The Pure and Applied Mathematics **2** (1995), No 1, pp. 31-34
J. Korea Soc. of Math. Edu. (Series B)

# SOME RELATIONS BETWEEN FUNCTION SPACES ON $\mathbb{R}^n$

SEUNG HYUN SHIN

## 1. Introduction

Let $\mathbb{R}^n$ be n-th Euclidean space. Let be the n-th spere embeded as a subspace in $\mathbb{R}^{n+1}$ centered at the origin.

In this paper, we are going to consider the function space

$$F = \{f | f : S^n \to S^n\}$$

metrized by as follow

$$D(f, g) = d(f(x),\ g(x))$$

where $f,\ g \in F$ and $d$ is the metric in $S^n$. Finally we want to find certain relation these spaces.

In Section 2, we shown on the subsqpces on $S^n$. Of cause the members of $F$ need not be continuous ; we use the word mapping when continuity is implies. And then we shown several fundamental theorems in order to understand main results.

In Section 3, for the main our assertion i.e., Theorem 4 ; Suppose $f \in F_p$. Then $f \in M_p$ iff $f$ is compact. Theorem 5 ; The space $F_p - Q_p$ is both open and dense in $F_p$.

## 2. Fundamental Theorems

In this Section we will show several fundamental theorems to make easy main theorems.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

At first we consider the subspaces

$$M = \{f|f \in F,\ f \text{ continuous on } S^n\}$$
$$G = \{f|f \in M, \text{ and onto }\}$$
$$H = \{f|f \in M, \text{ and one to one}\}$$

as well as $M_p$, $G_p$, and $H_p$. The space $M$ is a closed, hence complete, subspace of $F$. The topology on $M$ induced by the metric $D$ is identical with the compact-open topology. It is an immediate consequence that the subspaces $M$ inclduding point $m$ and $G$ are closed, hence complete, for in each case the complement is member of the sub-base of the co-topology.

The first theorem below is an immediate consequence of the fact that $S^n$ is not homeomorphic with a proper subset of itself. However here we derive this result from theorem 1, which is proved from the Borsuk-Ulam theorem : if $f$ maps $S^n$ into $R^n$, then $f$ maps a pair of antipodal point into the same point.

**Theorem 2.1.** $H \subset G$.

*Proof.* If $f \in (M - G)$, then there is a point $m$ such that $m \in (S^n - f(S^n))$. Let $\pi$ denote a homeomorphism on $S^n - \{m\}$ onto $R^n$. Then $\pi f : S^n \to R^n$ is continuous and by the Borsuk theorem there is a point.

**Corollary 2.2.** If $h : S^n \to S^n$ is a homeomorphism, then $h$ is onto.

Corresponding the any $f \in M$ and any $\epsilon > 0$, it is easy to find a function $g \in F - M$ such that $D(f, g) < \epsilon$; thus $F - M$ is dense in $F$. Furthermore the example of a ratation through an arbitrarily small angle shows that the set $M - M_p$ is dense in $M$. On the other hand, it is a conseqence of Borsuk-Vlam theorem that the set $M - G$ is not dense in $M$. For if $f$ is any map in $M - G$ there is an $x$ such that $f(x) = f(-x)$; thus $D(f$, identity$)> 1$, so the identity map is an interior point of $G$.

With attention restricted now to subspace of $M_p$ , it follows as above that $G_p$ is closed in $M_p$ and that $G_p$ has nonempty interior.

By Theorem 1, we may consider $H_p$ a subspace of $G_p$, and again it is easy to constrct examples which show, in this case, that $G_p - H_p$ is dense in $G_p$ but that $H_p$ is not closed in $G_p$.

Necertheless, using an argument involving $\epsilon$ -mapping [2] p57, it is follows that $H_p$ is a $G_\delta$-subset of $G_p$ hence is topologically complete.

**Theorem 2.3.** Let $f : R^n \to R^n$ be a Homeomorphism. Then $f$ is onto iff the image of each unbounded sequence is unbounded.

*Proof.* Suppose first that $f(R^n) = R^n$, that $\{x_n\}$ is an unbounded sequence, and that the image sequense $\{y_i\}$ is bounded. We may assume that $\{x_n\}$ has no limit point. There exists a subsequence $\{y_i'\}$ of $\{y_i\}$ converging to a point $y$ and a point $x$ such that $y = f(x)$. But $f^{-1}$ is continuous at $y$, hence $\{x_n\}$ must have a limit point. it is contradiction to the hypothesis.

Next suppose that the image of each unbounded sequence is unbounded. By Brower's theorem on invariance of domain [2] p95, $f(R^n)$ is open in $R^n$. If $y$ is any limit point in $R^n$ of $f(R^n)$, then there is a bounded sequence $\{y_i\}$ of distinct points in $f(R^n)$ converging to $y$.

If $x_i = f^{-1}(y_i)$, then $\{x_n\}$ is bounded and hence contains a convergent subsequence with limit $x$. But $f$ is continuous at $x$, hence $y \in f(R^n)$. Since $f(R^n)$ is both open and closed in $R^n$, $f(R^n) = R^n$.

**Theorem 2.4.** The set $\Phi$ of all homeomorphism on $R^n$ onto itself can be topologized as a complete metric space homeomorphic to $H_p$.

*Proof.* By Theorem 2.3, there is a natural 1-1 correspondence, via the stereographic projection $\pi : S^n \to \{p\} \to R^n$ with center $p$, between $\Phi$ and $M_p$. For suppose $f : S^n \to S^n$ is a function which is onto, one to one, and $p$ fixed. Then if $f$ is continuous at all $x \neq p$, $f$ is continuous also at $p$ and hence $f \in H_p$. The correspondence is then $f = \pi^{-1}\phi\pi$, where $\phi \in \Phi$ and $f \in H_p$.

## 3. Main Theorem.

In this section we consider the subspaces

$$F_p = \{f \in F | f(p) = p\},$$
$$T_p = \{f \in F_p | f \text{ continuous at each } x = p\},$$
$$Q_p = \{f \in T_p | f(S^n) \text{ is dense in } S^n\}.$$

Finally we show certain relations between these spaces and those considered in Section 2.

If $f \in F - F_p$ and if $d(p, f(p)) = \epsilon$, then $D(f,g) < \epsilon$ implies $g \in F - F_p$ hence $F_p$ is closed in $F$. Again, the example of a rotation through a sufficiently small angle shows that $F - F_p$ is dense in $F_p$.

The space $M_p$ is a subset of $T_p$. By extending slightly the useful notation of compact mapping [6], this subspace has a simple characterization. Define $f \in F$ to be compact iff for each set $K$ closed in $S^n$ the set $f^{-1}(K)$ is closed in $S^n$.

**Theorem 3.1.** Suppose $f \in T_p$. Then $f \in M_p$ iff $f$ is compact.

*Proof.* It $f \in M_p$ then $f$ is continuous on $S^n$. On the other hand, if $f \in T_p - M_p$ there is a sequence $\{x_i\}$ such that $x_i \to p$, $f(x_i) \neq p$, and $p$ is not a limit point of $\{f(x_i)\}$. If $K$ is the close of the set $\{f(x_i)\}$ is not since it fails to contain $p$.

Since $M_p$ is a closed subset of $F$, it follows that $M_p$ is closed in $T_p$. The example of a small rotation except for a discontinuity at $p$(which is fixed) shows that $T_p - H_p$ is dence in $T_p$.

**Theorem 3.2.** The space $T_p - Q_p$ is both open and dense in $T_p$.

*Proof.* If $f \in F_p - Q_p$ then there is a spherical neighborhood of radius $\epsilon > 0$ contained in $S^n - f(S^n)$. For each $g \in T_p$, if $D(g,f) < \epsilon/3$ then $S^n - g(S^n)$ contains a spherical neighborhood, hence $g \in T_p - Q_p$. Now let $s(\epsilon)$ denote spherical neighborhood about $p$ of radius $\epsilon$ and let $g$ be defined by

$$g(x) = \begin{cases} p \text{ if } x = p \\ x \text{ if } x \in S^n - s(\epsilon) \ uniforml\ shinks\ s(\epsilon) - \{p\}\ onto\ the \\ annular\ region\ s(\epsilon) - s(\epsilon/2) \end{cases}$$

Then if $f \in Q_p$, $\rho(f, g^\circ f) < \epsilon$ but $g^\circ f \in T_p - Q_p$.

## References

1. I.M.Gelfand, *Generalized Functions Aca.Press.*
2. T.Husain, *Topology and Maps*, Plenum Press (1977).
3. J.L.Kelly, *General Topology*, Von Nostrand (1955).
4. H.Sharp, *Strongly topological imbedding of $F_\sigma$-subset of $E^n$*, Amer,J.Math. (1953).
5. G.T.Whyburn, *Analytic Topology*, Amer.Math. Soc.Coll. (1982).
6. G.T.Whyburn, *Topological Analysis*, Princston Univ. (1958).

Department of Mathematics
Graduate School of Dankook University Seoul, 140-714, Korea