obtained in the previous section, it is possible to find the distribution function of T_{β}, U say, which of course depends on what H is. Hence the distribution function of T^{*}, V say, is given by

$$V(t) = \int_0^t U(t-u)\lambda e^{-\lambda u} du \tag{4.2}$$

and the renewal function of the embedded renewal process is given by

$$W(t) = \sum_{n=1}^{\infty} V^{(n)}(t), \tag{4.3}$$

where $V^{(n)}$ is the n-fold recursive Stieltjes convolution of H.

Now, notice that the state of the system is over zero at time t if and only if the initial T_{β} is larger than t or there is a renewal in the embedded renewal process at $u \in (0, t]$ and the succeeding T_{β} is larger than $t-u$, and hence

$$1 - F(0,t) = 1 - U(t) + \int_0^t \{1 - U(t-u)\} dW(u). \tag{4.4}$$

Equation (4.4) implies

$$F(0,t) = U(t) - \int_0^t \{1 - U(t-u)\} dW(u). \tag{4.5}$$

By applying the key renewal theorem to the equation (4.5), we obtain

$$\begin{aligned} F(0) &= \lim_{t\to\infty} F(0,t) \\ &= 1 - \frac{1}{E[T^{*}]} \int_0^{\infty} \{1 - U(t)\} dt \\ &= \frac{1}{\lambda E[T_{\beta}] + 1}, \end{aligned} \tag{4.6}$$

since $E[T^{*}] = E[T_{\beta}] + \frac{1}{\lambda}$ and $\int_0^{\infty}\{1 - U(t)\}dt = E[T_{\beta}]$, where $E[T_{\beta}]$ can be calculated by differentiating the Laplace transform of T_{β} given in Section 3.

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