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On the Performance of Empirical Bayes Simultaneous Interval Estimates for All Pairwise Comparisons †

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ABSTRACT

The goal of this article is to study the performances of various empirical Bayes simultaneous interval estimates for all pairwise comparisons. The considered empirical Bayes interval estimates are those based on unbiased estimate, a hierarchical Bayes estimate and a constrained hierarchical Bayes estimate. Simulation results for small sample cases are given and an illustrative example is also provided.

KEYWORDS: Empirical Bayes, Simultaneous interval estimates, Hierarchical Bayes, Constrained Bayes estimation

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1. INTRODUCTION

When experimenters compare several treatments, their main goal is often not to decide whether the treatments are all equal or not. In such situations, the so-called multiple comparisons procedures are known to be useful statistical tools to detect the real differences between the treatments.

For the purpose of all pairwise comparisons, Tukey's(1953) simultaneous confidence intervals are the best known, and can often be found in many statistical package programs. Some optimal properties of Tukey's have been studied by Gabriel(1970), Genizi and Hochberg(1978), and Kunte and Rattihalli(1984) among many others. Further detailed references including modifications and extensions of Tukey's can be found in excellent monographs by Miller(1981), and Hochberg and Tamhane(1987).

Bayesian decision theoretic approach has been taken by Duncan(1961, 1965), Waller and Duncan(1969,1972), and Dixon and Duncan(1975). In these Bayesian decision theoretic studies, they have assumed an additive loss under which the simultaneous comparison problem can be resolved into several pairwise comparison problems. Recognizing such a discomforting aspect, they have suggested some methods by which the assumed loss structure can be related to the level of homogeneity hypotheses testing. This approach, however, loses the flavor of *simultaneous* comparison.

A Bayesian approach without losing the flavor of simultaneous comparison is certainly to control the posterior *simultaneous* coverage probability. Such an approach has been noted in the literature (see, for example, Hochberg and Tamhane(1987)). Kim and Hwang(1991) have adopted such an approach, and have studied empirical Bayes simultaneous confidence intervals in which the prior parameters are estimated from the data. Their main goal is to study the asymptotic coverage probabilities of the proposed procedure.

This article treats the empirical Bayes simultaneous interval estimates for all pairwise comparisons. The considered empirical Bayes interval estimates are those based on unbiased estimate, a hierarchical Bayes estimate and a constrained hierarchical Bayes estimate. These are called the UE, the HBE, and the CHBE interval estimate, respectively. Simulation results for small sample cases show that the CHBE interval estimates dominate the UE and the HBE interval estimates. An illustrative example is provided with comparison of Tukey's(1953) in Bayesian sense.

2. EMPIRICAL BAYES SIMULTANEOUS INTERVAL ESTIMATES

For the purpose of comparing several means, many classification models can be reduced to the following :

$$\begin{cases} X_i \sim N(\theta_i, \sigma_n^2) & \text{independently} \quad (i = 1, \ldots, k), \\ rs^2 \sim \sigma^2 \chi^2(r) & \text{independent of } X_i\text{'s}, \end{cases} \tag{2.1}$$

where $\sigma_n^2 = \sigma^2/n$ and s^2 is an estimate of the unknown variance σ^2.

It is assumed that, a priori,

$$\theta_i \sim N(\mu_\pi, \sigma_\pi^2) \quad \text{independently}(i = 1, \ldots, k) \tag{2.2}$$

for *a presumably specified* μ_π and σ_π^2. Further, we assume the following non-informative prior for the nuisance parameter σ^2 :

$$\pi(\sigma^2) = 1/\sigma^2 \tag{2.3}$$

It should be pointed out at this point that the joint prior by (2.2) and (2.3) is not a conjugate prior. As pointed out by many researchers (see, for example, Berger(1985) p.288), the conjugate prior in this case is unattractive in the sense that the prior variance of the conjugate prior should be a multiple of the sample variance σ_n^2. The prior (2.2) is believed to be more natural eventhough the specification of μ_π and *particularly* σ_π^2 would be difficult in practice.

For the prior (2.2) and (2.3), the posterior distribution of θ_i's, given σ^2 and data $(\mathbf{x}, s^2)$, is easily seen to be independent normal. More precisely, given the data $(\mathbf{x}, s^2), (\mu_\pi, \sigma_\pi^2)$ and σ^2, θ_i's are independent and

$$\theta_i \sim N\left(\frac{a}{a + s^2/\sigma^2}\mu_\pi + \frac{s^2/\sigma^2}{a + s^2/\sigma^2}x_i, \frac{s^2/n}{a + s^2/\sigma^2}\right), \tag{2.4}$$

where $a = s^2/(n\sigma_\pi^2)$.

And the posterior density of

$$\xi = rs^2/\sigma^2$$

, given the data $(\mathbf{x}, s^2)$ and (μ_π, σ_π^2), is proportional to

$$(\frac{\xi/r}{a+\xi/r})^{k/2} \exp\{-\frac{b}{2}\frac{a\xi/r}{a+\xi/r}\}\xi^{\frac{r}{2}-1} \exp\{-\frac{\xi}{2}\}, \tag{2.5}$$

where $b = n\sum_{i=1}^k (x_i - \mu_\pi)^2/s^2$.

Based on this posterior, one would naturally propose the followings as Bayesian simultaneous interval estimates :

$$\theta_i - \theta_j \in \mathrm{E}_1(\theta_i - \theta_j) \pm q\sqrt{\mathrm{Var}_1(\theta_i - \theta_j)} \quad (i \neq j) \tag{2.6}$$

with E_1 and Var_1 denoting the posterior mean and variance, respectively, given $(\mathbf{x}, s^2)$ and (μ_π, σ_π^2) and some quantile q.

Besides the accurate specification of (μ_π, σ_π^2), it is extremely diffcult to compute the *simultaneous* coverage probability of (2.6) for such θ_i's with general correlation structure. In fact, it follows from (2.4) and (2.5) that the posterior moments are given by

$$\begin{cases} \mathrm{E}_1(\theta_i) &= \mathrm{E}\,(\frac{\xi/r}{a+\xi/r})(x_i - \mu_\pi) + \mu_\pi, \\ \mathrm{Var}_1(\theta_i) &= \mathrm{Var}(\frac{a}{a+\xi/r})(x_i - \mu_\pi)^2 + \mathrm{E}\,(\frac{1}{a+\xi/r})\frac{s^2}{n}, \\ \mathrm{Cov}_1(\theta_i, \theta_j) &= \mathrm{Var}(\frac{a}{a+\xi/r})(x_i - \mu_\pi)(x_j - \mu_\pi) \quad (i \neq j). \end{cases} \tag{2.7}$$

Therefore it is natural to search for an approximation of (2.7) in meaningful cases. We regard the case of large sample information relative to prior information σ_π^{-2} as meaningful one. This corresponds to the approximation of (2.7) as $\sigma_\pi^2 \to \infty$, i.e., $a = s^2/(n\,\sigma_\pi^2) \to 0$.

Such an approximation can be done by noting that the posterior density $\pi(\xi|a)$ in (2.5) is uniformly approximated by the chi-square density with r degrees of freedom, i.e.,

$$\pi(\xi|\,a) = \frac{1}{\Gamma(\frac{r}{2})2^{\frac{r}{2}}}\xi^{\frac{r}{2}-1}\mathrm{e}^{-\frac{\xi}{2}}\{1 + O(a)\},$$

as $a \to 0$. Therefore (2.7) can be approximated as follows :

$$\begin{cases} \mathrm{E}_1(\theta_i - \theta_j) &= \mathrm{E}\,(\frac{\xi/r}{a+\xi/r})(x_i - x_j), \\ \mathrm{Var}_1(\theta_i - \theta_j) &= 2\mathrm{E}\,(\frac{1}{a+\xi/r})s^2/n + O(a^2), \\ \mathrm{Cov}_1(\theta_i - \theta_j) &= O(a^2) \quad (i \neq j). \end{cases} \tag{2.8}$$

Thus, keeping terms up to the order of $O(a)$, we find that the interval estimates (2.6) can be approximated by

$$\begin{cases} \mathrm{E}_1(\theta_i - \theta_j) &= \mathrm{E}\,(\frac{\xi/r}{a+\xi/r})(x_i - x_j), \\ q\sqrt{\mathrm{Var}_1(\theta_i - \theta_j)} &\doteq Q_{k,r}^{(\alpha)} \sqrt{\frac{r-2}{r}\mathrm{E}\,(\frac{1}{a+\xi/r})}\,\frac{s}{\sqrt{n}}, \end{cases}$$

where $Q_{k,r}^{(\alpha)}$ denotes the upper α quantile of the Studentized range distribution with parameter k and r degrees of freedom. Hence approximate Bayesian simultaneous interval estimates for $\theta_i - \theta_j\ (i \neq j)$ are given as follows : For all $i \neq j$,

$$\theta_i - \theta_j \ \in\ \mathrm{E}\,(\frac{\xi}{\xi + ar})(x_i - x_j) \pm Q_{k,r}^{(\alpha)} \sqrt{\mathrm{E}\,(\frac{r-2}{\xi + ar})}\,\frac{s}{\sqrt{n}} \tag{2.9}$$

At this point it should be remarked that (2.9) is a refinement of the Bayesian solution

$$\theta_i - \theta_j \ \in\ x_i - x_j \pm Q_{k,r}^{(\alpha)}\, s/\sqrt{n} \quad (i \neq j)$$

with respect to the improper prior $\sigma_\pi^2 = \infty$, which is formally the same as Tukey's(1953). Even with the approximation (2.9), one needs to specify (μ_π, σ_π^2) to implement it. The explicit dependence of (2.9) on (μ_π, σ_π^2) can be expressed as follows : With $\sigma_n^2 = \sigma^2/n$,

$$\begin{cases} \mathrm{E}\,(\frac{\xi}{\xi+ar}) &= \mathrm{E}\,(\frac{\sigma_\pi^2}{\sigma_\pi^2+\sigma_n^2}\,|\,\mathbf{x}, s^2, \mu_\pi, \sigma_\pi^2), \\ \mathrm{E}\,(\frac{r-2}{\xi+ar})\frac{s^2}{n} &= \frac{r-2}{r}\,\mathrm{E}\,(\frac{\sigma_\pi^2\sigma_n^2}{\sigma_\pi^2+\sigma_n^2}\,|\,\mathbf{x}, s^2, \mu_\pi, \sigma_\pi^2). \end{cases} \tag{2.10}$$

The accurate specification of (μ_π, σ_π^2) being difficult, one needs to estimate (μ_π, σ_π^2) based on X_i's and s^2. Kim and Hwang(1991), in a little bit different setting, used an idea of plugging in unbiased estimates of $\sigma_\pi^2 + \sigma_n^2$ and σ_n^2 from the marginals of X_i's and s^2 which are independent $N(\mu_\pi, \sigma_\pi^2 + \sigma_n^2)$ and $\sigma^2\chi^2(r)/r$. Such an idea results in the following estimate of (2.9) :

$$\theta_i - \theta_j \in (1 - \hat{B}_{\mathrm{UE}})(x_i - x_j) \pm Q_{k,r}^{(\alpha)} \sqrt{\frac{r-2}{r}(1 - \hat{B}_{\mathrm{UE}})}\, \frac{s}{\sqrt{n}},$$

where

$$\begin{aligned} \hat{B}_{\mathrm{UE}} &= \text{estimate of } \sigma_n^2(\sigma_\pi^2 + \sigma_n^2)^{-1} \\ &= \min\{1, (n\sum_{i=1}^{k}(x_i - \bar{x})^2/((k-1)s^2))^{-1}\}. \end{aligned}$$

Modifying such an estimate, the interval estimates become

$$\theta_i - \theta_j \in (1 - \hat{B}_{\mathrm{UE}})(x_i - x_j) \pm Q_{k,r}^{(\alpha)} \sqrt{1 - \hat{B}_{\mathrm{UE}}}\, s/\sqrt{n}\, (i \neq j), \tag{2.11}$$

which will be called *unbiased estimation*(UE) method simultaneous interval estimates.

Smoother estimate of prior parameters can be obtained by the so-called hierarchical Bayes method, which considers the second stage prior for σ_π^2. Note that, marginally, X_i's are independent and

$$X_i \sim N(\mu_\pi, (\sigma_\pi^2/\sigma_n^2 + 1)\sigma_n^2),$$

and that μ_π and $(\sigma_\pi^2/\sigma_n^2 + 1)^{1/2}$ are the location and scale parameters, respectively, eventhough the scale parameter $(\sigma_\pi^2/\sigma_n^2 + 1)^{1/2}$ has a restricted range $(\sigma_\pi^2/\sigma_n^2 + 1)^{1/2} > 1$. Thus it seems natural to consider the invariant second stage prior

$$\pi(\mu_\pi, \sigma_\pi^2) = (\sigma_\pi^2/\sigma_n^2 + 1)^{-1}\, \sigma_n^{-2} \tag{2.12}$$

given $\sigma_n^2 = \sigma^2/n$. But, because of the restricted range, the estimation problem of (2.10) can not be treated as an equivariant estimation. Thus we simply take the posterior means given the data $(\mathbf{x}, s^2) = (x_1, \ldots, x_k, s^2)$, i.e.,

$$\begin{cases} \hat{B}_{\mathrm{HB}} = \mathrm{E}\left(\frac{\sigma_n^2}{\sigma_\pi^2 + \sigma_n^2} \,|\, \mathbf{x}, s^2\right) \\ \hat{C}_{\mathrm{HB}} = \frac{r-2}{r}\frac{n}{s^2}\mathrm{E}\left(\frac{\sigma_\pi^2\sigma_n^2}{\sigma_\pi^2 + \sigma_n^2} \,|\, \mathbf{x}, s^2\right). \end{cases}$$

The resulting simultaneous interval estimates are given by

$$\theta_i - \theta_j \in (1 - \hat{B}_{\text{HB}})(x_i - x_j) \pm Q_{k,r}^{(\alpha)} \sqrt{\hat{C}_{\text{HB}}}\, s/\sqrt{n}\, (i \neq j), \quad (2.13)$$

which will be referred as *hierarchical Bayes estimation*(HBE) method simultaneous interval estimates.

Before presenting the explicit form of posterior means, it should be mentioned that similar idea has been used in many empirical or robust Bayesian study. Box and Tiao(1973), Morris(1983), Berger(1985), Berger and Deely(1988), and Berger and Fong(1989), among many others, can be cited for the similar idea. Morris(1983) and Berger(1985) have used a flat prior $\pi(\mu_\pi, \sigma_\pi^2) = 1$ in point estimation when $k \geq 4$. But such a flat prior results in improper posterior for $k \leq 3$.

The explicit forms of $\hat{B}_{\text{HB}}$ and $\hat{C}_{\text{HB}}$ can be found from the posterior distribution of $(\mu_\pi, \sigma_\pi^2, \sigma_n^2)$, given the data $(\mathbf{x}, s^2)$, which can be described as follows :

$$\begin{cases} \mu_\pi \sim N(\bar{x}, (\sigma_\pi^2 + \sigma_n^2)/k), \text{ given } (\sigma_\pi^2, \sigma_n^2), \\ \chi^2 = \sum_{i=1}^k (x_i - \bar{x})^2 (\sigma_\pi^2 + \sigma_n^2)^{-1} + (rs^2/n)\sigma_n^{-2} \sim \chi^2(k + r - 1), \\ F_{\text{tr}} = f_o\, \sigma_n^2(\sigma_\pi^2 + \sigma_n^2)^{-1} \sim \text{truncated } F_{(k-1,r)}, \end{cases}$$

where χ^2 and F_{tr} are independent, and f_o denotes the usual F statistic for the homogeneity test, i.e.,

$$f_o = n \sum_{i=1}^k (x_i - \bar{x})^2 / ((k-1)s^2).$$

Representing $\sigma_n^2(\sigma_\pi^2 + \sigma_n^2)^{-1}$ and $\sigma_\pi^2\sigma_n^2(\sigma_\pi^2 + \sigma_n^2)^{-1}$ by χ^2 and F_{tr}, we have

$$\sigma_n^2(\sigma_\pi^2 + \sigma_n^2)^{-1} = f_o^{-1} F_{\text{tr}},$$
$$\sigma_\pi^2\sigma_n^2(\sigma_\pi^2 + \sigma_n^2)^{-1} = (rs^2/n)(1 - f_o^{-1} F_{\text{tr}})(1 + \frac{k-1}{r} F_{\text{tr}})/\chi^2.$$

Thus simple but tedious calculations with the density of F distribution lead to the following :

$$\begin{cases} \hat{B}_{\mathrm{HB}} &= \frac{r}{r-2} f_o^{-1} F_{k+1,r-2}(\frac{r-2}{r}\frac{k-1}{k+1} f_o)/F_{k-1,r}(f_o), \\ \hat{C}_{\mathrm{HB}} &= \{F_{k-1,r-2}(\frac{r-2}{r} f_o) - \frac{r}{r-4} f_o^{-1} F_{k+1,r-4}(\frac{r-4}{r}\frac{k-1}{k+1} f_o)\}/F_{k-1,r}(f_o), \end{cases} \tag{2.14}$$

which can be easily calculated with routine statistical packages.

The HBE interval estimates (2.13) have been devised by considering the first order approximation with respect to σ_π^{-2} of the Bayesian interval estimates (2.6). It should be, however, mentioned that the same type interval estimates can be derived from Berger's(1985, p.565) argument. Berger's idea is to replace the posterior of θ_i's, given $(\mathbf{x}, s^2)$ and σ^2, by the independent normal distribution with estimated mean

$$\hat{B}_{\mathrm{HB}}\, \bar{x} \; + \; (1 - \hat{B}_{\mathrm{HB}}) x_i$$

and estimated variance

$$\mathrm{E}\,((\sigma_\pi^{-2} + \sigma_n^{-2})^{-1} | \, \mathbf{x}, s^2) \sigma^2 / (r s^2/(r+2)).$$

With such an idea, the resulting interval estimates become

$$\theta_i - \theta_j \;\in\; (1 - \hat{B}_{\mathrm{HB}})(x_i - x_j) \pm Q_{k,r}^{(\alpha)} \sqrt{\frac{r+2}{r-2} \hat{C}_{\mathrm{HB}}}\, \frac{s}{\sqrt{n}},$$

which are the same as (2.13) except the constant term $\sqrt{(r+2)/(r-2)}$.

In fact, the HBE interval estimates use the estimates

$$\hat{\theta}_i \;=\; \bar{x} + (x_i - \bar{x})(1 - \hat{B}_{\mathrm{HB}}) \quad (i = 1, \ldots, k) \tag{2.15}$$

to locate the center $(x_i - x_j)(1 - \hat{B}_{\mathrm{HB}})$ of the interval for $\theta_i - \theta_j$. With $c(f_o) = f_o\, \hat{B}_{\mathrm{HB}}/(k-3)$ for $k > 3$, the estimate (2.15) can be written as

$$\hat{\theta}_i \;=\; \bar{x} + (1 - c(f_o)\, \frac{k-3}{f_o})\, (x_i - \bar{x}), \tag{2.16}$$

with $c(f_o)$ having the following expression :

$$c(f_o) = \frac{\frac{f_o}{k-3}\int_0^1 t^{\frac{k-1}{2}}(1+\frac{(k-1)f_o}{r}t)^{-\frac{k+r-1}{2}}dt}{\int_0^1 t^{\frac{k-3}{2}}(1+\frac{(k-1)f_o}{r}t)^{-\frac{k+r-1}{2}}dt} = \frac{1}{k-3}\frac{r}{r-2}\left[1-\frac{2}{k-1}\{\int_0^1 t^{\frac{k-3}{2}}(1+f_o(1-t))^{\frac{r-2}{2}}dt\}^{-1}\right],$$

which implies that $0 \le c(f_o) \le 2$ and $c(f_o)$ is non-decreasing in f_o for $k \ge 4$. It is well known (see, for example, Lehmann(1983), p.303-307) that the estimate (2.16) with such a $c(f_o)$ dominates the usual estimate x_i of θ_i. Thus the HBE interval estimates use minimax shrinkage estimates of θ_i as the center.

Finally, we consider the interval estimates using other point estimates of θ_i's in (2.6). Louis(1984) proposed new estimators so that the sample mean and variance of the estimates match the conditional expectation and variance for parameters. Ghosh(1992) generalized this idea and called the estimates the constrained Bayes estimates, which minimize

$$E[\sum_{i=1}^{k}(\theta_i - \hat{\theta}_i)^2|x]$$

within the class of $\hat{\theta} = (\hat{\theta}_1, \ldots, \hat{\theta}_k)$ that satisfy

$$\begin{cases} E(\theta_.|x) = \hat{\theta}_. \\ E[\sum_{i=1}^k(\theta_i - \theta_.)^2|x] = \sum_{i=1}^k(\hat{\theta}_i - \hat{\theta}_.)^2 \end{cases}$$

where $\theta_. = k^{-1}\sum_{i=1}^k \theta_i$, $\hat{\theta}_. = k^{-1}\sum_{i=1}^k \hat{\theta}_i$ and $x = (x_1, \ldots, x_k)$. Denoting $\hat{\theta}_i^B = E(\theta_i|x)$ and $\hat{\theta}_.^B = k^{-1}\sum_{i=1}^k \hat{\theta}_i^B$, the constrained Bayes estimate $\hat{\theta}_i^{CB}$ of θ_i in Ghosh(1992) is shown to be

$$\hat{\theta}_i^{CB} = A\hat{\theta}_i^B + (1-A)\hat{\theta}_.^B, \quad (2.17)$$

where $A^2 = 1 + H_1(x)/H_2(x)$, $H_1(x) = \sum_{i=1}^k Var((\theta_i - \theta_.)^2|x)$ and $H_2(x) = \sum_{i=1}^k(\hat{\theta}_i^B - \hat{\theta}_.^B)^2$.

Computing $H_1(x)$ and $H_2(x)$ from (2.4), (2.5) and (2.7), we have

$$\begin{cases} H_1(x) = (k-1)E(\frac{r}{\xi+ar}|x)\frac{s^2}{n} + O(a^2) \\ H_2(x) = \sum_{i=1}^{k}(x_i - \bar{x})^2 E(\frac{\xi}{\xi+ar}|x)^2. \end{cases}$$

Keeping terms up to the order of $O(a)$ and estimating the prior parameters by the hierarchical Bayes method, the estimate of A^2 can be shown to be

$$\hat{A}^2_{HB} = 1 + \frac{1}{f_o}\frac{\frac{r}{r-2}\hat{C}_{HB}}{(1-\hat{B}_{HB})^2}.$$

Considering that the variance of the constrained Bayes estimates is $A(>1)$ times the one of the Bayes estimates, the resulting simultaneous interval estimates are given by

$$\theta_i - \theta_j \in \hat{A}_{HB}(1-\hat{B}_{\rm HB})(x_i - x_j) \pm Q^{(\alpha)}_{k,r}\sqrt{\hat{A}_{HB}\hat{C}_{\rm HB}}\, s/\sqrt{n}\ (i \neq j), \quad (2.18)$$

which will be referred as *constrained hierarchical Bayes estimation*(CHBE) method simultaneous interval estimates.

Following the idea and the arguments in Kim and Hwang(1991), as $k \to \infty$, $1/f_o$ and $\hat{B}_{HB} \to \sigma_n^2/(\sigma_\pi^2 + \sigma_n^2)$ *a.s.* and $\hat{C}_{HB} \to \sigma_\pi^2/(\sigma_\pi^2 + \sigma_n^2)$ *a.s.* Then $\hat{A}^2_{HB} \to (\sigma_\pi^2 + \sigma_n^2)/\sigma_\pi^2$ *a.s.* Thus for large k, one may recommand use of $\hat{A}_{HB} = 1/\sqrt{1-\hat{B}_{HB}}$.

3. SIMULATION RESULTS AND EXAMPLE

To see the performances of the proposed simultaneous interval estimates, it is clear that we can not talk about the probability if the average is taken with respect to the improper prior $1/\sigma^2$ of the nuisance parameter. Hence we consider the empirical Bayes simultaneous coverage probability for a fixed but arbitrary σ^2. In this situation, the setting (2.1) and (2.2) can be represented as follows :

$$\begin{cases} \theta_i = \{B\mu_\pi + (1-B)x_i\} - \sqrt{1-B}\,\sigma_n Z_i, \\ x_i = \mu_\pi + \sqrt{B}/\sigma_n W_i, \\ s^2 = \sigma^2\chi^2/r, \end{cases} \tag{3.1}$$

where $Z_i, W_i\, (i = 1, \ldots, k)$ and χ^2 are, respectively, independent $N(0,1)$, $N(0,1)$ and $\chi^2(r)$ random variables, and

$$B = \frac{\sigma_n^2}{\sigma_n^2 + \sigma_\pi^2}. \tag{3.2}$$

Simple but tedious calculations from the representation (3.1) lead to the following expression for the empirical coverage probability of the procedures (2.11), (2.13) and (2.18) :

$$P\{\sqrt{1-B}(Z_i - Z_j) \in (B-\hat{B})(W_i - W_j)/\sqrt{B} \pm Q_{k,r}^{(\alpha)}\sqrt{\hat{C}}\,\sqrt{\chi^2/r}\,(i \neq j)\}, \tag{3.3}$$

where $\hat{B}$ and $\hat{C}$ are computed from $W_i/\sqrt{B\chi^2}$'s. Thus the true empirical coverage probability can be computed by Monte Carlo method.

We conducted a Monte Carlo simulation study, because it is not easy to see the performances of the proposed simultaneous interval estimates due to the complexity of the probability in (3.3). In our simulation study, the routines RNNOF and RNGAM in IMSL were employed to generate random samples from normal and chi-square distributions, respectively. We used 1,000 replications in obtaining empirical Bayes coverage probabilities for interval estimates in (3.3). Because the probability in (3.3) is $k+1$ dimensional integration, we evaluated the probability in (3.3) through the method of Monte Carlo integration using 1,000 random samples. It is well known that Monte Carlo integration becomes preferable, since numerical integration is rarely optimal in three or more dimensions.

Figure 1 shows the empirical Bayes coverage probabilities of the UE, the HBE and the CHBE interval estimates for selected values of k and r. In observing Figure 1, it should be noted that smaller B corresponds to larger sample information relative to the prior information σ_π^{-2} . The CHBE interval estimates outperform the others for various k and r in viewpoint to keep the nominal level. The UE and the HBE interval estimates get a little bit better as k gets larger. The UE interval estimates are inferior to the others in large B .

Figure 2 shows the length ratio of the interval estimates relative to the Tukey's interval estimates. In comparison of the others, the length ratio for the UE intervals is very small in large B . This is considered to be due to the degeneracy of the intervals when $\hat{B}_{UE} = 1$.

As a real data example, consider the data in Table 1 reported by Dudeck and Peacock(1981) and cited in Mendenhall, Wackerly and Scheaffer(1989).

Table 1. Rolling distances

	Treatement					Mean
	A	B	C	D	E	
Block						
1	2.600	2.183	2.334	2.164	2.127	2.282
2	2.764	2.568	2.506	2.612	2.238	2.538
3	3.043	2.977	2.533	2.675	2.616	2.769
4	3.049	3.028	2.895	2.724	2.697	2.879
mean	2.864	2.689	2.567	2.544	2.420	

Example : An experiment was conducted to evaluate the performance of several cool season grasses for winter overseeding of golf greens in northern Florida. One of the variables of interest was the distance that a golf ball would roll on a green after being rolled down a ramp (used to induce a constant initial velocity to the ball). Because the distance the ball would roll was influenced by the slope of the green and the direction in which the grass was mowed, the experiment was set up in a randomized block design.

Suppose we choose a confidence level $1 - \alpha = 0.95$. Then $Q_{5,12}^{(0.05)} = 4.508$ for treatment. Interval estimates by Tukey's, the HBE, and the CHBE are given in Table 2 for treatment. The UE interval estimates are not reported because of its inferiority in simulation result.

From Table 2, it can be observed that Tukey's and the CHBE intervals are sharp, whereas the HBE intervals are conservative. But the interval length of the HBE and the CHBE is 90.8% and 94.2% of Tukey's, respectively.

Figure 3 shows the variations of posterior coverage probabilities for each interval estimates within some interested range of $\tilde{B} = (s^2/n)/(s^2/n + \sigma_\pi^2)$. Note that Tukey's intervals keep the nominal level in small B, whereas the HBE intervals keep the nominal level in a little bit large B. The CHBE intervals are moderate and keep the nominal level in longer range of B than the others.

Finally, it should be remarked that the results in Figure 3 are evaluated in the case of $k(\bar{x} - \mu_\pi)^2/(s^2/n + \sigma_\pi^2) = 0$. Similar results are obtained in the case of $k(\bar{x} - \mu_\pi)^2/(s^2/n + \sigma_\pi^2) = 4$, and thus are not reported.

Table 2 . Interval estimates for treatments

	Treatment means				Interval estimates of difference		containing zero
Tukey	A	B	2.864	2.689	-.088	.438	*
	A	C	2.864	2.567	.034	.560	
	A	D	2.864	2.544	.057	.583	
	A	E	2.864	2.420	.181	.707	
	B	C	2.689	2.567	-.141	.385	*
	B	D	2.689	2.544	-.118	.408	*
	B	E	2.689	2.420	.006	.532	
	C	D	2.567	2.544	-.240	.286	*
	C	E	2.567	2.420	-.116	.410	*
	D	E	2.544	2.420	-.139	.387	*
HBE	A	B	2.864	2.689	-.088	.389	*
	A	C	2.864	2.567	.016	.493	
	A	D	2.864	2.544	.036	.513	
	A	E	2.864	2.420	.143	.620	
	B	C	2.689	2.567	-.134	.343	*
	B	D	2.689	2.544	-.114	.363	*
	B	E	2.689	2.420	-.008	.469	*
	C	D	2.567	2.544	-.219	.258	*
	C	E	2.567	2.420	-.112	.365	*
	D	E	2.544	2.420	-.132	.345	*
CHBE	A	B	2.864	2.689	-.086	.409	*
	A	C	2.864	2.567	.027	.522	
	A	D	2.864	2.544	.048	.543	
	A	E	2.864	2.420	.163	.658	
	B	C	2.689	2.567	-.135	.360	*
	B	D	2.689	2.544	-.113	.382	*
	B	E	2.689	2.420	.001	.496	
	C	D	2.567	2.544	-.226	.269	*
	C	E	2.567	2.420	-.112	.383	*
	D	E	2.544	2.420	-.133	.362	*

4. CONCLUSIONS

This study has been done under the situation where similar types of problems are repeated. Different situation, the so-called compound decision problem, is when one should make a simultaneous inference about the compound problem based on current observation. Mathematical formulations for both situations are the same, but the goal and the interpretation of the inferential procedure should be different as pointed out by Berger(1985,p.96).

In multiple comparison problems, it should be remarked that experimenters are faced with the case of large sample information relative to prior information. In this case, Bayes estimates using posterior means shrink the observed data too far toward the prior mean, regardless of small prior information. Thus the expected sample variance of these Bayes estimates becomes only a fraction of the variance of the prior. The coverage probability of the HBE intervals using Bayes estimates falls below the nominal level in some interested range of the prior configuration.

The constrained Bayes estimates adjust the shrinkage parameter so that the sample mean and variance of the estimates match the conditional expectation and variance for parameters. The CHBE intervals using the constrained Bayes estimates resolve the drawback of the HBE intervals by adjusting the center of the intervals and enlarging the length. This merit of the CHBE intervals does not become clear as prior information becomes large relative to sample information, since the HBE intervals having shorter length than the CHBE performs well. But this case is rarely in multiple comparison, thus it may be considered to be no problem.

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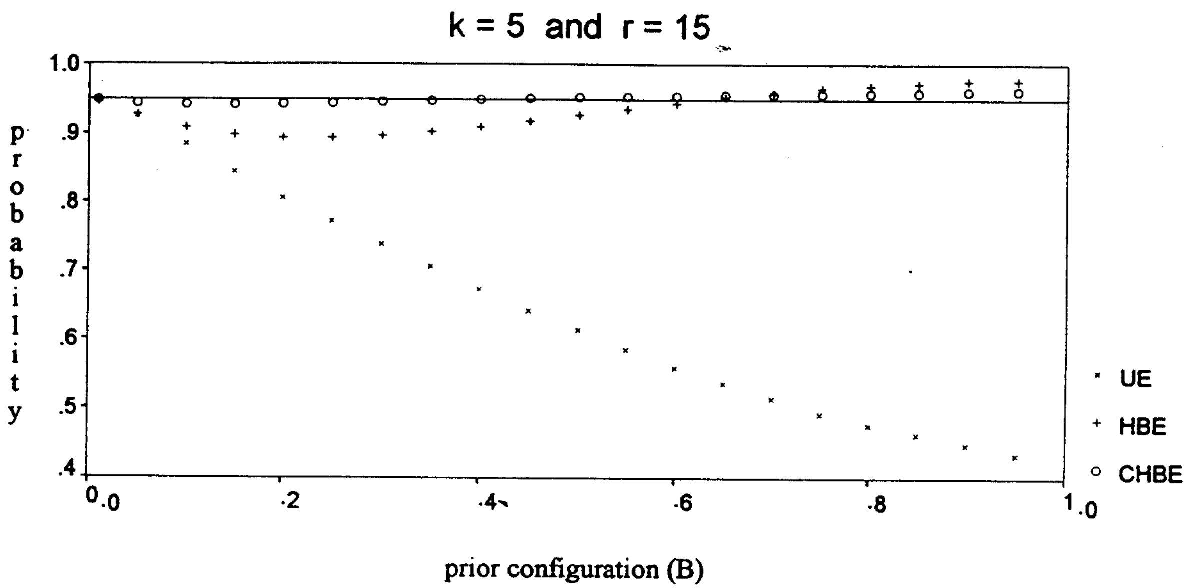


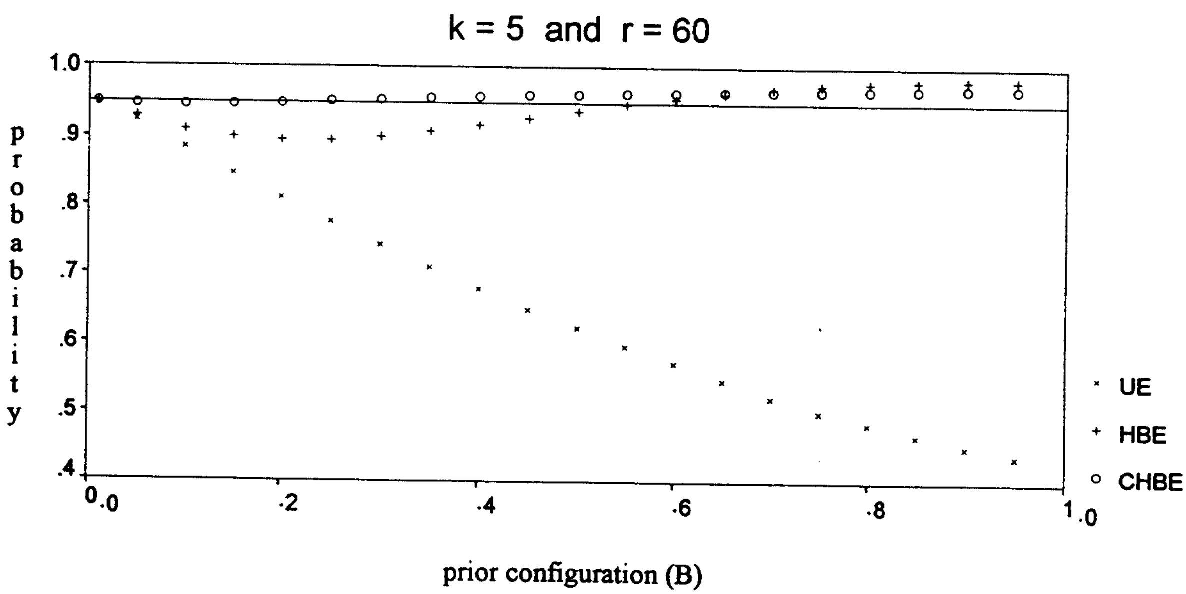
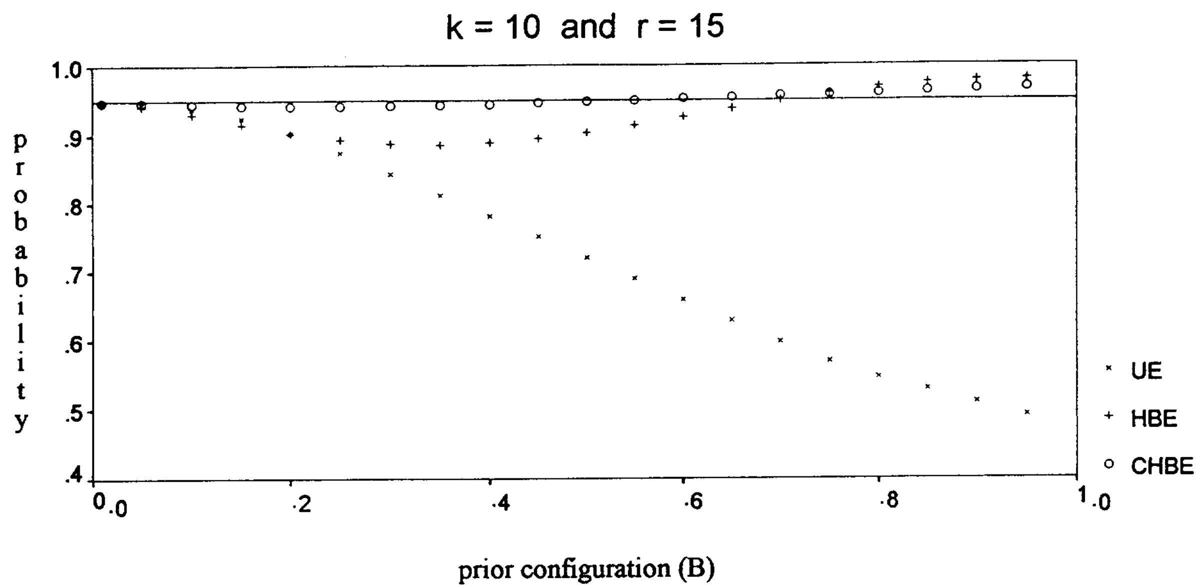


Figure 1. (a) Empirical Bayes coverage probabilities nominal coverage = 0.95



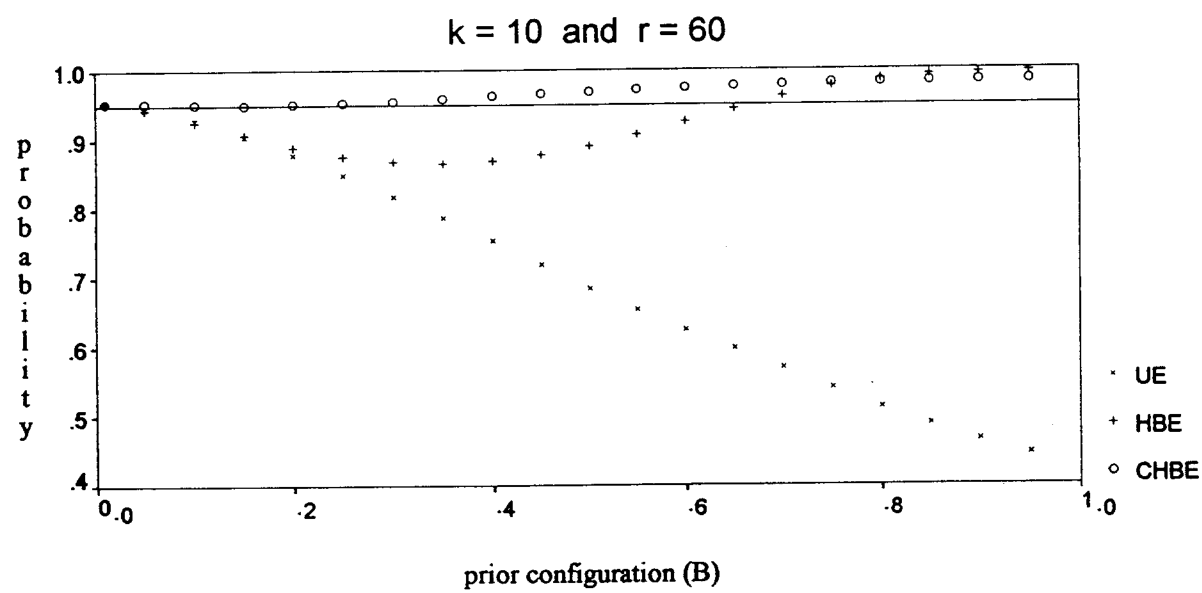


Figure 1. (b) Empirical Bayes coverage probabilities nominal coverage = 0.95

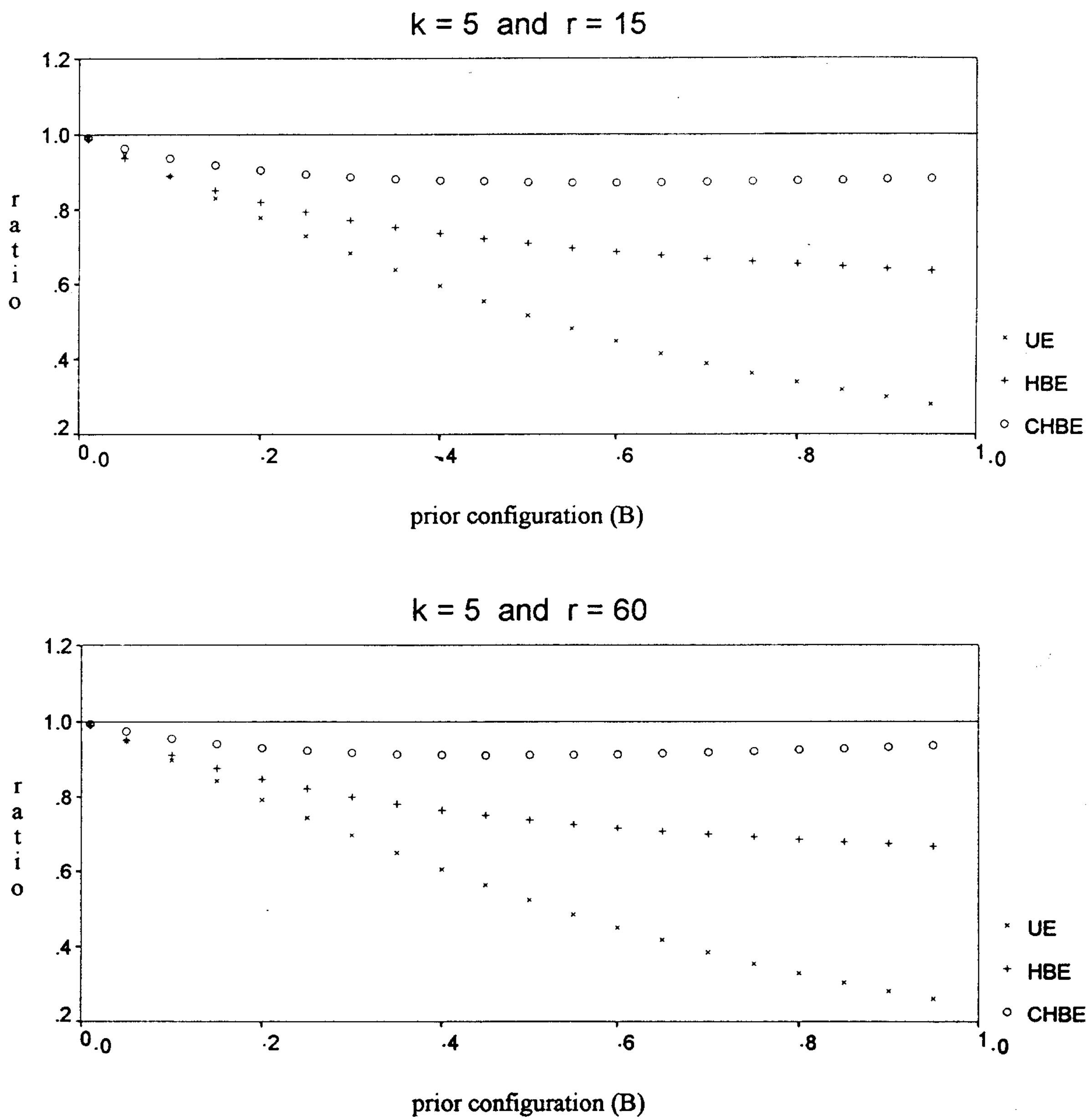
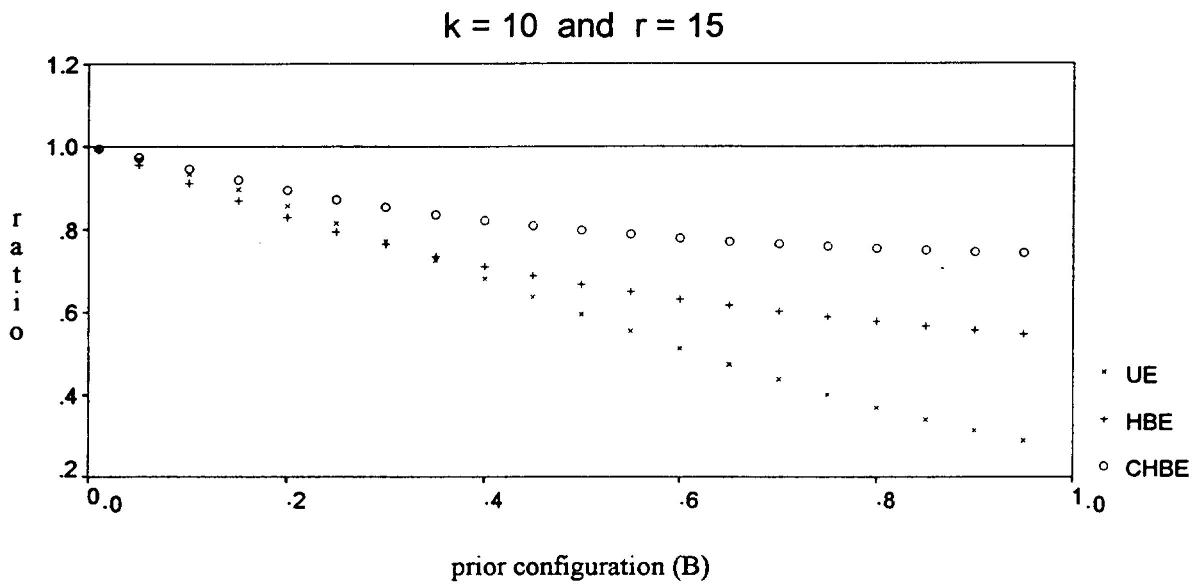


Figure 2. (a) Length ratio relative to Tukey
nominal coverage = 0.95



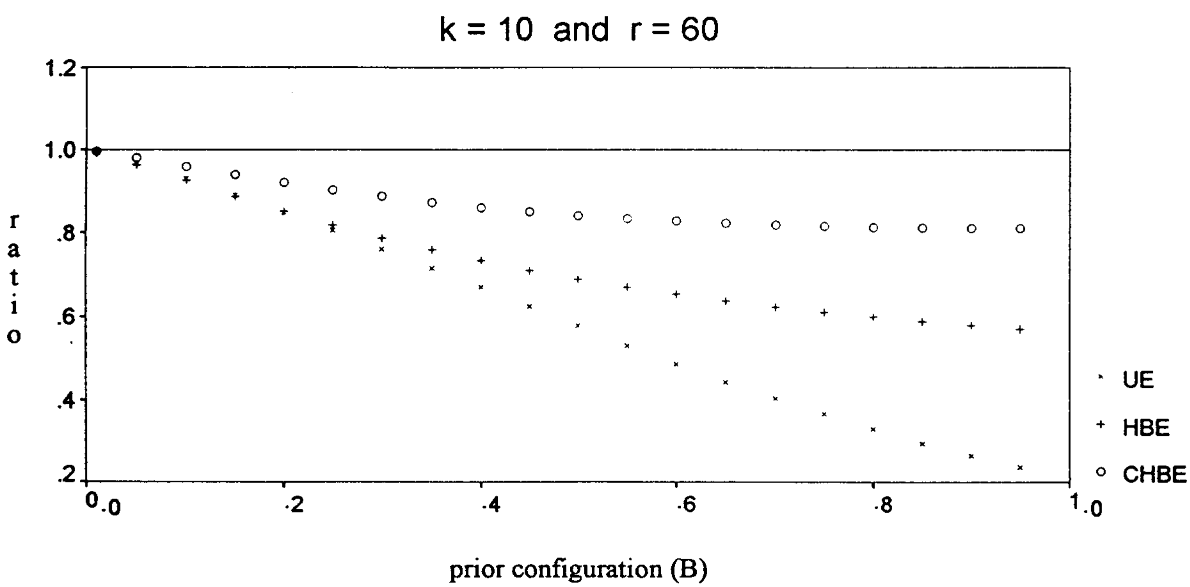


Figure 2. (b) Length ratio relative to Tukey
nominal coverage = 0.95

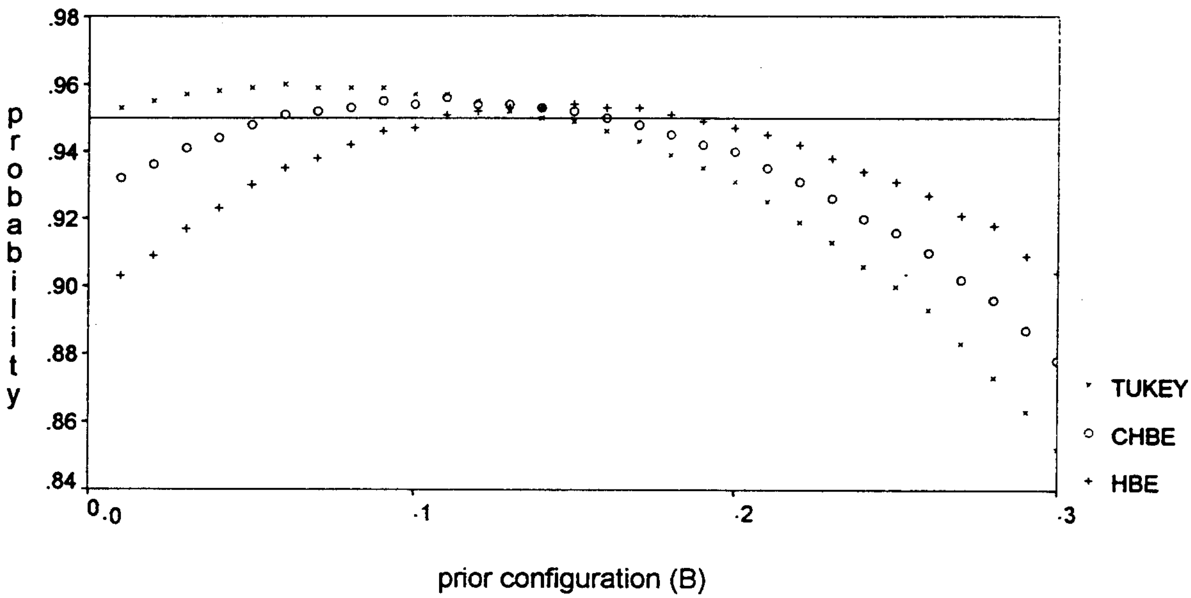


Figure 3. Posterior coverage probabilities