THE COMPLETION OF SOME METRIC SPACE OF FUZZY NUMBERS

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1. Introduction

D. Dubois and H. Prade introduced the notions of fuzzy numbers and defined its basic operations [3]. R. Goetschel, W. Voxman, A. Kaufmann, M. Gupta and G. Zhang [4, 5, 6, 9] have done much work about fuzzy numbers.

Let \( \mathbb{R} \) be the set of all real numbers and \( F^*(\mathbb{R}) \) all fuzzy subsets defined on \( \mathbb{R} \). G. Zhang [8] defined the fuzzy number \( \tilde{a} \in F^*(\mathbb{R}) \) as follows:

1. \( \tilde{a} \) is normal,
2. for every \( \lambda \in (0, 1] \), \( a_{\lambda} = \{ x \mid \tilde{a}(x) \geq \lambda \} \) is a closed interval, denoted by \( [a_{\lambda}^-, a_{\lambda}^+] \).

Now, let us denote the set of all fuzzy numbers defined by G. Zhang as \( F(\mathbb{R}) \).

The purpose of this paper is to prove that the metric space \( (F(\mathbb{R}), \delta) \) can be completed by using the equivalence classes of Cauchy sequences, where \( \delta \) is defined by \( \delta(\tilde{a}, \tilde{b}) = \sup_{0 \leq \lambda \leq 1} d(a_{\lambda}, b_{\lambda}) \). In section 2, we quote basic definitions and theorems from [1] which will be needed in the proof of main theorem. In section 3, after defining the isometry and the completion concepts, we prove main theorem:

The metric space \( (F(\mathbb{R}), \delta) \) has a completion \( (\hat{F}(\mathbb{R}), \hat{\delta}) \) which has a subspace \( X \) that is isometric with \( F(\mathbb{R}) \) and is dense in \( \hat{F}(\mathbb{R}) \). This space \( (\hat{F}(\mathbb{R}), \hat{\delta}) \) is unique except for isometries.

2. Basic definitions and results of fuzzy numbers
In this section, we quote basic definitions and theorems from [1] which will be needed in the proof of main theorem.

Let $\mathbb{R}$ be the set of all real numbers and $F^*(\mathbb{R})$ all fuzzy subsets defined on $\mathbb{R}$.

**Definition 2.1.** Let $\tilde{a} \in F^*(\mathbb{R})$. $\tilde{a}$ is called a fuzzy number if $\tilde{a}$ has the properties:

1. $\tilde{a}$ is normal, i.e., there exists $x \in \mathbb{R}$ such that $\tilde{a}(x) = 1$,
2. whenever $\lambda \in (0, 1]$, $a_{\lambda} = \{x \mid \tilde{a}(x) \geq \lambda\}$ is a closed interval, denoted by $[a^-_{\lambda}, a^+_{\lambda}]$.

Let $F(\mathbb{R})$ be the set of all fuzzy numbers on the real line $\mathbb{R}$.

If we define $\tilde{a}(x)$ by

$$\tilde{a}(x) = \begin{cases} 1 & \text{for } x = k, \\ 0 & \text{for } x \neq k \quad (k \in \mathbb{R}), \end{cases}$$

then $\tilde{a} \in F(\mathbb{R})$ and $a_{\lambda} = [k, k]$. In here, we can see that any real number is a fuzzy numbers.

**Definition 2.2.** Let $\tilde{a}, \tilde{b}, \tilde{c} \in F(\mathbb{R})$. We say that $\tilde{c} = \tilde{a} + \tilde{b}$ if for every $\lambda \in (0, 1]$, $c^-_{\lambda} = a^-_{\lambda} + b^-_{\lambda}$ and $c^+_{\lambda} = a^+_{\lambda} + b^+_{\lambda}$. We say that $\tilde{c} = \tilde{a} - \tilde{b}$ if for every $\lambda \in (0, 1]$, $c^-_{\lambda} = a^-_{\lambda} - b^+_{\lambda}$ and $c^+_{\lambda} = a^+_{\lambda} - b^-_{\lambda}$. For every $k \in \mathbb{R}$ and $\tilde{a} \in F(\mathbb{R})$, we say that $\tilde{c} = k\tilde{a}$ if for every $\lambda \in (0, 1]$, $c^-_{\lambda} = ka^-_{\lambda}$, $c^+_{\lambda} = ka^+_{\lambda}$ for $k \geq 0$, and $c^-_{\lambda} = ka^+_{\lambda}$, $c^+_{\lambda} = ka^-_{\lambda}$ for $k < 0$.

Note that we can find in [7] the definitions of multiplication, division, maximal and minimal operations of the fuzzy numbers.

**Definition 2.3.** Let $\tilde{a}, \tilde{b} \in F(\mathbb{R})$. We say that $\tilde{a} \leq \tilde{b}$ if for every $\lambda \in (0, 1]$, $a^-_{\lambda} \leq b^-_{\lambda}$ and $a^+_{\lambda} \leq b^+_{\lambda}$. We say that $\tilde{a} < \tilde{b}$ if $\tilde{a} \leq \tilde{b}$ and there exists $\lambda \in (0, 1]$ such that $a^-_{\lambda} < b^-_{\lambda}$ or $a^+_{\lambda} < b^+_{\lambda}$. We say that $\tilde{a} = \tilde{b}$ if $\tilde{a} \leq \tilde{b}$ and $\tilde{b} \leq \tilde{a}$.

**Definition 2.4.** For two closed intervals $a_{\lambda} = [a^-_{\lambda}, a^+_{\lambda}], b_{\lambda} = [b^-_{\lambda}, b^+_{\lambda}],$ we define a metric (distance) $d$ of $a_{\lambda}, b_{\lambda}$ as follows:

$$d(a_{\lambda}, b_{\lambda}) = \max(|a^-_{\lambda} - b^-_{\lambda}|, |a^+_{\lambda} - b^+_{\lambda}|).$$
Definition 2.5. A metric (distance) $\delta$ of $F(\mathbb{R})$ is a function $\delta : F(\mathbb{R}) \times F(\mathbb{R}) \rightarrow \mathbb{R}$ with the properties:

1. $\delta(\tilde{a}, \tilde{b}) \geq 0$, $\tilde{a} = \tilde{b}$ if and only if $\delta(\tilde{a}, \tilde{b}) = 0$,
2. $\delta(\tilde{a}, \tilde{b}) = \delta(\tilde{b}, \tilde{a})$,
3. for every $\tilde{c} \in F(\mathbb{R})$, we have $\delta(\tilde{a}, \tilde{b}) \leq \delta(\tilde{a}, \tilde{c}) + \delta(\tilde{c}, \tilde{b})$.

When $\delta$ is a metric of $F(\mathbb{R})$, we call $(F(\mathbb{R}), \delta)$ a metric space of $F(\mathbb{R})$ with the metric $\delta$.

We define

$$\delta(\tilde{a}, \tilde{b}) = \sup_{0 < \lambda \leq 1} d(a_\lambda, b_\lambda). \tag{\star}$$

Theorem 2.6. $\delta(\tilde{a}, \tilde{b})$ defined by the equality (\star) is a metric of $F(\mathbb{R})$.

Theorem 2.7. For every $\tilde{a}, \tilde{b}, \tilde{c} \in F(\mathbb{R})$, $k \in \mathbb{R}$, we have

1. $\delta(\tilde{a} + \tilde{b}, \tilde{a} + \tilde{c}) = \delta(\tilde{b}, \tilde{c})$,
2. $\delta(\tilde{a} - \tilde{b}, \tilde{a} - \tilde{c}) = \delta(\tilde{b}, \tilde{c})$,
3. $\delta(k\tilde{a}, k\tilde{b}) = |k| \delta(\tilde{a}, \tilde{b})$,
4. If $\tilde{a} \leq \tilde{b} \leq \tilde{c}$, then $\delta(\tilde{a}, \tilde{b}) \leq \delta(\tilde{a}, \tilde{c})$, $\delta(\tilde{b}, \tilde{c}) \leq \delta(\tilde{a}, \tilde{c})$.

Definition 2.8. Let \{${\tilde{a}}_n$\} $\subset F(\mathbb{R})$, $\tilde{a} \in F(\mathbb{R})$. A sequence \{${\tilde{a}}_n$\} is said to converge to $\tilde{a}$ in the metric $\delta$, denoted by

$$\lim_{n \rightarrow \infty} \tilde{a}_n = \tilde{a} \quad \text{or} \quad \tilde{a}_n \rightarrow \tilde{a} \text{ as } n \rightarrow \infty,$$

if for any $\varepsilon > 0$ there exists an integer $N > 0$ such that $\delta(\tilde{a}_n, \tilde{a}) < \varepsilon$ for $n \geq N$.

Theorem 2.9. Let \{${\tilde{a}}_n$\}, \{${\tilde{b}}_n$\} $\subset F(\mathbb{R})$, $\tilde{a}, \tilde{b} \in F(\mathbb{R})$, $k \in \mathbb{R}$. If $\lim_{n \rightarrow \infty} \tilde{a}_n = \tilde{a}$ and $\lim_{n \rightarrow \infty} \tilde{b}_n = \tilde{b}$, then

1. $\lim_{n \rightarrow \infty} (\tilde{a}_n \pm \tilde{b}_n) = \tilde{a} \pm \tilde{b}$ (the same order of sign),
2. $\lim_{n \rightarrow \infty} (k\tilde{a}_n) = k\tilde{a}$.

Theorem 2.10. Let $\lim_{n \rightarrow \infty} \tilde{a}_n = \tilde{a}$, $\lim_{n \rightarrow \infty} \tilde{b}_n = \tilde{b}$. Then

$$\lim_{n \rightarrow \infty} \delta(\tilde{a}_n, \tilde{b}_n) = \delta(\tilde{a}, \tilde{b}).$$
Definition 2.11. A sequence \( \{\hat{a}_n\} \) of \( F(\mathbb{R}) \) is said to be a Cauchy sequence if for every \( \varepsilon > 0 \) there exists an integer \( N > 0 \) such that \( \delta(\hat{a}_n, \hat{a}_m) < \varepsilon \) for \( n, m > N \).

If a metric space has the property that every Cauchy sequence converges, the space is called a complete metric space.

Theorem 2.12. The metric space \( (F(\mathbb{R}), \delta) \) is complete.

3. Main theorem

In this section, we prove that the metric space \( (F(\mathbb{R}), \delta) \) has a completion \( (\hat{F}(\mathbb{R}), \hat{\delta}) \).

Definition 3.1. [2] Let \( X_1 = (X_1, d_1) \), \( X_2 = (X_2, d_2) \) be metric spaces. Then,

(a) A mapping \( f \) of \( X_1 \) into \( X_2 \) is said to be isometric or an isometry if \( f \)
    preserves distances, that is, if for all \( x, y \in X_1 \), \( d_2(f(x), f(y)) = d_1(x, y) \),
    where \( f(x) \) and \( f(y) \) are the images of \( x \) and \( y \) respectively.
(b) The space \( X_1 \) is said to be isometric with the space \( X_2 \) if there exists a
    bijective isometry of \( X_1 \) onto \( X_2 \). The spaces \( X_1 \) and \( X_2 \) are then called
    isometric spaces.

Definition 3.2. [2] The complete metric space \( (X_1^*, d_1^*) \) is said to be a completion
    of the given metric space \( (X_1, d_1) \) if

1. \( (X_1, d_1) \) is isometric with a subspace \( (X_1, d_1^*) \) of \( (X_1^*, d_1^*) \),
2. \( X \) is dense in \( X_1^* \), i.e., the closure of \( X \), \( \overline{X} = X_1^* \).

MAIN Theorem. The metric space \( (F(\mathbb{R}), \delta) \) has a completion \( (\hat{F}(\mathbb{R}), \hat{\delta}) \) which
    has a subspace \( X \) that is isometric with \( F(\mathbb{R}) \) and is dense in \( \hat{F}(\mathbb{R}) \). This space
    \( (\hat{F}(\mathbb{R}), \hat{\delta}) \) is unique except for isometries, that is, if \( (\tilde{F}(\mathbb{R}), \tilde{\delta}) \) is another completion
    having a dense subspace \( Y \) isometric with \( F(\mathbb{R}) \), then \( \hat{F}(\mathbb{R}) \) and \( \tilde{F}(\mathbb{R}) \) are isometric.

Proof. The proof is somewhat lengthy. We divide it into four steps (a) to (d).

We construct :

(a) \( \hat{F}(\mathbb{R}) = (\hat{F}(\mathbb{R}), \hat{\delta}) \),

(b) an isometry \( f \) of \( F(\mathbb{R}) \) onto \( X \), where \( \overline{X} = \hat{F}(\mathbb{R}) \).
Then we prove:

c. completeness of \( \hat{F}(\mathbb{R}) \),

d. uniqueness of \( \hat{F}(\mathbb{R}) \) except for isometries.

(a). Construction of \( \hat{F}(\mathbb{R}) = (\hat{F}(\mathbb{R}), \hat{\delta}) \).

Let \( \{\tilde{x}_n\} \) and \( \{\tilde{x}_n'\} \) be Cauchy sequences in \( F(\mathbb{R}) \). Define \( \{\tilde{x}_n\} \) to be equivalent to \( \{\tilde{x}_n'\} \) written \( \tilde{x}_n \sim \{\tilde{x}_n'\} \), if

\[
\lim_{n \to \infty} \delta(\tilde{x}_n, \tilde{x}_n') = 0. \tag{1}
\]

Let \( \hat{F}(\mathbb{R}) \) be the set of all equivalence classes \( \hat{x}, \hat{y}, \cdots \) of Cauchy sequences thus obtained. We write \( \tilde{x}_n \in \hat{x} \) to mean that \( \{\tilde{x}_n\} \) is a member of \( \hat{x} \) (a representative of the class \( \hat{x} \)). We now set

\[
\hat{\delta}(\hat{x}, \hat{y}) = \lim_{n \to \infty} \delta(\tilde{x}_n, \tilde{y}_n) \tag{2}
\]

where \( \{\tilde{x}_n\} \in \hat{x} \) and \( \{\tilde{y}_n\} \in \hat{y} \). We show that this limit exists. We have

\[
\delta(\tilde{x}_n, \tilde{y}_n) \leq \delta(\tilde{x}_n, \tilde{x}_m) + \delta(\tilde{x}_m, \tilde{y}_m) + \delta(\tilde{y}_m, \tilde{y}_n),
\]

hence we obtain

\[
\delta(\tilde{x}_n, \tilde{y}_n) - \delta(\tilde{x}_m, \tilde{y}_m) \leq \delta(\tilde{x}_n, \tilde{x}_m) + \delta(\tilde{y}_m, \tilde{y}_n)
\]

and a similar inequality with \( m \) and \( n \) interchanged. Together,

\[
|\delta(\tilde{x}_n, \tilde{y}_n) - \delta(\tilde{x}_m, \tilde{y}_m)| \leq \delta(\tilde{x}_n, \tilde{x}_m) + \delta(\tilde{y}_m, \tilde{y}_n). \tag{3}
\]

Since \( \{\tilde{x}_n\} \) and \( \{\tilde{y}_n\} \) are Cauchy, we can make the right side as small as we please. This implies that the limit in (2) exists because \( (F(\mathbb{R}), \delta) \) is complete.

We must also show that the limit in (2) is independent of the particular choice of representatives. In fact, if \( \{\tilde{x}_n\} \sim \{\tilde{x}_n'\} \) and \( \{\tilde{y}_n\} \sim \{\tilde{y}_n'\} \), then by (1), (3),

\[
|\delta(\tilde{x}_n, \tilde{y}_n) - \delta(\tilde{x}_n', \tilde{y}_n')| \leq \delta(\tilde{x}_n, \tilde{x}_n') + \delta(\tilde{y}_n, \tilde{y}_n') \to 0
\]
as \( n \to \infty \), which implies the assertion
\[
\lim_{n \to \infty} \delta(\tilde{x}_n, \tilde{y}_n) = \lim_{n \to \infty} \delta(\tilde{x}'_n, \tilde{y}'_n).
\]

We prove that \( \hat{\delta} \) in (2) is a metric on \( \hat{F}(\mathbb{R}) \). Obviously, \( \hat{\delta} \) satisfies \( \hat{\delta}(\tilde{x}, \tilde{y}) \geq 0 \) (see Definition of \( \delta(\tilde{x}, \tilde{y}) \)) as well as \( \hat{\delta}(\tilde{x}, \tilde{x}) = 0 \) and \( \hat{\delta}(\tilde{x}, \tilde{y}) = \hat{\delta}(\tilde{y}, \tilde{x}) \). Furthermore,
\[
\hat{\delta}(\tilde{x}, \tilde{y}) = 0 \quad \Rightarrow \quad \{\tilde{x}_n\} \sim \{\tilde{y}_n\} \quad \Rightarrow \quad \tilde{x} = \tilde{y}
\]
gives \( \hat{\delta}(\tilde{x}, \tilde{y}) = 0 \Leftrightarrow \tilde{x} = \tilde{y} \), and the triangle inequality for \( \hat{\delta} \) follows from
\[
\delta(\tilde{x}_n, \tilde{y}_n) \leq \delta(\tilde{x}_n, \tilde{z}_n) + \delta(\tilde{z}_n, \tilde{y}_n)
\]
by letting \( n \to \infty \).

(b). Construction of an isometry \( f : F(\mathbb{R}) \to X \subset \hat{F}(\mathbb{R}) \).

With each \( \tilde{a} \in F(\mathbb{R}) \) we associate the class \( \tilde{a} \in \hat{F}(\mathbb{R}) \) which contains the constant Cauchy sequence \( \{\tilde{a}, \tilde{a}, \cdots\} \). This defines a mapping \( f : F(\mathbb{R}) \to X \) onto the subspace \( X = f(F(\mathbb{R})) \subset \hat{F}(\mathbb{R}) \). The mapping \( f \) is given by \( \tilde{a} \mapsto \tilde{a} = f(\tilde{a}) \), where \( \{\tilde{a}, \tilde{a}, \cdots\} \in \tilde{a} \). We see that \( f \) is an isometry since (2) becomes simply
\[
\hat{\delta}(\tilde{x}, \tilde{y}) = \delta(\tilde{a}, \tilde{b}),
\]
here \( \tilde{b} \) is the class of \( \{\tilde{y}_n\} \) where \( \tilde{y}_n = \tilde{b} \) for all \( n \). Any isometry is injective, and \( f : F(\mathbb{R}) \to X \) is surjective since \( f(F(\mathbb{R})) = X \). Hence \( X \) and \( F(\mathbb{R}) \) are isometric.

We show that \( X \) is dense in \( \hat{F}(\mathbb{R}) \). We consider any \( \tilde{x} \in \hat{F}(\mathbb{R}) \). Let \( \{\tilde{x}_n\} \in \tilde{x} \). For every \( \varepsilon > 0 \) there is an integer \( N > 0 \) such that
\[
\delta(\tilde{x}_n, \tilde{x}_N) < \varepsilon/2 \quad \text{for } n > N.
\]
Let \( \{\tilde{x}_N, \tilde{x}_N, \cdots\} \in \tilde{x}_N \). Then \( \tilde{x}_N \in X \). By (2),
\[
\hat{\delta}(\tilde{x}, \tilde{x}_N) = \lim_{n \to \infty} \delta(\tilde{x}_n, \tilde{x}_N) \leq \varepsilon/2 < \varepsilon.
\]
This shows that every \( \varepsilon \)-neighborhood of the arbitrary \( \tilde{x} \in \hat{F}(\mathbb{R}) \) contains an element of \( X \). Hence \( X \) is dense in \( \hat{F}(\mathbb{R}) \).
(c). Completeness of $\hat{F}(\mathbb{R})$.  
Let $\{\hat{x}_n\}$ be any Cauchy sequence in $\hat{F}(\mathbb{R})$. Since $X$ is dense in $\hat{F}(\mathbb{R})$, for every $\hat{x}_n \in \hat{F}(\mathbb{R})$ there is a $\hat{z}_n \in X$ such that
\[
\hat{\delta}(\hat{x}_n, \hat{z}_n) < \frac{1}{n}.
\]  
(4)
Hence, by the triangle inequality,
\[
\hat{\delta}(\hat{z}_m, \hat{z}_n) \leq \hat{\delta}(\hat{z}_m, \hat{x}_m) + \hat{\delta}(\hat{x}_m, \hat{x}_n) + \hat{\delta}(\hat{x}_n, \hat{z}_n)
< \frac{1}{m} + \hat{\delta}(\hat{x}_m, \hat{x}_n) + \frac{1}{n}
\]
and this is less than any given $\epsilon > 0$ for sufficiently large $m$ and $n$ because $\{\hat{x}_n\}$ is Cauchy. Hence $\{\hat{z}_m\}$ is Cauchy. Since $f : F(\mathbb{R}) \to X$ is isometric and $\hat{z}_m \in X$, the sequence $\{\hat{z}_m\}$, where $\hat{z}_m = f^{-1}(\hat{z}_m)$, is Cauchy in $F(\mathbb{R})$. Let $\hat{x} \in \hat{F}(\mathbb{R})$ be the class to which $\{\hat{z}_m\}$ belongs. We show that $\hat{x}$ is the limit of $\{\hat{x}_n\}$. By (4),
\[
\hat{\delta}(\hat{x}_n, \hat{x}) \leq \hat{\delta}(\hat{x}_n, \hat{z}_n) + \hat{\delta}(\hat{z}_n, \hat{x})
< \frac{1}{n} + \hat{\delta}(\hat{z}_n, \hat{x}).
\]  
(5)
Since $\{\hat{z}_m\} \in \hat{x} \in \hat{F}(\mathbb{R})$ and $\hat{z}_n \in X$, so that $\{\hat{z}_n, \hat{z}_n, \ldots\} \in \hat{z}_n$, the inequality (5) becomes
\[
\hat{\delta}(\hat{x}_n, \hat{x}) < \frac{1}{n} + \lim_{m \to \infty} \delta(\hat{z}_n, \hat{z}_m)
\]
and the right side is smaller than any given $\epsilon > 0$ for sufficiently large $n$. Hence the arbitrary Cauchy sequence $\{\hat{x}_n\}$ in $\hat{F}(\mathbb{R})$ has the limit $\hat{x} \in \hat{F}(\mathbb{R})$, and $\hat{F}(\mathbb{R})$ is complete.

(d). Uniqueness of $\hat{F}(\mathbb{R})$ except for isometries.

If $(\hat{F}(\mathbb{R}), \hat{\delta})$ is another completion with a subspace $Y$ dense in $\hat{F}(\mathbb{R})$ and isometric with $F(\mathbb{R})$, then for any $\hat{x}, \hat{y} \in \hat{F}(\mathbb{R})$ we have sequences $\{\hat{x}_n\}$, $\{\hat{y}_n\}$ in $Y$ such that $\hat{x}_n \to \hat{x}$ and $\hat{y}_n \to \hat{y}$. Hence we have
\[
\hat{\delta}(\hat{x}, \hat{y}) \leq \hat{\delta}(\hat{x}, \hat{x}_n) + \hat{\delta}(\hat{x}_n, \hat{y}_n) + \hat{\delta}(\hat{y}_n, \hat{y})
\]
for every $n$, where $\{\tilde{x}_n, \tilde{y}_n, \cdots\} \in \tilde{x}_n$ and $\{\tilde{y}_n, \tilde{y}_n, \cdots\} \in \tilde{y}_n$. Since it is true for every $n$, it is true in the limit as $n$ becomes infinite, which yields

$$\delta(\tilde{x}, \tilde{y}) \leq \lim_{n \to \infty} \delta(\tilde{x}_n, \tilde{y}_n).$$

But

$$\delta(\tilde{x}_n, \tilde{y}_n) \leq \delta(\tilde{x}_n, \tilde{x}) + \delta(\tilde{x}, \tilde{y}) + \delta(\tilde{y}, \tilde{y}_n)$$

which yields the reverse inequality. Hence

$$\delta(\tilde{x}, \tilde{y}) = \lim_{n \to \infty} \delta(\tilde{x}_n, \tilde{y}_n).$$

In a completely analogous manner, we can also show that

$$\delta(\tilde{x}, \tilde{y}) = \lim_{n \to \infty} \delta(\tilde{x}_n, \tilde{y}_n).$$

Consequently,

$$\hat{\delta}(\tilde{x}, \tilde{y}) = \delta(\tilde{x}, \tilde{y}),$$

that is, the distance on $\tilde{F}(\mathbb{R})$ and $\hat{F}(\mathbb{R})$ must be the same. Hence $\tilde{F}(\mathbb{R})$ and $\hat{F}(\mathbb{R})$ are isometric. □

REFERENCES


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