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A Note on the Strong Mixing Property for a Random Coefficient Autoregressive Process [†]

Sangyeol Lee[1]

ABSTRACT

In this article we show that a class of random coefficient autoregressive processes including the NEAR (New exponential autoregressive) process has the strong mixing property in the sense of Rosenblatt with mixing order decaying to zero. The result can be used to construct model free prediction interval for the future observation in the NEAR processes.

KEYWORDS : Random coefficient autoregresive process, NEAR (New exponential autoregressive) process, Strong mixing, Mixing order, Model free prediction interval.

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[1] Department of Statistics, Sookmyung Women's University, Yongsan-ku, Seoul, 140-742, Korea

1. INTRODUCTION

Many of statistical techniques available for time series have been developed under the Gaussian assumption, which usually leads to more tractable situations. For example, if $\{X_t\}$ is a Gaussian linear process, namely, $X_t = \sum_{j=0}^{\infty} a_j \varepsilon_{t-j}$, where $\{\varepsilon_t\}$ are i.i.d. Gaussian random variables and $\sum_{j=0}^{\infty} |a_j| < \infty$, the best predictor of X_{n+h} based on $X_1, \ldots, X_n$ in the mean square norm is given by a linear combination of those. In practice, however, there are a large number of cases where time series data does not show characteristics not shown by Gaussian processes. For example, the study of 21 economic time series by Nelson and Granger (1979) shows that Gaussian time series modelling fails even after suitable transformation.

In recent years, among others, the **NEAR** model introduced by Lawrance and Lewis (1985) has attracted much attention from researchers. It has been reported that the NEAR model fits well the time series data with positive and highly skewed distribution, e.g., wind velocity, the service in a queue, or the daily flows of river.

In the ordinary autoregressive time series model of the form $X_t = \beta_1 X_{t-1} + \cdots + \beta_p X_{t-p} + \varepsilon_t$, where ε_t are i.i.d. innovations with mean zero and finite variance, the best linear predictor of X_{n+1} based on $X_t, t = 1, \ldots, n$ in the mean square norm is the linear combination $\beta_1 X_n + \cdots + \beta_p X_{n-p+1}$, and thereby it is only required to estimate the parameters β_j, $j = 1, \ldots, p$ for the future prediction.

Like the ordinary case, the prediction in NEAR (p) processes requires suitable estimates for the autoregressive coefficients. However, as pointed out in Smith (1986, P. 252), there are several technical difficulties in estimating those parameters even in NEAR processes of order 2. Thus, instead of conventional methods, one can consider as an alternative the model free prediction approach that Cho and Miller (1987) has proposed for strictly stationary strong mixing processes. (See Butler (1982) for i.i.d. random sample). However, in order to utilize their method it is crucial to verify that a given NEAR process has a mixing order decaying exponentially to zero. For this reason, we focus on the question–whether the NEAR processes have the desired mixing order. Actually this question has been handled by Son and Cho (1988), but in this article somewhat different approach is taken to provide a lot simpler proof for the proposition.

In the next section we show that a class of random coefficient autoregressive time series including the NEAR (p) processes satisfy the strong mixing condition with mixing order decaying exponentially to zero. Here, the order p is not necessarily 1 or 2.

2. MAIN RESULTS

Assume that $\{X_t\}$ is a statinary process of the form:

$$X_t = \varepsilon_t + \begin{cases} \beta_1 X_{t-1} & \text{w.p. } \alpha_1 \\ \vdots & \\ \beta_p X_{t-p} & \text{w.p. } \alpha_p \\ 0 & \text{w.p. } \alpha_0 = 1 - \sum_{j=1}^{p} \alpha_j, \end{cases} \tag{2.1}$$

where $\{\varepsilon_t\}$ are i.i.d. random variables of distribution not necessarily exponential, $\alpha_j > 0, \sum_{j=0}^{p} \alpha_j = 1$ and $|\beta_j| < 1$. In addition, assume that initial random variable X_0 is independent of $\{\varepsilon_t; t \geq 1\}$. The above model with exponential distribution has been considered by Billard and Mohamed (1991).

Before we proceed we present here the definition of the strong mixing process.

Definition 1. A strictly stationary process $\{X_t; t \in \mathcal{Z}\}$ is a strong mixing process if there exists a sequence $\{\alpha(k)\}$ of positive real numbers such that for all A and B that belong to the σ-fields generated by $(\ldots, X_1, \ldots X_n)$ and $(X_{n+k}, X_{n+k+1}, \ldots)$, respectively,

$$|P(A \cap B) - P(A)P(B)| \leq \alpha(k) \to 0 \quad \text{as} \quad k \to \infty.$$

Theorem 1. $\{X_t\}$ in (2.1) is a strong mixing process with order $\gamma(k) = O(e^{-\theta k})$ for some $\theta > 0$.

Proof. Let $A \in \sigma(X_0, \ldots, X_n)$ and $B \in \sigma(X_{n+k}, \ldots, X_N), k > p$, where $\sigma(X_j, \ldots, X_{j'})$, $j \leq j'$, denotes the σ-fields generated by the random variables $X_j, \ldots, X_{j'}$. Introduce the i.i.d. random variables $\{V_t\}$ taking values $0, \ldots, p$, and independent of X_0 and ε_t, $t \geq 1$, such that $P(V_t = j) = \alpha_j$, $\beta_0 = 0$, and on the event $(V_t = j)$, $X_t = \varepsilon_t + \beta_j X_{t-j}$, viz. $(V_t = j) = (X_t = \varepsilon_t + \beta_j X_{t-j})$. Write $k - 1 = lp + q$, where $l \geq 1$, $0 \leq q \leq p - 1$, and let π be the event on which there are p consecutive V_t with t ranging from $n + 1$ to $n + lp$, such that $V_t = 0$.

Note first that on π, the events A and B are independent. Let π_1 denote the event $(V_{n+1} = \cdots = V_{n+p} = 0)$ and π_i, $2 \le i \le lp - p + 1$, be the event such that $V_{n+1} \ne 0, \ldots, V_{n+i-1} \ne 0$ and $V_{n+i} = \cdots = V_{n+i+p-1} = 0$. On π_i, it is obvious that $\{X_0, \ldots, X_n\}$ and $\{X_{n+i}, \ldots, X_N\}$ are independent. Then, by the independence of $\{V_t\}$ and $\{\varepsilon_t\}$, we have that

$$|P(A \cap B|\pi_i) - P(A)P(B|\pi_i)| = 0.$$

Since π_i are disjoint and their union equals to π, we obtain

$$\begin{aligned}&|P(A \cap B|\pi) - P(A)P(B|\pi)| \\ &\le \sum_{i=1}^{lp-p+1} |P(A \cap B|\pi_i) - P(A)P(B|\pi_i)|P(\pi_i)/P(\pi) = 0.\end{aligned}$$

Hence, combining with the following:

$$|P(A \cap B|\pi^c) - P(A)P(B|\pi^c)| \le 1,$$

we can write

$$\begin{aligned}|P(A \cap B) - P(A)P(B)| &\le P(\pi^c)|P(A \cap B|\pi^c) - P(A)P(B|\pi^c)| \\ &\le P(\pi^c),\end{aligned}$$

and it suffices to find the desired upper bound for $P(\pi^c)$.

Towards this end, consider the event π^* on which $V_{n+i_j} \ne 0$ for some $i_j \in [p(j-1)+1, pj], j = 1, \ldots, l$. Note that π^c is contained in π^* and $P(\pi^*) \le (1 - \alpha_0^p)^l$. This together with the inequality $l \ge k/p - 1$ yields that

$$|P(A \cap B) - P(A)P(B)| = O\Big((1 - \alpha_0^p)^{k/p}\Big),$$

which completes the proof. □

Thus far, we have shown that the process in (2.1) that includes the NEAR (p) processes is a strong mixing process with mixing order decaying to 0 exponentially. However, we did not investigate how the coefficients ϕ_i affect the mixing order. The remainder of this section deals with this problem.

From now on we restrict ourselves to the process

$$X_t = \varepsilon_t + \begin{cases} \phi X_{t-1} & \text{w.p. } \alpha \\ 0 & \text{w.p. } 1-\alpha \end{cases} \tag{2.2}$$

with $|\phi| < 1$ and initial random variable X_0 independent of $\varepsilon_t, t \geq 1$, beacuse from the result for the above process one can easily expect similar results for the processes in (2.1) with $p \geq 2$.

The following is a modification of the theorem of Gorodetskii (1977) for our own purpose.

Lemma 1. Suppose that $\{Y_t; t \in \mathcal{Z}\}$ is a stationary autoregressive process of order 1 such that

$$Y_t = \phi Y_{t-1} + \varepsilon_t, \quad |\phi| < 1,$$

where $E\varepsilon_t = 0$, $E\varepsilon_t^2 < \infty$. Further, assume that the density function g of ε_t satisfies

$$\int |g(x+\xi) - g(x)|dx \leq C|\xi| \text{ for some } C > 0. \tag{2.3}$$

Then $\{Y_t\}$ is a strong mixing process with mixing order $O(|\phi|^{\zeta k})$ for some $\zeta > 0$.

It is worthwhile to see that the condition (2.3) holds if $\int |g'(x)|dx < \infty$.

Theorem 2. Suppose that $\{X_t\}$ is the process in (2.2) with the errors ε_t satisfying the conditions of Lemma 1. Then $\{X_t\}$ is a strong mixing process with mixing order $\gamma(k) = O(|\phi|^{\zeta k}\alpha^k)$ for some $\zeta > 0$.

Proof. Let $A \in \sigma(X_0, \ldots, X_n)$ and $B \in (X_{n+k}, \ldots, X_N)(k > 1)$ and W_t be the iid random variables independent of X_0 and $\varepsilon_t, t \geq 1$, such that $(W_t = 1) = (X_t = \phi X_{t-1} + \varepsilon_t), (W_t = 0) = (X_t = \varepsilon_t)$, and further $P(W_t = 1) = \alpha$ and $P(W_t = 0) = 1 - \alpha$. Let π be the event such that on π, $W_{n+j} = 0$ for some $1 \leq j \leq k-1$. Then by Lemma 1,

$$\begin{aligned} &|P(A \cap B) - P(A)P(B)| \\ &\quad \leq |P(A \cap B|\pi) - P(A)P(B|\pi)|P(\pi)(=0) \\ &\quad +|P(A \cap B|\pi^c) - P(A)P(B|\pi^c)|P(\pi^c) \\ &\quad \leq |\phi|^{\zeta k}\alpha^{k-1} \end{aligned}$$

because on π^c, $\{X_t\}$ follows the autoregressive process scheme in (2.2). Hence, the theorem is established. □

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