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A Central Limit Theorem with Random Indices for Martingale Difference Sequences

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ABSTRACT

A central limit theorem with random indices is obtained for stationary martingale difference sequences.

KEYWORDS : Central limit theorem, Martingale difference sequence.

1. INTRODUCTION

Let $X_1, X_2, \cdots$ be a stationary ergodic stochastic sequence on $(\Omega, \mathcal{F}, \mathcal{P})$ with $E[X_i|X_1, X_2, \cdots, X_{i-1}] = 0$ almost surely(a.s.) and $EX_1^2 = 1$ and let $\nu_1, \nu_2, \cdots$ be a sequence of positive integer valued random variables. Let $S_n = X_1 + \cdots + X_n$ and let

$$\Phi(\alpha) = \int_{-\infty}^{\alpha} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx.$$

Billingsley(1961) has shown that

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$$\lim_{n\to\infty} P\{\frac{S_n}{\sqrt{n}} \le \alpha\} = \Phi(\alpha).$$

In this paper we prove the following theorem which is a generalization of Billingsely's result.

Theorem 1. If ν_n/n converges in probability to a positive random variable ν which is independent of $\{X_n, n \ge 1\}$, then

$$\lim_{n\to\infty} P\{\frac{S_{\nu_n}}{\sqrt{\nu_n}} \le \alpha\} = \Phi(\alpha) \qquad (-\infty < \alpha < \infty) \tag{1}$$

Remark. Blum, Hanson and Rosenblatt(1963) proved Theorem 1 on the sum of i.i.d. random variables without the assumption that ν is independent of $\{X_n, n \ge 1\}$.

2. RESULTS

Throughout this section, let $X_1, X_2, \cdots$ be a stationary ergodic stochastic sequence on $(\Omega, \mathcal{F}, \mathcal{P})$ with $E[X_i|X_1, X_2, \cdots, X_{i-1}] = 0$ a.s. and $EX_1^2 = 1$ and let $\nu_1, \nu_2, \cdots$ be a sequence of positive integer valued random variables. We will need the following lemmas in the proof of the theorem :

Lemma 1. (1) holds if ν_n/n converges in probability to some positive constant c.

Proof. Just follow the proof of Theorem 7.3.2. in Chung(1974) using Billingsely's result and Hàjek–Renyi inequality (Theorem 7.4.8, Chow and Teicher(1988)) instead of Kolmogorov's inequality.

Lemma 2. Let A be a subset of Ω for which $P(A) > 0$ and let $1_{\{A\}}$ be independent of $\{X_n\}$. Let $Q(\cdot) = P(\cdot|A) = P(\cdot, A)/P(A)$, then $\{X_n\}$ is a stationary ergodic martingale difference sequence with $EX_1^2 = 1$ with respect to Q.

Proof. Observing the following equalities

$$
\begin{aligned}
Q\{X_1 \in A_1, \cdots, X_n \in A_n\} \\
&= P\{X_1 \in A_1, \cdots, X_n \in A_n, A\}/P(A) \\
&= P\{X_1 \in A_1, \cdots, X_n \in A_n\} \\
&= P\{X_2 \in A_1, \cdots, X_{n+1} \in A_n\} \\
&= P\{X_2 \in A_1, \cdots, X_{n+1} \in A_n\}P(A)/P(A) \\
&= P\{X_2 \in A_1, \cdots, X_{n+1} \in A_n, A\}/P(A) \\
&= Q\{X_2 \in A_1, \cdots, X_{n+1} \in A_n\},
\end{aligned}
$$

the stationarity is proved. The ergodicity is obvious. Let $B \in \sigma(X_1, \cdots, X_{n-1})$. Then we have

$$
\begin{aligned}
\int_B X_n dQ &= \int_B X_n 1_{\{A\}} dP/P(A) \\
&= \int_B X_n dP = 0.
\end{aligned}
$$

Therefore $E[X_n|X_1, \cdots, X_{n-1}] = 0$ *a.s.* with respect to Q and similarly we have $\int X_1^2 dQ = \int X_1^2 dP = 1$.

The following lemma is easy to check.

Lemma 3. Suppose ν_n/n converges in probability to a positive random variable λ having a discrete distribution. For any α such that $P\{\lambda = \alpha\} > 0$, define $P_\alpha(\cdot) = P\{\cdot|\lambda = \alpha\} = \frac{P\{\cdot,\lambda=\alpha\}}{P\{\lambda=\alpha\}}$. Then for such α, ν_n/n converges in probability to α with respect to P_α.

Lemma 4. (1) holds if λ is a positive random variable having a discrete distribution which is independent of $\{X_n\}$ and ν_n/n converges in probability to λ.

Proof. Let $\iota_k (0 < \iota_1 < \iota_2 < \cdots)$ denote the values taken on by λ with positive probability. Then we have

$$
P\{\frac{S_{\nu_n}}{\sqrt{\nu_n}} \leq t\} = \sum_{k=1}^{\infty} P_{\iota_k}\{\frac{S_{\nu_n}}{\sqrt{\nu_n}} \leq t\} P(\lambda = \iota_k).
$$

As by Lemma 1,2 and 3

$$
\lim_{n\to\infty} P_{\iota_k}\{\frac{S_{\nu_n}}{\sqrt{\nu_n}} \leq t\} = \Phi(t) \quad for \quad k = 1, 2, \cdots,
$$

the assertion of Lemma 4 follows immediately.

The following lemma, due to Blum et al.(1963, Lemma 2), is used.

Lamma 5. Let $W_n, X_{m,n}, Y^{(j)}_{m,n}$,and $Z^{(j)}_{m,n}$ be random variables for $m, n = 1, 2, \cdots$ and $j = 1, \cdots, k$. Suppose

$$W_n = X_{m,n} + \sum_{j=1}^{k} Y^{(j)}_{m,n} Z^{(j)}_{m,n}$$

and

A) $\lim_{m\to\infty} \limsup_n P\{|Y^{(j)}_{m,n}| > \epsilon\} = 0$ for every $\epsilon > 0$ and $j = 1, \cdots, k$

B) $\lim_{M\to\infty} \limsup_m \limsup_n P\{|Z^{(j)}_{m,n}| > M\} = 0$ for $j = 1, \cdots, k$

C) the distributions of $\{X_{m,n}\}$ converge to the distribution function F for each fixed m.

Then the distribution of $\{W_n\}$ converge to F.

Proof of theorem. We use the same technique as in Blum, Hanson and Rosenblatt(1963). Only slight modifications are needed. Define $\mu_m = \frac{k}{2^m}$ when $\frac{k-1}{2^m} \le \nu < \frac{k}{2^m}$

$$\mu_{m,n} = \nu_n + [n(\mu_m - \nu)],$$

where $[x]$ stands for the greatest integer less than or equal to x. Note that μ_m is independent of $\{X_n, n \ge 1\}$, that μ_m is discrete for each m and that $\mu_{m,n}/n \to \mu_m$ in probability. Write

$$\begin{aligned}\frac{S_{\nu_n}}{\sqrt{\nu_n}} &= \frac{S_{\mu_{m,n}}}{\sqrt{\mu_{m,n}}} + \left(\frac{S_{\nu_n} - S_{\mu_{m,n}}}{\sqrt{n\mu_m}}\right)\left(\sqrt{\frac{n\mu_m}{\nu_n}}\right) + \left(\frac{\sqrt{\mu_{m,n}} - \sqrt{\nu_n}}{\sqrt{\nu_n}}\right)\left(\frac{S_{\mu_{m,n}}}{\sqrt{\mu_{m,n}}}\right) \\ &= X_{m,n} + Y^{(1)}_{m,n} Z^{(1)}_{m,n} + Y^{(2)}_{m,n} Z^{(2)}_{m,n}.\end{aligned}$$

It follows from Lemma 4 that the distribution of $X_{m,n} = Z^{(2)}_{m,n}$ converges to Φ for each m from which $Z^{(2)}_{m,n}$ satisfies condition B of Lemma 5. Also as in the proof of Theorem(Blum et al.1963), we can show that $Y^{(2)}_{m,n}$ satisfies condition A of Lemma 5 and $Z^{(1)}_{m,n}$ satisfies condition B of Lemma 5. Hence it sufficies to show that $\{\frac{S_{\nu_n}-S_{\mu_{m,n}}}{\sqrt{n\mu_m}}\}$ satisfies condition A to complete the proof of the theorem. We note that (see Blum et al.1963 P.392)

$$\limsup_m \limsup_n P\{|\frac{S_{\nu_n}-S_{\mu_{m,n}}}{\sqrt{n\mu_m}}|>\epsilon\}$$

$$\leq \limsup_m \sum_{k=m}^{m2^m} \limsup_n 2P\{max_{n(k-3)2^{-m}<r<n(k+3)2^{-m}}$$

$$|S_r - S_t| > \frac{\epsilon}{2}\sqrt{\frac{nk}{2^m}} | \frac{k-1}{2^m} \leq \nu < \frac{k}{2^m}\} P\{\frac{k-1}{2^m} \leq \nu < \frac{k}{2^m}\},$$

where $t=[n(k-3)2^{-m}]$. From Hàjek–Renyi inequality

$$P\{max_{n(k-3)2^{-m}<r<n(k+3)2^{-m}}|S_r - S_t| > \frac{\epsilon}{2}\sqrt{\frac{nk}{2^m}}\} \leq \frac{24n+2^{m+2}}{\epsilon^2 nk}$$

and $\limsup_n \frac{24n+2^{m+2}}{\epsilon^2 nk} \leq \frac{24}{\epsilon^2 m}$, for $k \geq m$. Noting that ν is independent of $\{X_n, n\geq 1\}$,

$$\limsup_m \sum_{k=m}^{m2^m} \frac{24}{\epsilon^2 m} P\{\frac{k-1}{2^m} \leq \nu < \frac{k}{2^m}\} \leq \limsup_m \frac{24}{\epsilon^2 m} = 0,$$

which completes the proof of the theorem.

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