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A Note on Stationary Linearly Positive Quadrant Dependent Sequences †

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ABSTRACT

In this note we prove an invariance principle for strictly stationary linear positive quadrant dependent sequences, satisfying some assumption on the covariance structure, $0 < \sum \mathrm{Cov}(X_1, X_j) < \infty$. This result is an extension of Burton, Dabrowski and Dehlings' invariance principle for weakly associated sequences to LPQD sequences as well as an improvement of Newman's central limit theorem for LPQD sequences.

KEYWORDS: Linearly positive quadrant dependence, invariance principle.

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1. INTRODUCTION

In recent years there has been growing interest in concepts of positive dependence for families of random variables. Lehmann[5] introduced a simple and natural definition of positive dependence. Random variables X, Y are said to be positive quadrant dependent(PQD) if for any real r, s

$$P\{X > r, Y > s\} \geq P\{X > r\}P\{Y > s\}.$$

A much stronger concept than PQD was considered by Esary, Proschan, and Walkup [4]. A sequence $\{X_j : j \in N\}$ of random variables is said to be associated if for any finite collection $\{X_{j(1)}, \cdots, X_{j(n)}\}$ and any real coordinatewise nondecreasing functions f, g on R^n

$$\mathrm{Cov}(f(X_{j(1)}, \cdots, X_{j(n)}), g(X_{j(1)}, \cdots, X_{j(n)})) \geq 0,$$

whenever the covariance is defined.

There exist a central limit theorem [6,7] and an invariance principle [8] for associated sequences. Most of these results, however, have not been appiled to weaker concepts of positive dependence. In stead of association Newman's original central limit theorem requires only that positive linear combinations of the random variables are PQD.

A sequence $\{X_j : j \in N\}$ of random variables is said to be linearly positive quadrant dependent (LPQD) if for any disjoint A, B and positive $r_j^{\prime}s$

$$\sum_{i \in A} r_i X_i \text{ and } \sum_{j \in B} r_j X_j \text{ are PQD.}$$

Newman[7] showed that this concept of positive dependence is between PQD and association and that the X_j's are jointly independent if and only if $\mathrm{Cov}(X_j, X_k) = 0$ for all $j \neq k$ (see Theorem 6 of [7]). The following central limit theorem for stationary LPQD sequences is due to Newman [7]:

Theorem A.(Newman(1984)) Let $\{X_j : j \in N\}$ be a strictly stationary LPQD sequence with $EX_j = 0, EX_j{}^2 < \infty$. Assume

$$0 < \sigma^2 = \mathrm{Var}(X_1) + 2\sum_{j=2}^{\infty} \mathrm{Cov}(X_1, X_j) < \infty. \tag{1.1}$$

Then $\{X_j : j \in N\}$ fulfills the central limit theorem, that is, ${\sigma_n}^{-1}S_n$ is asymptotically normally distributed where $S_n = X_1 + \cdots + X_n$, $\quad E{S_n}^2 = {\sigma_n}^2$.

Burton, Dabrowski, and Dehling [3] introduced a much weaker concept than association but a stronger concept than LPQD. A sequence $\{X_j : j \in N\}$ of random variables said to be weakly associated if for any finite collection $\{X_{j(1)}, \cdots, X_{j(n)}\}$ and any real coordinatewise nondecreasing functions f, g

$$\mathrm{Cov}(f(X_{j(r)} : r \in A), g(X_{j(s)} : s \in B)) \geq 0$$

whenever $A \cap B = \emptyset$ and $A \cup B = \{1, 2, \cdots, n\}$.

Theorem A can be extended to an invariance principle if instead of LPQD the stronger concept of weak association is required :

Theorem B.(Burton, Dabrowski, Dehling (1986)) Let $\{X_j : j \in N\}$ be a strictly stationary weak associated sequence with $EX_j = 0, E{X_j}^2 < \infty$. Assume that (1.1) holds. Then $\{X_j : j \in N\}$ fulfills an invariance principle, that is,

$$W_n(t) = {\sigma_n}^{-1} S_{[nt]}, \quad t \in [0, 1]$$

converges weakly to standard Brownian motion W on the set of all functions on [0,1] which have left hand limits and are continuous from the right.

It is the purpose of this note to show that Theorem B still holds if instead of weak association the weaker concept of LPQD is required without any further assumptions.

Some preliminary results for LPQD random variables are stated in Section 2. An invariance principle for an LPQD sequence is also given in Section 3.

2. PRELIMINARIES

In the following theorem we carry out that Theorem 2 of Newman and Wright[8] still holds for LPQD random variables.

Theorem 2.1. Let $\{X_j : j \in N\}$ be a strictly stationary LPQD sequence with $EX_j = 0, E{X_j}^2 < \infty$. Put

$$M_n = \max(S_1, S_2, \cdots, S_n).$$

Then we have

$$E({M_n}^2) \leq E({S_n}^2) \tag{2.1}$$

Proof. Define

$$\begin{aligned} K_m &= \min(X_2 + \cdots + X_m, X_3 + X_4 + \cdots + X_m, \cdots, X_m, 0), \\ L_m &= \max(X_2, X_2 + X_3, \cdots, X_2 + \cdots + X_m), \\ J_m &= \max(0, L_m). \end{aligned}$$

Then $K_m = (X_2 + \cdots + X_m) - J_m$ and K_m is a nondecreasing function of the X_i's so that $\text{Cov}(X_1, K_m) \geq 0$. From the facts that $J_m^2 \leq L_m^2$ pointwise, and that $M_m = X_1 + J_m$ we have

$$\begin{aligned} E(M_m^2) &= E((X_1 + J_m)^2) \\ &= E(X_1^2) + 2\text{Cov}(X_1, J_m) + E(J_m^2) \\ &= E(X_1^2) + 2\text{Cov}(X_1, X_2 + \cdots + X_m) \\ &\quad -2\text{Cov}(X_1, K_m) + E(J_m^2) \\ &\leq EX_1^2 + 2\text{Cov}(X_1, X_2 + \cdots + X_m) + E(L_m^2). \end{aligned} \tag{2.2}$$

The proof is completed by induction on m since the induction hypothesis implies $E(L_m^2) \leq E(X_2 + \cdots + X_m)^2$ which together with (2.2) yields (2.1).

Theorem 2.2. Let $\{X_j : j \in N\}$ be a strictly stationary LPQD sequence with $EX_j = 0, E{X_j}^2 < \infty$. Define ${S_n}^* = \max(0, S_1, \cdots, S_n)$. Then for $\lambda_1 < \lambda_2$,

$$P({S_n}^* \geq \lambda_2) \leq (1 - {\sigma_n}^2/(\lambda_2 - \lambda_1)^2)^{-1} P(S_n \geq \lambda_1), \tag{2.3}$$

$$P(\max(|S_1|, |S_2|, \cdots, |S_n|) \geq \lambda\sigma_n) \leq 2P(|S_n| \geq (\lambda - \sqrt{2})\sigma_n). \tag{2.4}$$

Proof. Note that for $\lambda_1 < \lambda_2$,

$$P({S_n}^* \geq \lambda_2) \;\leq\; P(S_n \geq \lambda_1) + P({S_{n-1}}^* \geq \lambda_2, {S_{n-1}}^* - S_n > \lambda_2 - \lambda_1)$$

$$\leq P(S_n \geq \lambda_1) + P(S_{n-1}{}^* \geq \lambda_2)P(S_{n-1}{}^* - S_n > \lambda_2 - \lambda_1)$$
$$\leq P(S_n \geq \lambda_1) + P(S_n{}^* \geq \lambda_2)E((S_{n-1}{}^* - S_n)^2)/(\lambda_2 - \lambda_1)^2,$$

where the second inequality follows from the fact that S^*_{n-1} and $S_n - S^*_{n-1}$ are PQD since $X_i's$ are LPQD random variables and the third inequality is obtained by Chebyshev's inequality. Now Theorem 2.1 with X_i replaced by $Y_i = -X_{n-i+1}$ yields that

$$E([S^*_{n-1} - S_n]^2) = E([max(Y_1, Y_1 + Y_2, \cdots, Y_1 + \cdots + Y_n)]^2) \leq E(S_n{}^2)$$

and thus we have, for $(\lambda_2 - \lambda_1)^2 \geq E(S_n{}^2) \equiv \sigma_n{}^2$, that (2.3) holds. By adding to (2.3) the analogous inequality with each X_i replaced by $-X_i$ and by choosing $\lambda_2 = \lambda\sigma_n, \lambda_1 = (\lambda - \sqrt{2})\sigma_n$, we obtain (2.4).

Note that (2.4) yields by the standard argument the needed tightness of the distribution of the $W_n's$ to obtain the desired convergence in distribution.

Remark. Note that Newman and Wright[8] showed that Theorem 2.2 holds for associated sequences.

Lemma 2.3.(Birkel(1993)) For each $n \geq 1$ let $X^{(n)}, Y^{(n)}$ be PQD random variables such that

$$(X^{(n)}, Y^{(n)}) \rightarrow_n (X, Y) \text{ in distribution.}$$

Then X, Y are PQD.

3. AN INVARIANCE PRINCIPLE

Theorem 3.1. Let $\{X_j : j \in N\}$ be a strictly stationary LPQD sequence with $EX_j = 0, EX_j{}^2 < \infty$. Assume that (1.1) holds. Then $\{X_j : j \in N\}$ fulfills an invariance principle.

Proof. First we will show that the finite dimensional distributions of W_n converge to those of the standard Wiener process W. Let X be a limit in distribution of a subsequence of $\{W_n : n \in N\}$. Then it suffices to show that X is distributed like W. By Theorem A $\{X_j : j \in N\}$ fulfills the central limit theorem and by simple estimates based on (1.1) we obtain for $t \in [0,1]$

$$W_n(t) \longrightarrow_n N(0,t) \text{in distribution.} \tag{3.1}$$

Hence the sets $\{W_n(t) : n \in N\}$ and $\{{W_n}^2(t) : n \in N\}$ are uniformly integrable. As

$$W_n(t) \longrightarrow_n X(t), \quad {W_n}^2(t) \longrightarrow_n X^2(t)$$

in distribution (for subsequence),

$$EX(t) = 0, \qquad EX^2(t) = t$$

by Theorem 5.4 of Billingsley [1]. According to Theorem 19.1 of Billingsley [1], X is distributed like W if X has independent increments, that is,

$$X(t_1) - X(t_0), \cdots, X(t_k) - X(t_{k-1}) \tag{3.2}$$

are independent for all $k \geq 1, 0 \leq t_0 \leq t_1 \leq \cdots \leq t_k \leq 1$.
To show (3.2), put

$$U_{n,i} = W_n(t_i) - W(t_{i-1}), \quad 1 \leq i \leq k.$$

Since the U_{ni} are PQD random variables and

$$(U_{n1}, \cdots, U_{nk}) \longrightarrow_n (X(t_1) - X(t_0), \cdots, X(t_k) - X(t_{k-1}))$$

in distribution (for a subsequence) according to Lemma 2.3 the $X(t_i) - X(t_{i-1})$ are LPQD. It would follow from (3.1) by simple estimates that for $0 \leq t_0 \leq t_1 \leq \cdots \leq t_k \leq 1$,

$$W_n(t_i) - W_n(t_{i-1}) \longrightarrow_n N(0, t_i - t_{i-1}) \text{ in distribution,} \tag{3.3}$$

$1 \leq i \leq k$ and then by simple estimates based on (1.1) that for $i \neq j$,

$$\lim_{n\in N}\mathrm{Cov}(U_{ni},U_{nj})=0. \tag{3.4}$$

Using Theorem 5.4 of Billingsley [1] and (3.4), we get, for $i\neq j$,

$$\mathrm{Cov}(X(t_i)-X(t_{i-1}),X(t_j)-X(t_{j-1}))=\lim_{n\in N}\mathrm{Cov}(U_{ni},U_{nj})=0.$$

Hence the $X(t_i)-X(t_{i-1})$ are uncorrelated, PQD random variables and thus independent by the statement preceding Theorem A. This proves (3.2). It remains to show the needed tightness, that is applying (2.4) to the LPQD random variables involved in this theorem, we have, for $\lambda > 2\sqrt{2}$,

$$P\{\max_{i\le n}|S_i|\ge\lambda\sigma_n\}\le 2P\{|S_n|\ge\frac{1}{2}\lambda\sigma_n\}.$$

By the central limit theorem (Theorem A),

$$P\{|S_n|\ge\frac{1}{2}\lambda\sigma_n\}\longrightarrow_n P\{|N|\ge\frac{1}{2}\lambda\}\le\frac{8}{\lambda^3}E|N|^3.$$

Therefore, if ε is positive, we have

$$\limsup_{n\in N}P\{\max_{i\le n}|S_i|\ge\lambda\sigma_n\}<\frac{\varepsilon}{\lambda^2}$$

for λ sufficiently large. Tightness now follows by Theorem 8.4 of Billingsley [1]. Thus the proof of this theorem is complete.

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