한국통계학회논문집 제2권 1호
1995년 4월 pp. 201-208

Some Dependence Structures of Multivariate Processes

Jong -Il Baek[1)]

1. Introduction

In the last years there has been growing interest in concepts of positive dependence for families of random variables such that concepts are considerable us in deriving inequalities in probability and statistics.

Lehman[12] introduced various concepts of positive dependence for bivariate random variables.A much stronger notions of positive dependence were later considered by Esary,Proschan,and Walkup[8].Ahmed et al[1] and Ebrahimi and Ghosh[5] also obtained multivariate versions of various bivariate positive dependence as descrived by Lehman[12].See also Block al[3]. Glaz and Johnson[10] and Barlow and Proschan[2] and the references there.

Multivariate processes arise when instead of observing a single process we observe several processes, say $X_1(t), \cdots, X_n(t)$ simultaneously. For example, in an engineering context we may want to study the simultaneous variation of current and voltage, or temperature, pressure and volume over time.In economics we may be interested in studying inflation rates and money supply,unemployment and interest rates. We could of course, study each quantity on its own and treat each as a separate univariate process. Although this would give us some information about each quantity it could never give information about the interrelationship between various quantities. This leads us to introduce some concepts of positive and for multivariate stochastic processes.The concepts of positive dependence have subsequently been extended to stochastic processes in different directions by many authors.

In section 2, we introduce the various concepts of multivariate processes,namely, positively upper(lower)orthant dependent(PUOD(PLOD)), associated, right corner set increasing (RCSI), right tail increasing in sequence (RTIS).In section 3, we study the properties of these concepts and derive their relationships among them .

Finally,we show that the stochastic processes present a sampling of useful examples and applications of the theory developed in Sections 2 and 3.

2. Definitions and Notations

Suppose that $\{X_1(t)|t \geq 0\}, \cdots, \{X_n(t)|t \geq 0\}$ are stochastic processes. The state space of

1) Department of Statistics, Won-Kwang University, Iri-City, 570-749, KOREA

$X_i(t)$ will be taken to be a subset E_i of the real line R, $i=1,2,\cdots,n$. For any state $a_i \in E_i$, $i=1,2,\ldots,n$, we define the random times as follows

$$T_i(a_i)=\infty\{t \mid X_i(t) \geq a_i, 0 \leq t \leq \infty\}, \tag{2.1}$$

that is, $T_i(a_i)$ is the first hitting time that the process $X_i(t)$ reaches a_i.

Definition 2.2. Stochastic processes $\{X_1(t) \mid t \geq 0\}, \cdots, \{X_n(t) \mid t \geq 0\}$ are said to be positively upper orthant dependent(PUOD) if

$$P(T_1(a_1) > t_1, \cdots, T_n(a_n) > t_n) \geq P(T_1(a_1) > t_1) \cdots P(T_n(a_n) > t_n)$$

for all t_i and a_i, $i=1,2,\ldots,n$.

Definition 2.3. Stochastic processes $\{X_1(t) \mid t \geq 0\}, \cdots, \{X_n(t) \mid t \geq 0\}$ are said to be positively lower orthant dependent(PLOD) if

$$P(T_1(a_1) \leq t_1, \cdots, T_n(a_n) \leq t_n) \geq P(T_1(a_1) \leq t_1) \cdots P(T_n(a_n) \leq t_n)$$

for all t_i and a_i, $i=1,2,\ldots,n$. Moreover, stochastic processes $\{X_1(t) \mid t \geq 0\}, \cdots, \{X_n(t) \mid t \geq 0\}$ are said to be positively orthant dependent (POD)} if they satisfy both PUOD and PLOD.

Definition 2.4. Stochastic processes $\{X_1(t) \mid t \geq 0\}, \cdots, \{X_n(t) \mid t \geq 0\}$ are said to be associated
if

$$Cov(f(T_1(a_1), \cdots, T_n(a_n)), g(T_1(a_1), \cdots, T_n(a_n))) \geq 0$$

for all increasing functions f and g for which the covariance exists.

Definition 2.5. Stochastic processes $\{X_1(t) \mid t \geq 0\}, \cdots, \{X_n(t) \mid t \geq 0\}$ are said to be right corner set increasing(RCSI) if

$$P(T_1(a_1) > t_1, \cdots, T_n(a_n) > t_n \mid T_1(a_1) > t_{1'}, \cdots, T_n(a_n) > t_{n'})$$

is increasig in $t_{1'}, \cdots, t_{n'}$ for every choice of $t_1, \cdots, t_n$.

Finally, we defined the concepts of right tail increasing in sequence of processes as follows.

Definition 2.6. Stochastic processes $\{X_1(t) \mid t \geq 0\}, \cdots, \{X_n(t) \mid t \geq 0\}$ are said to be right tail increasing in sequence (RTIS) if for all t_i, $i = 2, \cdots, n$,

$$P(T_i(a_i) > t_i \mid T_1(a_1) > t_1, \cdots, T_{i-1}(a_{i-1}) > t_{i-1})$$

is increasing in $t_1, \cdots, t_{i-1}$.

Now,we introduce some properties of positive dependence of multivariate processes and relationships among them

3. Some properties and relationships

Theorem 3.1. If $\{X_1(t) \mid t \geq 0\}, \cdots, \{X_n(t) \mid t \geq 0\}$ are POD and if $g_1, \cdots, g_n$ are nonnegative increasing functions, then $\{g_1(X_1(t)) \mid t \geq 0\}, \cdots, \{g_n(X_n(t)) \mid t \geq 0\}$ are POD.

Proof. We prove this result for PUOD .
Let $W_i(a_i) = \inf\{s \mid g_i(X_i(s)) \geq a_i\}$ and $T_i(b_i) = \inf\{t \mid X_i(t) \geq b_i\}$, $i = 1, \cdots, n$.
Then

$$\begin{aligned}
P\ & (W_1(a_1) > t_1, \cdots, W_n(a_n) > t_n) \\
&= P[\ (\inf\{s \mid g_1(X_1(s)) \geq a_1\}) > t_1, \cdots, (\inf\{s \mid g_n(X_n(s)) \geq a_n\}) > t_n] \\
&= P[\ (\inf\{s \mid X_1(s) \geq g_1^{-1}(a_1)\}) > t_1, \cdots, (\inf\{s \mid X_n(s) \geq g_n^{-1}(a_n)\}) > t_n)] \\
&= P[\ T_1(g_1^{-1}(a_1)) > t_1, \cdots, T_n(g_n^{-1}(a_n)) > t_n] \\
&\geq \prod_{i=1}^{n} P(T_i(g_i^{-1}(a_i)) > t_i) \\
&= \prod_{i=1}^{n} P(W_i(a_i) > t_i) \text{ for every } t_1, \cdots, t_n,\ a_1, \cdots, a_n
\end{aligned}$$

The proof of PLOD is similar to that proof of PUOD.
Next, we now show that RCSI implies RTIS and RCSI implies POD.

Theorem 3.2. If the processes $X_1(t), X_2(t), \cdots, X_n(t)$ are RCSI, then they are RTIS.

Proof. By the definition of RCSI,
$P(T_1(a_1)>t_1, \cdots, T_n(a_n)>t_n | T_1(a_1)>t_{1'}, \cdots, T_n(a_n)>t_{n'})$ is increasing in $t_{1'}, \cdots, t_{n'}$. By taking $t_1 \to 0, \cdots, t_{n-1} \to 0$ and $t_{n'} \to 0$,
$P(T_n(a_n)>t_n | T_1(a_1)>t_{1'}, \cdots, T_{n-1}(a_{n-1})>t_{n-1'})$ is increasing in $t_{1'}, \cdots, t_{n-1'}$ for all $a_1, \cdots, a_n$ and t_n, so that $X_1(t), \cdots, X_n(t)$ are RTIS.

Theorem 3.3. If the processes $X_1(t), X_2(t), \cdots, X_n(t)$ are RTIS, then they are POD.

proof. We prove this result for RTIS implies PUOD.

$$
\begin{aligned}
&P(T_1(a_1)>t_1, \cdots, T_n(a_n)>t_n) \\
&= P(T_1(a_1)>t_1 | T_2(a_2)>t_2, \cdots, T_n(a_n)>t_n) P(T_2(a_2)>t_2, \cdots, T_n(a_n)>t_n) \\
&\geq P(T_1(a_1)>t_1) \prod_{i=2}^{n} P(T_i(a_i)>t_i \bigcap_{j=1}^{n-1} T_j(a_j)>t_j) \quad \textit{by RTIS} \\
&= \prod_{i=1}^{n} P(T_i(a_i)>t_i), \text{ by taking } t_j \to 0 (j=1, \cdots, i-1)
\end{aligned}
$$

The proof of the PLOD is similar to that of PUOD.

We show that the next theorem demonstrates preservation of the POD among the random times under limits.

Theorem 3.4. Let $\{X_{1n}(t) | t \geq 0\}, \cdots, \{X_{pn}(t) | t \geq 0\}$ be POD processes with distribution functions H_n such that $H_n \to^w H$ as $n \to \infty$ where H is the distribution functions of stochastic processes $\{X_1(t) | t \geq 0\}, \cdots, \{X_p(t) | t \geq 0\}$. Then $\{X_1(t) | t \geq 0\}, \cdots, \{X_p(t) | t \geq 0\}$ are POD.

Proof. We prove this result for PUOD.

$$
\begin{aligned}
&P(T_1(a_1) > t_1, \cdots, T_p(a_p) > t_p) \\
&= \lim_{n\to\infty} [\ P(T_{1n}(a_{1n}) > t_{1n}, \cdots, P(T_{pn}(a_{pn}) > t_{pn})] \\
&\geq \lim_{n\to\infty} \prod_{i=1}^{p} P(T_{in}(a_{in}) > t_{in}) \\
&= \prod_{i=1}^{p} \lim_{n\to\infty} P(T_{in}(a_{in} > t_{in}) \\
&= \prod_{i=1}^{p} P(T_i(a_i) > t_i).
\end{aligned}
$$

The proof of the PLOD is similar to that of PUOD.

Theorem 3.5. Let $Z_1(t) = \sum_{j=1}^{N(t)} X_{1j}, \cdots, Z_k(t) = \sum_{j=1}^{N(t)} X_{kj}$ and let $\{(X_{1n}, \cdots, X_{kn}); n \geq 1\}$ be a k-variate processes.

(a) $(X_{11}, \cdots, X_{k1}), (X_{12}, \cdots, X_{k2}), \cdots$ are independent

(b) $X_{1j}, \cdots, X_{kj}$ are POD, $j = 1, 2, \cdots$

(c) $N(t)$ is a Poisson process which is independent of $X_{1j}'s, X_{2j}'s, \cdots, X_{kj}'s,\ j = 1, 2, \cdots$

Then $\{Z_1(t) \mid t \geq 0\}, \cdots, \{Z_k(t) \mid t \geq 0\}$ are POD.

Proof. We prove this result for PLOD.

$$
\begin{aligned}
&P(T_1(a_1) \leq t_1, \cdots, T_k(a_k) \leq t_k) \\
&= P[\ \{\sum_{j=1}^{N(s)} X_{1j} \geq a_1, t_1 \leq s < \infty\}, \cdots, \{\sum_{j=1}^{N(s)} X_{kj} \geq a_k, t_k \leq s < \infty\}] \\
&= P[\ (\sum_{j=1}^{N(t_1)} X_{1j} \geq a_1), \cdots, (\sum_{j=1}^{N(t_k)} X_{kj} \geq a_k)] \\
&= \sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} [\ P(\sum_{j=1}^{k_1} X_{1j} \geq a_1, \cdots, \sum_{j=1}^{k_n} X_{kj} \geq a_k \mid N(t_1) = k_1, \cdots, N(t_k) = k_n)] \\
&\quad \cdot [\ P(N(t_1) = k_1, \cdots, N(t_k) = k_n)]
\end{aligned}
$$

$$= \sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} [\ P(\sum_{j=1}^{k_1} X_{1j} \geq a_1, \cdots, \sum_{j=1}^{k_n} X_{kj} \geq a_n)[\ P(N(t_1)=k_1, \cdots, N(t_k)=k_n)] \quad \text{by } (c)$$

$$\geq \sum_{k_1=1}^{\infty} \cdots \sum_{k_n=0}^{\infty} [\ P(\sum_{j=1}^{k_1} X_{1j} \geq a_1) P(\sum_{j=1}^{k_n} X_{kj} \geq a_n) \quad P(N(t_1)=k_1, \cdots, N(t_k)=k_n)] \quad \text{by } (a) \text{ and } (b)$$

$$= [\ \sum_{k_1=1}^{\infty} P(\sum_{j=1}^{k_1} X_{1j} \geq a_1 | (N(t_1)=k_1)) P(N(t_1=k_1))] \cdots$$

$$[\ \sum_{k_n=0}^{\infty} P(\sum_{j=1}^{k_n} X_{kj} \geq a_n | (N(t_k)=k_n) P(N(t_k)=k_n))]$$

$$= P(\{ \sum_{j=1}^{N(s)} \} X_{1j} \geq a_1, t_1 \leq s < \infty\}) \cdots P(\{ \sum_{j=1}^{N(s)} X_{kj} \geq a_k, t_k \leq s < \infty\})$$

$$= P(T_1(a_1) \leq t_1) \cdots P(T_k(a_k) \leq t_k)$$

The proof of the PUOD is similar to that of PLOD

4. Examples

Example 4.1. Consider a system with 3 components which is subjected to shocks. Let $N(t)$ be the number of shocks received by time t and let $W_1(t) = \sum_{i=1}^{N(t)} X_i$, $W_2(t) = \sum_{i=1}^{N(t)} Y_i$, $W_3(t) = \sum_{i=1}^{N(t)} Z_i$ be total damages to components 1, 2 and 3 by time t respectively, where X_i, Y_i and Z_i are damages to components 1, 2 and 3 by shocks respectively. Let $X_i, Y_i, Z_i, i=1,2,\cdots$ be POD and let $(X_1, Y_1, Z_1), (X_2, Y_2, Z_2)\cdots$ be independent. Then by Theorem 3.5 $\{W_1(t) | t \geq 0\}$, $\{W_2(t) | t \geq 0\}$, $\{W_3(t) | t \geq 0\}$ are POD.

Application 4.2. Consider a system with n associated components. Assume that the component i fails if the total damages to the component exceeds a threshold a_i, $i=1,\cdots,k$. Let $X_i(t)$ be the total damages to the i-th component at time t.
Then, We get the useful bound

$$P(\min_{1 \leq i \leq n} T_i > t) \geq \Pi_{i=1}^{n} P(T_i > t) \text{ and } P(\max_{1 \leq i \leq n} T_i \leq t) \geq \Pi_{i=1}^{n} P(T_i \leq t),$$

for all $t \geq 0$ where the hitting time $T_i = T_i(a_i)$ is the defined in (2.1).

Before stating any further applications of POD stochastic processes, note that let

$X_1, X_2, \cdots$ are independent sequence and identically distributed nonnegative continuous random variables with distribution F. We say that a record occurs at time $n, n>0$, and X_n is called a record value if

$$X_n > \max(X_1, \cdots, X_{n-1}) \quad \text{where} \quad X_0 = -\infty,$$

that is a record occurs each time a new high is reached.

Application 4.3. If we have a trivariate random vector (X, Y, Z), where X has cumulative distribution function c.d.f. F, Y has c.d.f. G and Z has c.d.f. H, respectively. Now, if we take a sequence of random sample from F, a sequence of random sample G and a sequence of random sample H, respectively, then $T_1(x)$, the first record value greater than or equal to x and $T_2(y)$,the first record value greater than or equal to y,and $T_3(z)$, the first record value greater than or equal to z,are POD if X,Y and Z are POD.

Ackonwledgement

We would like to thank to the editor and the referees for comments and suggestions in improving this paper.

References

[1] Ahmed, A. N., Langberg, N. A. Leon, R. and Proschan, F.(1978). Two concepts of multivariate positive dependence, with applications in multivariate analysis. *Tech.Report* 78-6, Department of statistics, Florida University.

[2] Barow,R.D. and Proschan,R.(1975). *Statistical Theory of Reliability and Life Testing Probability Models,* Holt, Rinehart and Winston, New York.

[3] Block,H.W., and Ting M.(1981). Some concepts of multivariate dependence, *Commun. Statist. -Theor. Meth., A* 10(8), 749-762.

[4] Cox D.R. and Isham,V.(1980). *Point Rpocesses,* Chapman and Hall, London.

[5] Ebrahimi.N. and Ghosh,M.(1981). Elements of applied Stochastic processes, *Commun. Statist.,* A 10, 307-337.

[6] Ebrhimi, N. and ramalinggam,T.(1988). On the dependence structure of hitting times of univariate processes, *J. Appli. Prob.,* 25, 355-362.

[7] Esary,J.D. and Proschan,R.(1972). Relationships among some concepts of bivariate

dependence, *Ann. Math. Statist.,* 43, 651-655.

[8] Esary,J.D.,Proschan,F. and Wlakup,D.W.(1967). Association of randomvariables,with applications. *Ann. Math. Statist.,* 38,1466-1474.

[9] Friday,D.S.(1981). Dependence concepts for stochastic processes, *Proc. NATO A. Stst. Inst Series,* 5, 349-361

[10] Glaz,J. and Johnson,B.M.(1982). *Probability inequalities for multivariate distributions dependence structures,* Technical Report, University of Connecticut.ar

[11] Karlin,S. and Taylor,H.(1975). *A first course in Stochastic Processes,* Academic Press, New York.

[12] Lehaman,E.L.(1966). Some concepts of dependence. *Ann. Math. Statist.,* 37, 1137-1153

[13] Ross,S.M.(1983). *Stochastic Processes,* Wiley, New York.