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THE PETTIS INTEGRABILITY OF BOUNDED WEAKLY MEASURABLE FUNCTIONS ON FINITE MEASURE SPACES

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1. Introduction

Since the concept of Pettis integral was introduced in 1938 [10], the Pettis integrability of weakly measurable functions has been studied by many authors [5, 6, 7, 8, 9, 11]. It is known that there is a bounded function that is not Pettis integrable [10, Example 10. 8]. So it is natural to raise the question : when is a bounded function Pettis integrable ?

In this paper we study the Pettis integrability of bounded weakly measurable functions on finite measure spaces. We will show that the Pettis integrability of those functions is deeply related to the core of the function and the sets $S(\pi, B)$.

Throughout this paper, (Ω, Σ, μ) denotes a finite measure space and f a function from Ω into a real Banach space X with continuous dual X^*. For each $x^* \in X^*$, $x^*f : \Omega \to \mathbb{R}$ is the scalar function defined by $(x^*f)(t) = x^*(f(t))$, $t \in \Omega$. For any $A \subset X$, $\overline{\text{co}}A$ denotes the closed convex hull of A.

A function $f : \Omega \to X$ is said to be *weakly measurable* if for each $x^* \in X^*$, the function x^*f is measurable. This function $f : \Omega \to X$ is said to be *strongly measurable* if there is a sequence $\{f_n\}$ of measurable simple functions such that

$$\lim_{n\to\infty} ||f_n - f|| = 0 \text{ a. e.}$$

Two functions $f, g : \Omega \to X$ are said to be *weakly equivalent* if $x^*f = x^*g$ a. e. for all $x^* \in X^*$.

A weakly measurable function $f : \Omega \to X$ is said to be *Pettis integrable* if for every set $E \in \Sigma$, there is an element of X, denoted $\int_E f d\mu$, such that

$$x^* \left(\int_E f d\mu \right) = \int_E x^* f d\mu$$

for all $x^* \in X^*$. $\int_E f d\mu$ is called the *Pettis integral* of f.

Remark 1. If $x^* f$ is integrable for all $x^* \in X^*$, then for every set $E \in \Sigma$, there exists an element $x^{**} \in X^{**}$ such that $x^{**}(x^*) = \int_E x^* f d\mu$ [2, Lemma 1, II. 3]. This $x^{**} = \int_E f d\mu$ is called the *Dunford integral* of f.

Hence f is Pettis integrable if for all $E \in \Sigma$, the Dunford integral of f over E lies in X.

A strongly measurable function f is *Bochner integrable* if there is a sequence of simple functions $\{f_n\}$ such that

$$\lim_{n \to \infty} \int ||f_n - f|| d\mu = 0.$$

In this case, we denote $\int_E f d\mu = lim_{n \to \infty} \int f_n d\mu$.

For Bochner integrable function, the following is true :

Theorem [3, Corollary 8, II. 2]. Let $f : \Omega \to X$ be Bochner integrable, then for each $E \in \Sigma$ with $\mu(E) > 0$,

$$\int_E f d\mu \in \overline{\text{co}}(f(E))\mu(E).$$

Motivated by the above theorem, we consider the case of Pettis integrable functions.

Let $f : \Omega \to X$ be Pettis integrable and $B \in \Sigma$. Let π be any finite partition of B, that is π is a finite set of disjoint measurable subsets E of B such that $\bigcup_{E \in \pi} E = B$. Then

$$\int_B f d\mu = \sum_{E \in \pi} \int_E f d\mu \in \sum_{E \in \pi} \overline{\text{co}} f(E)\mu(E).$$

We denote the closure of $\sum_{E \in \pi} \overline{\text{co}} f(E)\mu(E)$ by $S(\pi, B)$.

Remark 2. Note that

$$\int_B f d\mu \in \sum_{E\in\pi} \overline{\text{co}} f(E)\mu(E) \subset S(\pi, B)$$

for all finite partition π of B. And so

$$\int_B f d\mu \in \bigcap_{\pi} \subset S(\pi, B),$$

where the intersection is taken over all partitions π of B.

Let $f : \Omega \to X$ and let $E \in \Sigma$. The *core* of f over E, denoted by $\text{Cor}_f(E)$, is the subset of X given by

$$\text{Cor}_f(E) = \bigcap_{\mu(A)=0} \overline{\text{co}} f(E \setminus A).$$

If $\text{Cor}_f(E) \neq \emptyset$ for each set E of positive measure, then we say that f *has nonempty core.*

2. Main Results

The following theorem asserts that the average value of a Pettis integrable function lies in its core.

Theorem 2.1. If $f : \Omega \to X$ is Pettis integrable, then for each set E in Σ,

$$\text{Cor}_f(E) = \overline{\text{co}} \left\{ \frac{1}{\mu(B)} \int_B f d\mu \ : \ B \subset E, \mu(B) > 0 \right\}.$$

Proof. Let $B \subset E$ be of positive measure. Let A be a null set and $x^* \in X^*$. We have

$$\begin{aligned} x^*\left(\int_B f d\mu\right) &= \int_B x^* f d\mu = \int_{B\setminus A} x^* f d\mu \\ &\geq \int_B \inf[x^* f(E \setminus A)] d\mu \\ &= \inf[x^* f(E \setminus A)]\mu(B). \end{aligned}$$

Hence for all $x^* \in X^*$, the inequality

$$x^*\left[\frac{1}{\mu(B)}\int_B f d\mu\right] \geq \inf[x^* f(E \setminus A)]$$

holds. By the Hahn-Banach theorem,

$$\frac{1}{\mu(B)}\int_B f d\mu \in \frac{1}{\mu(B)}\int_B f d\mu \in \overline{\text{co}} f(E \setminus A).$$

Since A was an arbitrary null set, we see that

$$\frac{1}{\mu(B)}\int_B f d\mu \in \text{Cor}_f E = \bigcap_{\mu(A)=0} \overline{\text{co}} f(E \setminus A).$$

Moreover, since $\text{Cor}_f E$ is closed and convex,

$$\overline{\text{co}}\left\{\frac{1}{\mu(B)}\int_B f d\mu \;:\; B \subset E, \mu(B) > 0\right\} \subset \text{Cor}_f E.$$

Let $x \in \text{Cor}_f(E)$ and $||x^*|| \leq 1$. For any $\varepsilon > 0$, choose a countable partition π of E and a function α, constant on the set π such that the inequality

$$|x^* f(t) - \alpha(t)| < \frac{\varepsilon}{4}$$

holds for all $t \in E$. If B is any set in π with positive measure and $t \in B$, then

$$\left|x^* f(t) - \frac{1}{\mu(B)}\int_B x^* f d\mu\right| < \frac{\varepsilon}{2}$$

since α is constant on B.

Let A be the union of the null sets in π. Then $x \in \overline{\text{co}} f(E \setminus A)$ and hence there is a finite convex sum $\Sigma \lambda_i f(t_i)$ such that $t_i \in E \setminus A$ and

$$\left|\left|x - \sum \lambda_i f(t_i)\right|\right| < \frac{\varepsilon}{2}.$$

We have
$$\left|x^*(x) - \sum \lambda_i x^* f(t_i)\right| < \frac{\varepsilon}{2}.$$
For each i, let B_i be the set in π containing t_i. Then
$$\left|x^*(x) - \sum \lambda_i \frac{1}{\mu(B_i)} \int_{B_i} x^* f d\mu\right| < \varepsilon.$$
Since ε was arbitrary, we can see that
$$x^*(x) \geq \inf\left[x^*\left\{\frac{1}{\mu(B)} \int_B x^* f d\mu \ : \ B \subset E, \mu(B) > 0\right\}\right].$$
By Hahn-Banach theorem, we conclude that
$$x \in \overline{\mathrm{co}}\left\{\frac{1}{\mu(B)} \int_B x^* f d\mu \ : \ B \subset E, \mu(B) > 0\right\}.$$
This shows that
$$\mathrm{Cor}_f(E) \subset \overline{\mathrm{co}}\left\{\frac{1}{\mu(B)} \int_B x^* f d\mu \ : \ B \subset E, \mu(B) > 0\right\}.$$
This completes the proof.

Corollary 2.2. If f is weakly equivalent to a strongly measurable function g, then f has nonempty core.

Proof. Let E be any measurable set of positive measure. Then there exists a subset B of E such that $g\chi|_B$ is bounded and Bochner integrable. Since $f\chi|_B$ is weakly equivalent to a Bochner integrable function $g\chi|_B$, it is Pettis integrable (see [2]). Hence $\mathrm{Cor}_f(B) \neq \emptyset$. □

Next, we will show that the sets $S(\pi, B)$ play very important role in the study of the Pettis integrability.

Theorem 2.3. A bounded weakly measurable function $f : \Omega \to X$ is Pettis integrable if and only if for each measurable set B, $\bigcap_\pi S(\pi, B) \neq \emptyset$, where the intersection is taken over all finite partitions of B.

Proof. Let f be Pettis integrable. By Remark 2,

$$\int_B f d\mu \in \bigcap_\pi S(\pi, B).$$

Conversely, assume that $\bigcap_\pi S(\pi, B) \neq \emptyset$ for all $B \in \Sigma$. We will show that $\bigcap_\pi S(\pi, B) = \{\int_B f d\mu\}$ for all $B \in \Sigma$. Fix $B \in \Sigma$ and $x \in \bigcap_\pi S(\pi, B)$. First, we will show that $x^*(x) = \int_B x^* f d\mu$ for all $x^* \in X^*$. Fix $x^* \in X^*$ and $\varepsilon > 0$. Since $x^* f$ is bounded and measurable, it is the uniform limit of a sequence of simple functions. Hence there is a finite partition π of B such that

$$|x^* f(t_1) - x^* f(t_2)| < \varepsilon$$

for all $E \in \pi$ and $t_1, t_2 \in E$.

Let $E \in \pi$ then for any $t \in E$, we have

$$\left| \frac{1}{\mu(E)} \int_E x^* f d\mu - x^* f(t) \right| < \varepsilon,$$

and

$$\left| \int_E x^* f d\mu - x^* f(t) \mu(E) \right| < \varepsilon \mu(E).$$

If $\sum \alpha_i f(t_i)$ is a convex sum with each $t_i \in E$, then

$$\left| \int_E x^* f d\mu - x^* (\sum \alpha_i f_i(t)) \mu(E) \right| < \varepsilon \mu(E).$$

It follows that if $x_E \in \overline{\text{co}} f(E)$,

$$\left| \int_E x^* f d\mu - x^*(x_E) \mu(E) \right| < \varepsilon \mu(E).$$

Summing over all the sets $E \in \pi$, we have

$$\left| \int_B x^* f d\mu - x^*(x) \right| < \varepsilon \mu(B).$$

Since ε was arbitrary, we have

$$x^*(x) = \int_B x^* f d\mu,$$

and this means that $x = \int_B f d\mu$ where the integral is the Pettis integral.

Theorem 2.4. A bounded weakly measurable function $f : \Omega \to X$ is Pettis integrable if and only if for each set B in Σ, there is a weakly compact set $W \subset X$ such that $W \bigcap S(\pi, B) \neq \emptyset$ for every finite partition π of B.

Proof. Suppose that f is Pettis integrable. Let W be the weak closure of the set $\{\int_E f d\mu \ : \ E \in \Sigma\}$. Then by Corollary 9, II. 3 in [3], W is weakly compact. Since $\int_B f d\mu \in W \cap S(\pi, B)$, $W \cap (\pi, B) \neq \emptyset$ for all finite partition π of B.

Conversely, let $B \in \Sigma$ and π and π' be finite partitions of B such that π' refines π. Then $S(\pi', B) \subset S(\pi, B)$ and hence the family of sets $\{W \cap S(\pi, B) : \pi$ is a finite partition of $B\}$ has the finite intersection property. Since W is weakly compact and these sets are weakly closed, this family has the nonempty intersection. Since $B \in \Sigma$ was arbitrary, f is Pettis integrable by Theorem 2.3.

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