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A Uniform CLT for Continuous Martingales

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ABSTRACT

An eventual uniform equicontinuity condition is investigated in the context of the uniform central limit theorem (UCLT) for continuous martingales. We assume the usual integrability condition on metric entropy. We establish an exponential inequality for a martingales. Then we use the chaining lemma of Pollard (1984) to prove an eventual uniform equicontinuity which is a sufficient condition of UCLT. We apply the result to approximate a stochastic integral with respect to a martingale to that of a Brownian motion.

KEY WORDS: Continuous martingales, Eventual uniform equicontinuity, Uniform central limit theorem, Chaining argument.

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1. INTRODUCTION AND MAIN RESULTS

Let $A = (\Omega, (\mathcal{F})_{t\geq 0}, \mathcal{F}, P)$ be a stochastic base and (U, d) be a metric space. For each $u \in U$ and $n \in N$, we assume that $X_n(u,t)$ is a martingale in $0 \leq t \leq 1$ relative to A.

This paper discuss an eventual uniform equicontinuity condition in the following sense: $\forall \epsilon > 0,\ \exists \delta > 0$ such that

$$\limsup_n P\{\sup_{d(u,v)\leq\delta}\sup_{0\leq t\leq 1}|X_n(u,t) - X_n(v,t)| \geq \epsilon\} \leq \epsilon. \tag{1.1}$$

We will use some notations: The quadratic variation process of the process X, denoted $[X,X] = ([X,X]_t)_{t\geq 0}$, is defined by $[X,X] = X^2 - 2\int X_- dX$ where X_- is the process whose value at s is given by $(X_-)_s = \lim_{u\to s\ u<s} X_u$, $(X_-)_0 = 0$. We simply denote $[X]$ to mean $[X,X]$. The jump process associated the process X, denoted $\Delta X = (\Delta X_t)_{t\geq 0}$, is defined by $\Delta X_t = X_t - X_{t-}$. For a metric space (U,d) and $u > 0$ we define a packing number by

$\nu(u,U,d) = \max\{m : \textit{There exists a subset } \{u_1, \cdots, u_m\} \subseteq U \textit{ satisfying } d(u_i,u_j) \geq u \textit{ for } i \neq j\}$.

We will prove the following

Theorem 1. Assume

(a) $\int_0^1 [\ln \nu(u,U,d)]^{1/2} du < \infty$,

(b) $P\{X_n(\cdot,\cdot)$ *is continuous*$\} = 1$ *for each* n,

and

(c) $P\{\sup_{(u,v)\in U\times U} \frac{[X_n(u)-X_n(v)]_1}{d^2(u,v)} \geq 1\} \to 0$ *as* $n \to \infty$.

Then the condition (1.1) is satisfied.

Remark. If we further assume the following eventual uniform equicontinuity condition in t for every u: $\forall \epsilon > 0,\ \exists \delta > 0$ such that

$$\limsup_n P\{\sup_{|s-t|\leq\delta}|X_n(u,s) - X_n(u,t)| \geq \epsilon\} \leq \epsilon, \tag{1.2}$$

then we have the following eventual uniform equicontinuity condition: $\forall \epsilon > 0,\ \exists \delta > 0$ such that

$$\limsup_n P\{\sup_{\rho((u,s),(v,t))\leq\delta}|X_n(u,s) - X_n(v,t)| \geq \epsilon\} \leq \epsilon,$$

where $\rho((u,s),(v,t)) = \max\{d(u,v), |s-t|\}$.

The following corollary may be considered as a uniform central limit theorem for continuous martingales.

Corollary 1. Assume (a) ,(b) ,(c) and (1.2). Assume further that

$$[X_n(u) - X_n(v)]_t \to C_{u,v}(t),$$

in probability for each $(u,v) \in U \times U$ and $t \geq 0$ where $C_{u,v}(t)$ is deterministic and continuous in t and $(C_{u_j,v_k}(t) - C_{u_j,v_k}(s))_{1\leq j,k\leq m}$ is positive definite matrix for each choice of $m > 0$, u_j, v_k and $0 \leq s < t$. Then X_n converges in distribution to G, where $G(u,\cdot)$ has Gaussian independent increments and is continuous a.s. in u and s.

Proof. The martingale CLT (See Ethier and Kurtz ,1986, Theorem 1.4 p.339) provides the finite dimensional convergence of X_n to G. The result of Theorem 1 together with (1.2) implies the eventual uniform equicontinuity condition of X_n. These two conditions implies the result. □

2. PROOF OF THE THEOREM

We will prove the theorem by using the usual chaining argument. For this purpose we introduce some more notations. Define a covering integral

$$J(\delta) = \int_0^\delta [2\ln \frac{N^2(u)}{u}]^{1/2} du,$$

where $N(u)$ equals the smallest m for which there exist points $t_1, \cdots, t_m$ with $\min_{1\leq i\leq m} d(t,t_i) \leq u$ for every t in U. The following chaining lemma appears in Pollard (1984, p.144).

Lemma 1. Let $\{Z(u) : u \in U\}$ be a stochastic process with continuous sample paths whose index set has a finite covering integral $J(\cdot)$. Suppose there exists a constant D such that, for all u and v,

$$P\{|Z(u) - Z(v)| > \eta d(u,v)\} \leq 2\exp\{-\frac{\eta^2}{2D^2}\} \quad for\ \eta > 0.$$

Then for $\delta > 0$

$$P\{\sup_{d(u,v)\leq\delta} |Z(u) - Z(v)| > 26DJ(\delta)\} \leq 2\delta.$$

In order to apply Lemma 1 in the proof of the theorem we need an exponential inequality. The following exponential inequality is more general than we need.

Lemma 2. Let X be a local martingale such that $X(0) = 0$, $|\Delta X| \leq M$ *a.s.* and $[X]_t \leq K$ where M and K are finite constants and T is a finite valued stopping time. If $0 \leq \epsilon \leq \frac{3K}{2M}$, then

$$P\{\sup_{0\leq s\leq T} |X_s| \geq \epsilon\} \leq 4\exp\{-\frac{\epsilon^2}{6K}\}.$$

Proof of Lemma 2. We consider the stochastic differential equation

$$Z_t = 1 + \int_0^t Z_{s-}d(\theta X_s),$$

where $0 \leq \theta \leq \frac{1}{2M}$ is fixed. According to Doleans-Dade formula we have

$$Z_t = \{\exp(\theta X_t - \frac{1}{2}\theta^2[X,X]_t^c)\}\{\Pi_{s\leq t}(1 + \theta\Delta X_s)\exp(-\theta\Delta X_s)\},$$

where the process $[X,X]^c$ denotes the path by path continuous part of $[X,X]$. Z_t is a positive local martingale, and hence, by Fatou lemma, it is a supermartingale. In particular, $EZ_t \leq 1\ \forall t$. Since $\ln(1+x) - x \geq -x^2$ for $|x| \leq \frac{1}{2}$ and $[X,X]_t = [X,X]_t^c + \sum_{s\leq t}(\Delta X_s)^2$ we have for each $0 < t \leq T$:

$$\begin{aligned} Z_t &\geq \exp\{\theta X_t - \frac{1}{2}\theta^2[X,X]_t^c - \theta^2\sum_{s\leq t}(\Delta X_s)^2\} \\ &\geq \exp\{\theta X_t - \frac{3}{2}\theta^2[X,X]_t\} \\ &\geq \exp\{\theta X_t - \frac{3}{2}\theta^2 K\}. \end{aligned}$$

Next we can calculate as follows:

$$P\{\sup_{s\le T} X_s \ge \epsilon\} \le P\{\sup_{s\le T} Z_s \ge \exp(\theta\epsilon - \frac{3}{2}\theta^2 K)\}$$
$$\le (E\sup_{s\le T} Z_s)\exp(-\theta\epsilon + \frac{3}{2}\theta^2 K)$$

By Doob's inequality for the super-martingale $Z(T\wedge t)$, we have

$$E\sup_{s\le T} Z_s \le EZ_0 + \sup_{s\le T} EZ_s^-$$
$$\le EZ_0 + EZ_0$$
$$\le 2$$

By choosing $\theta = \frac{\epsilon}{3K}$ with $\epsilon \in [0, \frac{3K}{2M}]$ we have

$$P\{\sup_{s\le T} X_s \ge \epsilon\} \le 2\exp\{-\frac{\epsilon^2}{6K}\}.$$

By working with $-X$ instead of X we see that: If $0 \le \epsilon \le \frac{3K}{2M}$, then

$$P\{\sup_{s\le T} |X_s| \ge \epsilon\} \le 4\exp\{-\frac{\epsilon^2}{6K}\}.$$

Remark. If X has continuous sample paths so that $M = 0$ then condition on ϵ in Lemma 2 disappears.

Proof of Theorem 1. First we note that the condition (a) imply the finiteness of the covering integral $J(\cdot)$. Define a stopping time

$$\tau_n := 1 \wedge \inf\{0 \le t : [X_n(u) - X_n(v)]_t > d^2(u,v) \ for\ some\ (u,v) \in U \times U\}$$

Note that $[X_n(u) - X_n(v)]_{\tau_n} \le d^2(u,v)$. By Lemma 2 we have, for each $\eta > 0$,

$$P\{\sup_{0\le t<\tau_n} |X_n(u,t) - X_n(v,t)| \ge \eta d(u,v)\} \le 4\exp\{-\frac{\eta^2}{6}\}.$$

Also we get from (c) that $P\{\tau_n < 1\} \to 0$. So it suffices to prove our theorem for $X_n(\cdot, t\wedge\tau_n)$. By applying Lemma 1 with $D = \sqrt{3}$, we have

$$P\{\sup_{d(u,v)\le\delta}\sup_{0\le t<\tau_n}|X_n(u,t)-X_n(v,t)|>26DJ(\delta)\}\le 4\delta.$$

Let $\epsilon>0$. Choose $\delta>0$ small enough to have $26DJ(\delta)\wedge 4\delta<\epsilon$. Then

$$\limsup_n P\{\sup_{d(u,v)\le\delta}\sup_{0\le t<\tau_n}|X_n(u,t)-X_n(v,t)|>\epsilon\}\le\epsilon.$$

This completes the proof of the theorem. □

3. CONVERGENCE OF STOCHASTIC INTEGRALS

Let $\{M_n\}$ be a sequence of continuous martingales in t. Assume $[M_n]_1\to 1$ in probability. Let $\mathcal{F}$ be a collection of real valued left continuous functions defined on $[0,1]$ which is uniformly bounded by 1. We define a metric d on $\mathcal{F}$ by $d^2(f,g)=\sup_{t\in[0,1]}|f(t)-g(t)|$. We assume that $\int_0^1[\ln\nu(u,\mathcal{F},d)]^{1/2}du$ is finite. We consider $X_n(f,t)=\int_0^t f(s)dM_n(s)$, $f\in\mathcal{F}$, $t\in[0,1]$. Then it follows that, for each fixed $f\in\mathcal{F}$, $X_n(f,t)$ is also a martingale in t. We observe that(See Protter (1990, Theorem 29, p.68))

$$\begin{aligned}
&P\{\sup_{(f,g)\in\mathcal{F}\times\mathcal{F}}\frac{[X_n(f)-X_n(g)]_1}{d^2(f,g)}\ge 1\}\\
&= P\{\sup_{(f,g)\in\mathcal{F}\times\mathcal{F}}\frac{\int_0^1\{f(s)-g(s)\}^2d[M_n]_s}{d^2(f,g)}\ge 1\}\\
&\le P\{\sup_{(f,g)\in\mathcal{F}\times\mathcal{F}}\frac{\sup_{t\in[0,1]}|f(t)-g(t)|\int_0^1|f(s)-g(s)|d[M_n]_s}{d^2(f,g)}\ge 1\}\\
&= P\{\sup_{(f,g)\in\mathcal{F}\times\mathcal{F}}\int_0^1|f(s)-g(s)|d[M_n]_s\ge 1\}\\
&\le P\{\sup_{(f,g)\in\mathcal{F}\times\mathcal{F}}2[M_n]_1\ge 1\}\\
&= P\{[M_n]_1\ge\frac{1}{2}\}\to 0.
\end{aligned}$$

By applying the theorem we have an eventual uniform equicontinuity of $\{X_n\}$: $\forall\epsilon>0$, $\exists\delta>0$ such that

$$\limsup_n P\{\sup_{d(f,g)\le\delta}\sup_{0\le t\le 1}|X_n(f,t)-X_n(g,t)|\ge\epsilon\}\le\epsilon.$$

If we also assume the condition similar to (1.2), then we have the eventual uniform equicontinuity condition: $\forall\epsilon>0,\ \exists\delta>0$ such that

$$\limsup_n P\{\sup_{\rho((f,s),(g,t))\le\delta}|X_n(f,s)-X_n(g,t)|\ge\epsilon\}\le\epsilon$$

where $\rho((f,s),(g,t))=\max\{d(f,g),|s-t|\}$.

Then by the corollary we get that X_n converges in distribution to G where $G(f,t)=\int_0^t f(s)dB$ is a.s. continuous in (f,t) and B is standard Brownian motion.

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