Fuzzy Almost Continuous Mappings and Fuzzy Almost Quasi-Compact Mappings

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ABSTRACT

In this paper we introduce the class of fuzzy almost continuous mappings. It contains the class of fuzzy continuous mappings and is contained in the class of fuzzy weakly continuous mappings. In section 3 we discuss various properties of such mappings. In section 4 we also introduce the notion of fuzzy almost quasi-compact mappings and give relations between fuzzy almost quasi-compact mappings and the mappings which are introduced in section 2 and 3.

I. Introduction

Fuzzy topological spaces were first introduced by Chang [3] who studied a number of the basic conepts, including fuzzy continuous mappings and compactness.

Fuzzy topological spaces are natural generalization of topological spaces in the following sense. By the one-to-one and onto mapping between the family of all subsets of a set X and the set of all characteristic mappings, whose domain is X, a topology on X can be regarded as a family of characteristic mappings, if we replace the usual set operations of $\subseteq$, $\cup$, $\cap$ and complementation with the mapping operations of $\leq$, $\vee$, $\wedge$ and $1-\mu$, respectively. A fuzzy topological space allows more general mappings to be members of the topology.

In this paper we introduce the class of fuzzy almost continuous mappings.

In section 2, we study properties of fuzzy continuous mappings and fuzzy regularly open sets.

The class of fuzzy almost continuous mappings contains the class of fuzzy continuous mappings and is contained in the class of fuzzy weakly continuous mappings. We discuss various properties of such mappings.

In section 4, we continue the study of fuzzy almost open and fuzzy almost closed mapping and give relations between fuzzy almost quasi-compact mappings and the mappings which are introduced in section 2 and 3.

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Ⅱ. Preliminaries

Throughout this paper, we denote I the unit interval [0, 1] of the real line. Let X be an ordinary nonempty set, we denote I^X the collection of all mappings from X into I. We call a member μ of I^X a *fuzzy sct* of X. We also denote 0_X and 1_X constant mappings taking whole of X to 0 and 1, respectively. We now define the union, intersection and complement of fuzzy sets as follows:

$$\left(\bigcup_{j\in J} \mu_j\right)(x) = \bigvee_{j\in J} \mu_j(x) = \sup_{j\in J} \mu_j(x), \qquad \text{for all } x\in X.$$

$$\left(\bigcap_{j\in J} \mu_j\right)(x) = \bigwedge_{j\in J} \mu_j(x) = \inf_{j\in J} \mu_j(x), \qquad \text{for all } x\in X.$$

$$\mu^c(x) = (1_X - \mu)(x) = 1 - \mu(x), \qquad \text{for all } x\in X.$$

Let μ and ν be members of I^X. $\mu \geq \nu$ if and only if $\mu(x) \geq \nu(x)$ for all $x\in X$, and in this case, ν is said to be contained in μ. We now define a fuzzy set of $X\times Y$. Let μ be a fuzzy set of X and let ν be a fuzzy set of Y. Then $\mu\times\nu$ is a fuzzy set of $X\times Y$, defined by $(\mu\times\nu)(x, y) = \inf(\mu(x), \nu(y))$, for each $(x, y)\in X\times Y$.

Let $f: X\rightarrow Y$ be a mapping and let μ be a fuzzy set of X. We define $f(\mu)$ as follows:

$$f(\mu)(y) = \sup_{x\in f^{-1}(y)} \mu(x) \qquad \text{if } f^{-1}(y) \neq 0 \quad \text{for each } y\in Y$$
$$= 0 \qquad \text{otherwise.}$$

Let ν be a fuzzy set of Y. We define $f^{-1}(\nu)$ to $f^{-1}(\nu)(x) = (\nu \circ f)(x)$, for each $x\in X$. The identity $i_X: X\rightarrow X$ on X is defined by $i_X(x) = x$, for all $x\in X$. Let $f_1: X_1\rightarrow Y_1$ and $f_2: X_2\rightarrow Y_2$ be mappings. The product $f_1\times f_2: X_1\times X_2\rightarrow Y_1\times Y_2$ is defined by $(f_1\times f_2)(x_1, x_2) = (f_1(x_1), f_2(x_2))$ for each $(x_1, x_2)\in X_1\times X_2$. And, the graph $g: X\rightarrow X\times Y$ of f is defined by $g(x) = (x, f(x))$, for each $x\in X$, where $f: X\rightarrow Y$.

We now state the following elementary results which we will use later.

LEMMA 2. 1 *Let f be a mapping from X to Y. Then.*

(1) $(f(\mu))^c \leq f(\mu^c)$ *for any fuzzy set* μ *in* X.

(2) $f^{-1}(\nu^c) = (f^{-1}(\nu))^c$ *for any fuzzy set* ν *in* Y.

(3) $\mu_1 \leq \mu_2$ *implies* $f(\mu_1) \leq f(\mu_2)$, *where* μ_1 *and* μ_2 *are fuzzy sets in* X.

(4) $\nu_1 \leq \nu_2$ *implies* $f^{-1}(\nu_1) \leq f^{-1}(\nu_2)$, *where* ν_1 *and* ν_2 *are fuzzy sets in* Y.

(5) $\mu \leq f^{-1}(f(\mu))$ *for any fuzzy set* μ *in* X.

(6) $f(f^{-1}(\nu)) \leq \nu$ *for any fuzzy set* ν *in* Y.

(7) $f\left(\bigvee_{j\in J} \mu_j\right) = \bigvee_{j\in J} f(\mu_j)$, *where each* μ_j *is fuzzy set in* X.

(8) $f(\mu_1 \wedge \mu_2) \leq f(\mu_1) \wedge f(\mu_2)$, *where* μ_1 *and* u_2 *are fuzzy sets in* X.

(9) $f^{-1}\left(\bigvee_{j\in J} \nu_j\right) = \bigvee_{j\in J} f^{-1}(\nu_j)$, *where each* ν_j *is fuzzy set in* Y.

(10) $f^{-1}\left(\bigwedge_{j\in J} \nu_j\right) = \bigwedge_{j\in J} f^{-1}(\nu_j)$, *where each* ν_j *is fuzzy set in* Y.

(11) *If f is a one-to-one mapping then, for any fuzzy set* μ *in* X, $f^{-1}(f(\mu)) = \mu$.

(12) *If f is an onto mapping then, for any fuzzy set ν in Y,* $f(f^{-1}(\nu))=\nu$.

(13) *If f is a one-to-one and onto mapping then, for any fuzzy set μ in X,* $(f(\mu))^c=f(\mu^c)$.

(14) *Let g be a mapping from Y to Z. Then $g(f(\mu))=g\circ f(\mu)$ for any set μ in X. Also,* $(g\circ f)^{-1}(\delta)=f^{-1}(g^{-1}(\delta))$ *for any fuzzy set δ in Z.*

LEMMA 2. 2 [1] *If μ is a fuzzy set in X and ν is a fuzzy set in Y then* $1-\mu\times\nu=\mu^c\times 1\vee 1\times\nu^c$.

LEMMA 2. 3 [1] *Let $f_j:X_j\to Y_j$ be a mapping and let ν_j be a fuzzy sets in Y_j for $(j=1,2)$. Then* $(f_1\times f_2)^{-1}(\nu_1\times\nu_2)=f_1^{-1}(\nu_1)\times f_2^{-1}(\nu_2)$.

LEMMA 2. 4 [1] *Let $g:X\to X\times Y$ be the graph of $f:X\to Y$. If μ is a fuzzy set in X and ν is a fuzzy set in Y, then* $g^{-1}(\mu\times\nu)=\mu\wedge f^{-1}(\nu)$.

A subfamily T_X of I^X is called a *fuzzy topology* on X iff (1) 0_X and 1_X belong to T_X, (2) any union of members of T_X is in T_X, and (3) a finite intersection of members of T_X is in T_X. Members of T_X are called *fuzzy open sets* of X and their complements are called *fuzzy closed sets.* The pair (X, T_X) is a *fuzzy topological space*(abbreviated as *fts*).

Let μ be a fuzzy set of X. We define the *closure* $\overline{\mu}$ and the *interior* μ° of μ as follows;

$$\overline{\mu}=\inf\{\nu\mid\nu\geq\mu,\ \nu^c\in T\}$$

and

$$\mu^\circ=\sup\{\nu\mid\nu\leq\mu,\ \nu\in T\}.$$

Let X and Y be *fts*. The *fuzzy product space* of X and Y is the Cartesian product $X\times Y$ of sets X and Y together with the fuzzy topology $T_{X\times Y}$ generated by the family $\{\pi_1^{-1}(\mu_i),\ \pi_2^{-1}(\nu_j)\mid\mu_i\in T_X,\ \nu_j\in T_Y$, where π_1 and π_2 are projections of $X\times Y$ onto X and Y, respectively$\}$. Since $\pi_1^{-1}(\mu_i)=\mu_i\times 1$, $\pi_2^{-1}(\nu_j)=1\times\nu_j$ and $\mu_i\times 1\wedge 1\times\nu_j=\mu_i\times\nu_j$, the family $B=\{\mu_i\times\nu_j\mid\mu_i\in T_X,\ \nu_j\in T_Y\}$ forms a basis for the fuzzy product topology $T_{X\times Y}$ on $X\times Y$.

LEMMA 2. 5 [1] *Let $\{\mu_j\}_{j\in J}$ be a family of fuzzy sets of a fts X. Then* $\bigvee_j\overline{\mu_j}\leq\left(\overline{\bigvee_j\mu_j}\right)$ *and* $\bigvee_j\mu_j^\circ\leq\left(\bigvee_j\mu_j\right)^\circ$. *In particular, if J is finite then* $\bigvee_j\overline{\mu_j}=\left(\overline{\bigvee_j\mu_j}\right)$

LEMMA 2. 6 [10] *Let μ be a fuzzy set of a fts X. Then* $1-\mu^\circ=\overline{1-\mu}$, *and* $1-\overline{\mu}=(1-\mu)^\circ$

LEMMA 2. 7 [1] *Let μ be a fuzzy closed set of a fts X and ν be a fuzzy closed set of a fts Y. Then $\mu\times\nu$ is a fuzzy closed set of the fuzzy product space $X\times Y$.*

REMARK 2. 8 *In general topological spaces, it is well known that the closure of the product is the product of their closures. But it need not be true in fuzzy topological space as shown in* [1].

However, from LEMMA 2. 7, we have the following proposition.

PROPOSITION 2. 9 [1] *If μ is a fuzzy set of a fts X and ν is a fuzzy set of a fts Y, then $\bar{\mu}\times\bar{\nu}\geq\overline{\mu\times\nu}$. Also, we have $\mu^{\circ}\times\nu^{\circ}\leq(\mu\times\nu)^{\circ}$.*

DEFINITION 2. 10 [1] A *fts X* is *product related to a fts Y*, iff for any μ of X and ν of Y, whenever $\lambda^c\not\geq\mu$ and $\delta^c\not\geq\nu$ implies $\lambda^c\times 1\vee 1\times\delta^c\geq\mu\times\nu$, where $\lambda\in T_X$ and $\delta\in T_Y$, there exist $\lambda_1\in T_X$ and $\delta_1\in T_Y$ such that $\lambda_1^c\geq\mu$ or $\delta_1^c\geq\nu$ and $\lambda_1^c\times 1\vee 1\times\delta_1^c=\lambda^c\times 1\vee 1\times\delta^c$.

PROPOSITION 2. 11 [1] *Let X and Y be fts such that X is product related to Y. Then $\overline{\mu\times\nu}=\bar{\mu}\times\bar{\nu}$ and $(\mu\times\nu)^{\circ}=\mu^{\circ}\times\nu^{\circ}$, for a fuzzy set μ of X and a fuzzy set ν of Y.*

DEFINITION 2. 12 A mapping $f:(X, T_X)\rightarrow(Y, T_Y)$ from a *fts X* to a *fts Y* is called,

(1) a *fuzzy continuous mapping* iff $f^{-1}(\nu)\in T_X$ for each $\nu\in T_Y$;

(2) a *fuzzy open mapping* iff $f(\mu)\in T_Y$ for each $\mu\in T_X$;

(3) a *fuzzy closed mapping* iff $f(\lambda)$ is a fuzzy closed set of Y, for each fuzzy closed set λ of X.

THEOREM 2. 13 [3] (1) *The identity mapping $i_X:(X, T_X)\rightarrow(Y, T_X)$ is a fuzzy continuous mapping.*

(2) *If $f:(X, T_X)\rightarrow(Y, T_Y)$ and $g:(Y, T_Y)\rightarrow(Z, T_Z)$ are fuzzy continuous mapping, then $g\circ f:(X, T_X)\rightarrow(Z, T_Z)$ is a fuzzy continuous mapping.*

THEOREM 2. 14 [2] *Let $f:(X, T_X)\rightarrow(Y, T_Y)$ and $g:(Y, T_Y)\rightarrow(Z, T_Z)$ be fuzzy open(closed) mapping. Then $g\circ f:(X, T_X)\rightarrow(Y, T_Z)$ is also a fuzzy open(closed) mapping.*

THEOREM 2. 15 [10] *Let $f:(X, T_X)\rightarrow(Y, T_Y)$ be a mapping. Then the following statements are equivalent:*

(1) *f is a fuzzy continuous mapping.*

(2) *$f^{-1}(\nu)$ is a fuzzy closed set, for each fuzzy closed set ν in Y.*

(3) *$f^{-1}(\eta)$ is a neighborhood of μ, for each fuzzy set μ in X and every neighborhood η of $f(\mu)$.*

(4) *$f(\bar{\mu})\leq\overline{f(\mu)}$, for each fuzzy set μ in X.*

(5) *$\overline{f^{-1}(\nu)}\leq f^{-1}(\bar{\nu})$, for each fuzzy set ν in Y.*

(6) *The graph $g:X\rightarrow X\times Y$ of f is a fuzzy continuous mapping.*

THEOREM 2. 16 [2] *Let $f_1:X_1\rightarrow Y_1$ and $f_2:X_2\rightarrow Y_2$ be fuzzy continuous mappings. Then $f_1\times f_2:X_1\times X_2\rightarrow Y_1\times Y_2$ is fuzzy continuous.*

THEOREM 2. 17 *Let $f:(X, T_X)\rightarrow(Y, T_Y)$ be a mapping. Then, f is a fuzzy open mapping if and only if $f(\mu^{\circ})\leq(f(\mu))^{\circ}$, for each fuzzy sct μ in X.*

PROOF. Let μ be a fuzzy set in X. Then $\mu^{\circ}\leq\mu$. So $f(\mu^{\circ})\leq f(\mu)$. Now, since f is a fuzzy open mapping, $f(\mu^{\circ})$ is a fuzzy open set in $f(\mu)$. But $(f(\mu))^{\circ}$ is the largest fuzzy open set in $f(\mu)$. Therefore, $f(\mu^{\circ})\leq(f(\mu))^{\circ}$.

Conversely, let μ be a fuzzy open set of X. Then $\mu=\mu^{\circ}$ and, therefore, $f(\mu)\leq f(\mu^{\circ})\leq(f(\mu))^{\circ}\leq f(\mu)$, which implies that $f(\mu)=(f(\mu))^{\circ}$. Hence $f(\mu)$ is a fuzzy open set. ■

THEOREM 2. 18 *Let $f:(X, T_X)\rightarrow(Y, T_Y)$ be a mapping. Then f is a fuzzy closed mapping if and only if $\overline{f(\mu)}\leq f(\bar{\mu})$ for each fuzzy set μ in X.*

PROOF. Let μ be a fuzzy set in X. Since f is a fuzzy closed mapping, $f(\overline{\mu})$ is a fuzzy closed set in Y. Then $f(\overline{\mu}) = \overline{f(\overline{\mu})}$. Since $\mu \leq \overline{\mu}$, we have $f(\mu) \leq f(\overline{\mu})$. Consequently, $\overline{f(\mu)} \leq \overline{f(\overline{\mu})} = f(\overline{\mu})$.

Conversely, let μ be a closed set in X. Since $\mu = \overline{\mu}$, we have $f(\mu) \leq \overline{f(\mu)} \leq f(\overline{\mu}) \leq f(\mu)$. Therefore $f(\mu) = \overline{f(\mu)}$, and hence $f(\mu)$ is a fuzzy closed set. ■

THEORME 2. 19 *Let Y be a subset of Z and let $f:(X, T_X) \rightarrow (Y, T_Y)$ be a mapping. Then, $f:(X, T_X) \rightarrow (Y, T_Y)$ is a fuzzy continuous mapping if and only if $f:(X, T_X) \rightarrow (Z, T_Z)$ is a fuzzy continuous mapping.*

PROOF. Let δ be a fuzzy open set in Z and let $f:(X, T_X) \rightarrow (Y, T_Y)$ be a fuzzy continuous mapping. Then $\nu = \delta \wedge 1_Y$ is a fuzzy open set in Y. Therefore $f^{-1}(\delta): f^{-1}(\delta) \wedge 1_X = f^{-1}(\delta \wedge 1_Y) = f^{-1}(\nu)$ is a fuzzy open set in X. Hence $f:(X, T_X) \rightarrow (Z, T_Z)$ is a fuzzy continuous mapping.

Conversely, let $f:(X, T_X) \rightarrow (Z, T_Z)$ be a fuzzy continuous mapping and let ν be a fuzzy open set in Y. Then there is a fuzzy open set δ in Z such that $\nu = \delta \wedge 1_Y$. Then $f^{-1}(\nu) = f^{-1}(\delta \wedge 1_Y) = f^{-1}(\delta) \wedge 1_X = f^{-1}(\delta)$ is a fuzzy open set in X. Therefore $f:(X, T_X) \rightarrow (Y, T_Y)$ is a fuzzy continuous mapping. ■

DEFINITION 2. 20 A fuzzy set μ of a *fts* X is called (1) a *fuzzy regularly open set* of X iff $\overline{\mu}^\circ = \mu$, and (2) a *fuzzy regularly closed set* of X iff $\overline{\mu^\circ} = \mu$.

PROPOSITION 2. 21 *A fuzzy set μ of a fts X is fuzzy regularly open if and only if μ^c is fuzzy regularly closed.*

PROOF. Since μ is a fuzzy regularly open set in *fts* X, we have $\mu^c = 1 - \mu = 1 - \overline{\mu}^\circ = \overline{1 - \overline{\mu}} = \overline{(1-\mu)^\circ} = \overline{\mu^{c\circ}}$.

Conversely, since μ^c is a fuzzy regularly closed set in *fts* X, we have $\mu = 1 - \mu^c = 1 - \overline{\mu^{c\circ}} = (1 - \mu^{c\circ})^\circ = \overline{1-\mu^c}^\circ = \overline{\mu}^\circ$. ■

REMARK 2. 22 *It is clear that a fuzzy regularly open(closed) set is fuzzy open(closed). But the converse need not be true as shown in* [1].

PROPOSITION 2. 23 (1) *The intersection of two fuzzy regularly open sets is a fuzzy regularly open set.* (2) *The union of two fuzzy regularly closed sets is a fuzzy regularly closed set.*

PROOF. Proof of (1): Let μ and ν be two fuzzy regularly open sets of a *fts* X. Since $\mu \wedge \nu$ is a fuzzy open set, we have $\mu \wedge \nu \leq \overline{(\mu \wedge \nu)}^\circ$. Since $\overline{(\mu \wedge \nu)}^\circ \leq \overline{\mu}^\circ = \mu$ and $\overline{(\mu \wedge \nu)}^\circ \leq \overline{\nu}^\circ = \nu$, it implies $\overline{(\mu \wedge \nu)}^\circ \leq \mu \wedge \nu$. Consequently, $\overline{(\mu \wedge \nu)}^\circ = \mu \wedge \nu$.

Proof of (2): Let μ and ν be two fuzzy regularly open sets of a *fts* X. Then μ^c and ν^c are fuzzy regularly closed sets. Since $\mu \wedge \nu$ is a fuzzy regularly open set, $(\mu \wedge \nu)^c = \mu^c \vee \nu^c$ is a fuzzy regularly closed set. ■

The union(intersection) of two fuzzy regularly open(closed) sets need not be a fuzzy regularly open (closed) set[1].

PROPOSITION 2. 24 (1) *The closure of a fuzzy open set is a fuzzy regularly closed set.*

(2) *The interior of a fuzzy closed set is a fuzzy regularly open set.*

PROOF. Proof of (1): Let μ be a fuzzy open set of a *fts* X. Clearly $\mu^\circ \leq \overline{\mu}$, it implies that $\overline{\overline{\mu}^\circ} \leq \overline{\mu}$. Since μ is a fuzzy open set, we have $\overline{\mu} \leq \overline{\mu}^\circ$, and have $\overline{\mu} \leq \overline{\overline{\mu}^\circ}$. Thus $\overline{\mu}$ is a fuzzy regularly closed set.

Proof of (2): Let ν be a fuzzy closed set of a *fts* X. Clearly $\nu^\circ \leq \overline{\nu^\circ}$. Since ν is a fuzzy closed set, we have $\overline{\nu^\circ} \leq \nu$ and have $\overline{\nu^\circ}^\circ \leq \nu^\circ$. Thus ν° is a fuzzy regularly open set. ■

III. Fuzzy almost continuous mappings.

In this section we will investigate properties of fuzzy almost continuous mappings.

Definition 3. 1 A mapping $f:(X, T_X)\rightarrow(Y, T_Y)$ from a *fts* X to a *fts* Y is called,
(1) a *fuzzy almost continuous mapping* iff $f^{-1}(\nu)\in T_X$, for each fuzzy regularly open set ν in Y.
(2) a *fuzzy almost open mapping* iff $f(\mu)\in T_Y$ for each fuzzy regularly open set μ in X.
(3) a *fuzzy almost closed mapping* iff $f(\delta)$ is a fuzzy closed set of Y, for each fuzzy regularly closed set δ in X.

Remark 3. 2 *A fuzzy continuous mapping is a fuzzy almost continuous mapping. But the converse need not be true as shown in* [1].

Theorem 3. 3 [1] *Let $f:(X, T_X)\rightarrow(Y, T_Y)$ be a mapping. Then the following statements are equivalent:*
(1) *f is a fuzzy almost continuous mapping.*
(2) *$f^{-1}(\nu)$ is a fuzzy closed set, for each fuzzy regularly closed set ν in Y.*
(3) *$f^{-1}(\eta)$ is a neighborhood of μ, for each fuzzy set μ in X and every regularly open neighborhood η of $f(\mu)$.*
(4) *$f^{-1}(\nu)\leq(f^{-1}(\bar{\nu}^{\circ}))^{\circ}$, for each $\nu\in T_Y$.*
(5) *$\overline{f^{-1}(\bar{\delta}^{\circ})}\leq f^{-1}(\delta)$, for each fuzzy closed set δ in Y.*
(6) *The graph $g:X\rightarrow X\times Y$ of f is a fuzzy almost continuous mapping.*

Obviously, A one-to-one mapping is fuzzy almost open if and only if it is fuzzy almost closed. And also, every fuzzy open(closed) mapping is fuzzy almost open(closed). But the converse is not necessarily true as shown in [8].

Theorem 3. 4 [8] *Let f be a fuzzy almost continuous mapping from a fts X onto a fts Y, and g is a mapping from a fts Y onto a fts Z. If $g\circ f$ is fuzzy open(closed), then g is fuzzy almost open(closed).*

Theorem 3. 5 [8] *If f is a fuzzy almost open and fuzzy almost continuous mapping from a fts X onto a fts Y, then the inverse image of a fuzzy regularly open(closed) set is a fuzzy regularly open(closed) set.*

Theorem 3. 6 *Let f be a fuzzy open and fuzzy continuous mapping from a fts X into a fts Y, and let g be a mapping from a fts Y into a fts Z. Then, $g\circ f$ is fuzzy almost continuous if and only if g is fuzzy almost continuous.*

Proof. Let $g\circ f$ be fuzzy almost continuous and let δ be a fuzzy regularly open set in Z. Then $(g\circ f)^{-1}(\delta)$ is a fuzzy open set in X, that is, $f^{-1}(g^{-1}(\delta))$ is a fuzzy open set in X. Since f is a fuzzy open mapping, $f(f^{-1}(g^{-1}(\delta)))$ is fuzzy open set in Y. Thus $g^{-1}(\delta)$ is a fuzzy open set in Y and consequently g is fuzzy almost continuous.

Conversely, let g be fuzzy almost continuous and let ν be a fuzzy regularly open set in Z. Then $g^{-1}(\nu)$ is a fuzzy open set in Y. Since f is fuzzy continuous, $f^{-1}(g^{-1}(\nu))$ is a fuzzy open set in X, that is, $(g\circ f)^{-1}(\nu)$ is a fuzzy open set in X. Hence $g\circ f$ is fuzzy almost continuous. ■

Theorem 3. 7 *A restriction of a fuzzy almost continuous mapping is fuzzy almost continuous.*

Proof. Let A be a subset of *fts* X and let $f:(X, T_X)\rightarrow(Y, T_Y)$ be a fuzzy almost continuous mapping. We take a fuzzy regularly open set ν in *fts* Y. Then $f_A^{-1}(\nu)=1_A\wedge f^{-1}(\nu)$. Since $f^{-1}(\nu)$ is a fuzzy open set in X, $1_A\wedge f^{-1}(\nu)$ is a fuzzy open set in A. Therefore, a restriction f_A of a fuzzy almost continuous mapping f is fuzzy almost continuous. ■

Theorem 3. 8 *Let A and B be subsets of fts X with $A \cup B = X$. If $1_X = 1_A \vee 1_B$, where 1_A and 1_B are both fuzzy closed(or both fuzzy open) sets of X, and if $f:(X, T_X) \to (Y, T_Y)$ is a mapping such that both f_A and f_B are fuzzy almost continuous, then f is fuzzy almost continuous.*

Proof. Let ν be a fuzzy regularly closed set in *fts* Y. Since f_A and f_B are both fuzzy almost continuous, $f_A^{-1}(\nu)$ and $f_B^{-1}(\nu)$ are both fuzzy closed set in A and B, respectively. Since 1_A and 1_B are fuzzy closed sets of X, $f_A^{-1}(\nu)$ and $f_B^{-1}(\nu)$ are also fuzzy closed sets of X. Also, $f^{-1}(\nu) = f_A^{-1}(\nu) \vee f_B^{-1}(\nu)$. Thus $f^{-1}(\nu)$ is the union of two fuzzy closed sets and therefore, f^{-1} is a fuzzy closed set in X. Consequently, f is fuzzy almost continuous. ■

Theorem 3. 9 *Let A and B be subsets of fts X with $A \cup B = X$. If $f:(X, T_X) \to (Y, T_Y)$ is a mapping and if f_A and f_B are both fuzzy almost continuous on $A \cap B$, respectively, then $f_{(A \cap B)}$ is fuzzy almost continuous.*

Proof. For convenience, we denote $g = f_A$ and $h = f_B$. Let ν be a fuzzy regularly open set in *fts* Y. Then,

$$f_{(A\cap B)}^{-1}(\nu) = 1_{(A\cap B)} \wedge f^{-1}(\nu) = 1_{(A\cap B)} \wedge (g^{-1}(\nu) \vee h^{-1}(\nu))$$
$$= (1_{(A\cap B)} \wedge g^{-1}(\nu)) \vee (1_{(A\cap B)} \wedge h^{-1}(\nu)) = g_{(A\cap B)}^{-1}(\nu) \vee h_{(A\cap B)}^{-1}(\nu).$$

Since $g_{(A\cap B)}^{-1}(\nu)$ and $h_{(A\cap B)}^{-1}(\nu)$ are fuzzy open set in $A \cap B$, $f_{(A\cap B)}^{-1}(\nu) = g_{(A\cap B)}^{-1}(\nu) \vee h_{(A\cap B)}^{-1}(\nu)$ is fuzzy open set in $A \cap B$. Therefore $f_{(A\cap B)}$ is fuzzy almost continuous. ■

Theorem 3. 10 *Let $\{A_j \mid j \in J\}$ be a covering of fts X. If $\bigvee_{j\in J} 1_{A_j} = 1_X$, where each 1_{A_j} is a fuzzy open set of X, and if $f:(X, T_X) \to (Y, T_Y)$ is a mapping such that each f_{A_j} is fuzzy almost continuous, then f is fuzzy almost continuous.*

Proof. Let ν be a fuzzy regularly open set in *fts* Y. Since each f_{A_j} is fuzzy almost continuous, $f_{A_j}^{-1}(\nu)$ is a fuzzy open set in A_j for $j \in J$. Since 1_{A_j} is a fuzzy open sets of X, $f_{A_j}^{-1}(\nu)$ is also fuzzy open set in X. Notice that $f^{-1}(\nu) = \bigvee_{j\in J} f_{A_j}^{-1}(\nu)$. Thus $f^{-1}(\nu)$ is the union of fuzzy open sets and therefore fuzzy open set in X. Hence f is fuzzy almost continuous. ■

Theorem 3. 11 [1] *Let X_1, X_2, Y_1 and Y_2 be fts such that Y_1 is product related to Y_2, and let $f_1: X_1 \to Y_1$ and $f_2: X_2 \to Y_2$ be fuzzy almost continuous mappings. Then $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ is also fuzzy almost continuous.*

Theorem 3. 12 [1] *Let X, X_1 and X_2 be fts and $\pi_j : X_1 \times X_2 \to X_j$ be the projection of $X_1 \times X_2$ onto X_j for $j = 1$, 2. If $f: X \to X_1 \times X_2$ is a fuzzy almost continuous mapping, then $\pi_j \circ f$ is fuzzy almost continuous.*

Definition 3. 13 A mapping $f:(X, T_X) \to (Y, T_Y)$ from a *fts* X to a *fts* Y is called a *fuzzy weakly continuous mapping* iff for each fuzzy open set ν in Y, $f^{-1}(\nu) \leq (f^{-1}(\bar{\nu}))^\circ$.

From the definition, a fuzzy almost continuous mapping is obviously fuzzy weakly continuous. But the converse need not be true in general as shown in [1].

Theorem 3. 14 *Let $f:(X, T_X) \to (Y, T_Y)$ be a mapping. Then the following statements are equivalent:*
(1) *f is a fuzzy weakly continuous mapping.*
(2) $\overline{f^{-1}(\nu^\circ)} \leq f^{-1}(\nu)$, *for each fuzzy closed set ν in Y.*
(3) $\overline{f^{-1}(\delta)} \leq f^{-1}(\bar{\delta})$, *for each $\delta \in T_Y$*

PROOF. (1) implies (2): Let ν be a fuzzy closed set in Y. Then $\nu^c \in T_Y$. Since f is weakly continuous, we have $f^{-1}(\nu^c) \leq (f^{-1}(\overline{\nu^c}))^\circ$. By LEMMA 2. 6, we have $(f^{-1}(\nu))^c \leq (f^{-1}(\nu^\circ))^{c\circ} = \overline{f^{-1}(\nu^\circ)}^c$. Thus $\overline{f^{-1}(\nu^\circ)} \leq f^{-1}(\nu)$.

(2) implies (3): Let $\delta \in T_Y$. Then $\bar{\delta}$ is a fuzzy closed set and $\delta \leq \bar{\delta}^\circ$. Hence $\overline{f^{-1}(\delta)} \leq \overline{f^{-1}(\bar{\delta}^\circ)} \leq f^{-1}(\bar{\delta})$.

(3) implies (1): Let μ be a fuzzy closed set in Y. Then $\mu^\circ \in T_Y$ and $\overline{\mu^\circ} \leq \mu$. Since $\overline{f^{-1}(\mu^\circ)} \leq f^{-1}(\overline{\mu^\circ}) \leq f^{-1}(\mu)$, we have $(f^{-1}(\mu))^c \leq \overline{f^{-1}(\mu^\circ)}$. By LEMMA 2. 6, we have $f^{-1}(\mu^c) \leq (f^{-1}(\mu^{\circ c}))^\circ$. Therefore $f^{-1}(\mu) \leq (f^{-1}(\overline{\mu^c}))^\circ$ for a $\mu^c \in T_Y$. Hence f is fuzzy weakly continuous. ■

THEOREM 3. 15 [8] *If $f:(X, T_X) \to (Y, T_Y)$ is a fuzzy weakly continuous and fuzzy open mapping, then f is fuzzy almost continuous*

The following three theorems are analogous to Azad's theorem in [1].

THEOREM 3. 16 *Let X_1, X_2, Y_1 and Y_2 be fts such that Y_1 is product related to Y_2. Then, the product $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ of fuzzy weakly continuous mappings $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ is fuzzy weakly continuous.*

PROOF. For convenience, we denote $\lambda = \bigvee_{i,j} (\mu_i \times \nu_j)$, where μ_i's and ν_j's are fuzzy open sets of Y_1 and Y_2, respectively, and λ is a fuzzy open set of $Y_1 \times Y_2$. By LEMMA 2. 1, 2. 3, 2. 5 and PROPOSITION 2. 11, we have

$$(f_1 \times f_2)^{-1}(\lambda) = \bigvee_{i,j} (f_1^{-1}(\mu_i) \times f_2^{-1}(\nu_j)) \leq \bigvee_{i,j} \left((f_1^{-1}(\bar{\mu}_i))^\circ \times (f_2^{-1}(\bar{\nu}_j))^\circ \right)$$

$$\leq \left(\bigvee_{i,j} (f_1 \times f_2)^{-1} (\bar{\mu}_i \times \bar{\nu}_j) \right)^\circ = \left(\bigvee_{i,j} (f_1 \times f_2)^{-1} (\overline{\mu_i \times \nu_j}) \right)^\circ$$

$$\leq \left((f_1 \times f_2)^{-1} \left(\overline{\bigvee_{i,j} (\mu_i \times \nu_j)} \right) \right)^\circ = \left((f_1 \times f_2)^{-1}(\bar{\lambda}) \right)^\circ.$$

Thus $f_1 \times f_2$ is fuzzy weakly continuous. ■

THEOREM 3. 17 *Let X, X_1 and X_2 be fts and $\pi_j : X_1 \times X_2 \to X_j$ be the projection of $X_1 \times X_2$ onto X_j for $j = 1, 2$. If $f : X \to X_1 \times X_2$ is a fuzzy weakly continuous mapping, then $\pi_j \circ f$ is fuzzy weakly continuous.*

PROOF. Let μ be a fuzzy open set of X_j for $j = 1, 2$. Then we have

$$(\pi_j \circ f)^{-1}(\mu) = f^{-1}(\pi_j^{-1}(\mu)) \leq \left(f^{-1}(\overline{\pi_j^{-1}(\mu)}) \right)^\circ \leq \left(f^{-1}(\pi_j^{-1}(\bar{\mu})) \right)^\circ = \left((\pi_j \circ f)^{-1}(\bar{\mu}) \right)^\circ.$$

Hence $\pi_j \circ f$ is fuzzy weakly continuous. ■

THEOREM 3. 18 *Let X and Y be fts such that X is product related to Y and*

$f : X \to Y$ a mapping. Then, the graph $g : X \to X \times Y$ of f is fuzzy weakly continuous if and only if f is fuzzy weakly continuous.

PROOF. Let g be a fuzzy weakly continuous mapping and let μ be a fuzzy open set in Y. Then, by LEMMA 2. 4 and PROPOSITION 2. 11, we have

$$f^{-1}(\mu) = 1 \wedge f^{-1}(\mu) = g^{-1}(1 \times \mu) \leq \left(g^{-1}(\overline{1 \times \mu}) \right)^\circ = \left(g^{-1}(1 \times \bar{\mu}) \right)^\circ = \left(1 \wedge f^{-1}(\bar{\mu}) \right)^\circ = \left(f^{-1}(\bar{\mu}) \right)^\circ.$$

Thus f is fuzzy weakly continuous.

Conversely, let $\lambda = \bigvee_{i,j} (\mu_i \times \nu_j)$, where μ_i's and ν_j's are fuzzy open sets of X and Y, respectively, λ a fuzzy open set of $X \times Y$ and f a fuzzy weakly continuous mapping. Then by LEMMA 2. 1, 2. 4, 2. 5 and PROPOSITION 2. 11, we have

$$g^{-1}(\lambda) = g^{-1}\left(\bigvee_{i,j} (\mu_i \times \nu_j)\right) = \bigvee_{i,j} \left(\mu_i \wedge f^{-1}(\nu_j)\right)$$

$$\leq \bigvee_{i,j} \left(\mu_i \wedge \left(f^{-1}(\bar{\nu}_j)\right)^{\circ}\right) \leq \bigvee_{i,j} \left(\bar{\mu}_i \wedge f^{-1}(\bar{\nu}_j)\right)^{\circ} \leq \left(\bigvee_{i,j} g^{-1}(\bar{\mu}_i \times \bar{\nu}_j)\right)^{\circ}$$

$$= \left(g^{-1}\left(\bigvee_{i,j} (\bar{\mu}_i \times \bar{\nu}_j)\right)\right)^{\circ} \leq \left(g^{-1}\left(\overline{\bigvee_{i,j} (\mu_i \times \nu_j)}\right)\right)^{\circ} = \left(g^{-1}(\bar{\lambda})\right)^{\circ}.$$

Thus g is fuzzy weakly continuous. ■

Ⅳ. Fuzzy almost quasi-compact mappings.

We now give the following definitions and investigate its properties.

DEFINITION 4. 1 Let X be a *fts* and let Y be a set, and $p: X \to Y$ be an onto mapping. Then the family $T(p)$ of fuzzy subsets of Y defined by

$T(p) = \{\nu \in I^Y \mid p^{-1}(\nu)$ is a fuzzy open set in $X\}$

is a fuzzy topology on Y. We call $T(p)$ the *fuzzy identification topology* induced by p on Y.

From DEFINITION 4. 1, it is clear that the fuzzy identification topology induced by p is the largest fuzzy topology on Y making p fuzzy continuous. We should also note that the fuzzy identification topology can be completely described as follows:
$\mu \in I^Y$ is fuzzy closed in the fuzzy identification topology induced by p if and only if $g^{-1}(\mu)$ is fuzzy closed in X.

DEFINITION 4. 2 Let (X, T_X) and (Y, T_Y) be *fts*. An onto mapping $p: X \to Y$ is called a *fuzzy identification mapping* iff $T(p) = T_Y$. In case $T(p) \subseteq T_Y$, p is said to be *fuzzy quasi-compact.*

Notice that every fuzzy identification mapping is fuzzy quasi-compact.

EXAMPLE 4. 3 Let X_1 and X_2 be *fts*. Notice that $\pi_j^{-1}(\mu)$ is a fuzzy open set in $X_1 \times X_2$ if and only if μ is a fuzzy open set in X_j. Therefore the fuzzy identification topology $T(\pi_j)$ in the set $X_j (j = 1, 2)$, induced by the $\pi_j: X_1 \times X_2 \to X_j$, is precisely the original fuzzy topology in X_j. ■

But we notice that not every fuzzy continuous onto mapping is a fuzzy identification mapping. It is shown in the following EXAMPLE 4. 4.

EXAMPLE 4. 4 Let μ_1, μ_2 and μ_3 be fuzzy sets of I defined by

$$\mu_1(x)=x,\ \mu_2=1-x,\ \mu_3=x \qquad 0\le x\le \frac{1}{2}$$

$$=0 \qquad \frac{1}{2}<x\le 1. \quad \text{for each } x\in I.$$

Consider fuzzy topologies $T_1=\{0_I, \mu_1, \mu_2, \mu_3, \mu_1\wedge\mu_2, \mu_1\vee\mu_2, 1_I\}$ and $T_2=\{0_I, \mu_1, \mu_2, \mu_1\wedge\mu_2, \mu_1\vee\mu_2, 1_I\}$ on I, and the mapping $i_I:(I, T_1)\to(I, T_2)$ defined by $i_I(x)=x$, for all $x\in I$. Then i_I is a fuzzy continuous onto mapping. Also, $i_I^{-1}(\mu_3)=\mu_3\in T_1$, but $\mu_3\notin T_2$. Therefore i_I is not a fuzzy identification mapping. ■

THEOREM 4. 5 *Let X and Y be fts. If $p:X\to Y$ is a fuzzy continuous fuzzy open(closed) onto mapping, then p is a fuzzy identification mapping.*

PROOF. Let T_Y be the fuzzy topology in Y. Since $T(p)$ is the largest fuzzy topology on Y making p fuzzy continuous, $T_Y\subseteq T(p)$. If $\nu\in T(p)$, then $p^{-1}(\nu)$ is a fuzzy open set in X. Since p is a fuzzy open and onto mapping, $p(p^{-1}(\nu))=\nu$ in T_Y, hence $T(p)\subseteq T_Y$. Consequently, p is a fuzzy identification mapping. ■

THEOREM 4. 6 *Let X and Y be fts and let $p:X\to Y$ be a fuzzy continuous mapping. If $q:X\to Y$ is a fuzzy continuous mapping such that $p\circ q=i_Y$, the p is a fuzzy indentification mapping.*

PROOF. By LEMMA 2. 1, p is an onto mapping. Let ν be a fuzzy set in Y such that $p^{-1}(\nu)$ is a fuzzy open set in X. Then $q^{-1}(p^{-1}(\nu))$ is a fuzzy open set in Y. Since $q^{-1}(p^{-1}(\nu))=(p\circ q)^{-1}(\nu)=i_Y(\nu)=\nu$, ν is a fuzzy open set in Y. Thus p is a fuzzy identification mapping. ■

THEOREM 4. 7 *Let X, Y and Z be fts and let $p:X\to Y$ be a fuzzy identification mapping. Then $q:Y\to Z$ is fuzzy continuous if and only if $q\circ p:X\to Z$ is fuzzy continuous.*

PROOF. Necessity is clearly true.

Conversely, Let $q\circ p$ be fuzzy continuous and let δ be a fuzzy set in Z. Then $(q\circ p)^{-1}(\delta)=p^{-1}(q^{-1}(\delta))$ is a fuzzy open set in X. Since p is a fuzzy identification mapping, $q^{-1}(\delta)$ is a fuzzy open set in Y. Hence q is fuzzy continuous. ■

DEFINITION 4. 8 Let X be a *fts* and let Y be a set, and $p:X\to Y$ be an onto mapping. Then the family $T_r(p)$ of fuzzy subsets of Y defined by

$$T_r(p)=\{\nu\in I^Y \mid p^{-1}(\nu) \text{ is a fuzzy regularly open set in } X\}$$

is a fuzzy topology on Y. We call $T_r(p)$ the *fuzzy almost identification topology* induced by p on Y.

DEFINITION 4. 9 Let (X, T_X) and (Y, T_Y) be *fts*. An onto mapping $p:X\to Y$ is called a *fuzzy almost identification mapping* iff $T_r(p)=T_Y$. In case $T_r(p)\subseteq T_Y$, p is said to be *fuzzy almost quasi-compact.*

REMARK 4. 10 *Every fuzzy almost identification mapping is fuzzy almost quasi-compact. Also, if $p:X\to Y$ is a fuzzy almost quasi-compact mapping and $p^{-1}(\mu)$ is a fuzzy regularly closed set in X then μ is a fuzzy closed set in Y.*

THEOREM 4. 12 *Every fuzzy quasi-compact mapping is fuzzy almost quasi-compact.*

PROOF. Let (X, T_X) and (Y, T_Y) be *fts* and let $p:X\rightarrow Y$ be a fuzzy quasi-compact mapping. Let ν be a fuzzy set in Y such that $p^{-1}(\nu)$ is fuzzy regularly open. Then $p^{-1}(\nu)$ is fuzzy open in X. Since p is fuzzy quasi-compact, ν is fuzzy open in Y. Thus $T_r(p)\subseteq T_Y$. Consequently p is fuzzy almost quasi-compact. ■

But the converse need not be true as shown in EXAMPLE 4. 4.

THEOREM 4. 13 *Let X and Y be fts. Then, a mapping from X onto Y is fuzzy almost quasi-compact if and only if the image of every fuzzy regularly open inverse set is a fuzzy open set.*

PROOF. Let p be a fuzzy almost quasi-compact mapping, and let ν be a fuzzy set in Y such that $p^{-1}(\nu)$ is fuzzy regularly open. Then ν is a fuzzy open set in Y. Since p is an onto mapping, by LEMMA 2. 1, $p(p^{-1}(\nu))=\nu$ is a fuzzy open set in Y.

Conversely, let μ be a fuzzy set in Y such that $p^{-1}(\mu)$ is fuzzy regularly open. Since p is an onto mapping, $p(p^{-1}(\mu))=\mu$. Then $p(p^{-1}(\mu))=\mu$ is a fuzzy open set in Y. Hence p is a fuzzy almost quasi-compact mapping. ■

COROLLARY 4. 14 *Let X and Y be fts. Then, a mapping p from X onto Y is fuzzy almost quasi-compact if and only if the image of every fuzzy regularly closed inverse set is a fuzzy closed set.*

In the following theorem, the equivalence of statesments (1) (2) (4) is already known, [8]. In this paper we will give a relatively simple and different proof by putting statement (3).

THEOREM 4. 15 *Let X and Y be fts and let f be a one-to-one mapping X onto Y. Then the following statements are equivalent:*

(1) *f is a fuzzy almost open mapping.*

(2) *f is a fuzzy almost closed mapping.*

(3) *f is a fuzzy almost quasi-compact mapping.*

(4) *f^{-1} is a fuzzy almost continuous mapping.*

PROOF. (1) implies (2): Let μ be a fuzzy regularly closed set in X. Then μ^c is a fuzzy regularly open set in X. Therefore $f(\mu^c)$ is a fuzzy open set. Thus $(f(\mu^c))^c=f(\mu)$ is a fuzzy closed set in Y. Hence f is a fuzzy almost closed mapping.

(2) implies (3): Let ν be a fuzzy set in Y such that $f^{-1}(\nu)$ is fuzzy regularly closed. By COROLLARY 4. 14, $f(f^{-1}(\nu))=\nu$ is fuzzy closed. Hence f is a fuzzy almost quasi-compact mapping.

(3) implies (4): Let μ be a fuzzy regularly open set of X. Then $f^{-1}(f(\mu))=\mu$ is a fuzzy regularly open set. Hence $f(\mu)$ is a fuzzy open set. Thus $f(\mu)=(f^{-1})^{-1}(\mu)$ is fuzzy open set. Therefore f^{-1} is a fuzzy almost continuous mapping.

(4) implies (1): Let μ be a fuzzy regularly open set in X. Then $(f^{-1})^{-1}(\mu)=f(\mu)$ is a fuzzy open set in Y. Hence f is a fuzzy almost open mapping. ■

THEOREM 4. 16 *Let X, Y and Z be fts, and let $p:X\rightarrow Y$ and $q:Y\rightarrow Z$ be onto mappings. If p is a fuzzy almost continuous mapping and $q\circ p$ is a fuzzy quasi-compact mapping, then q is a fuzzy almost quasi-compact mapping.*

PROOF. Let p be a fuzzy almost continuous and let $q\circ p$ be a fuzzy quasi-compact mapping. Let $q^{-1}(\delta)$ be a fuzzy regularly open set in Y. Since p is fuzzy almost continuous, $p^{-1}(q^{-1}(\delta))=(q\circ p)^{-1}(\delta)$ is a fuzzy open set in X. Since $q\circ p$ is a fuzzy quasi-compact mapping, δ is a fuzzy open set in Z. Thus q is fuzzy almost quasi-compact. ■

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