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Preservation of Some Partial Ordering under Formation of k-out-of-n Systems of Like components

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Abstract In this paper, we shall convey preservation of some partial orderings and closures of some positive ageing classes under k-out-of-n systems of like components. That is, if the life time of a component A is larger than that of a component B in any of the NBU, DMRL, NBUE, HNBUE, NBUFR and NBUFRA orderings, then a k-out-of-n system formed by i.i.d. components of type A has larger life time, in that ordering, than that of a similar system consisting of n i.i.d. components type of B. And using these partial orderings, closures of positive aging classes(NBU, DMRL,NBUE, HNBUE, NBUFR and NBUFRA) under the coherent system like components.

Keywords : k-out-of n systems, Partial Orderings, Ageing Classes.

1. Introduction

By the aging of a mechanical unit, component, or some other physical or biological system. we mean the phenomenon by which an older system has a short remaining lifetime, n some stochastic sense, than a newer or younger one. Many criteria of aging have been developed in the literature. See, for example, Bryson and Siddiqui(1969), Barlow and Proschan(1975), Deshpande, Kochar and Singh(1986) and Kochar and Wiens(1987). Suppose that X and Y are nonnegative absolutely continuous random variables with probability density functions $f(x)$ and $g(x)$, respectively. Let F and G be the cumulative distribution functions of X and Y, and $F = 1 - F(x)$ and $G = 1 - G(x)$ are known in the literatures.

Deshpande, Kochar and Singh(1986) introduced various aspects of this concept which are described in terms of conditional probability distributions of residual life times, failure rates, equilibrium distributions. Gupta(1987) studied how the ageing properties IFR, NBU, NBUE and DMRL of the original distribution were

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transformed into the ageing properties of the distribution of the residual life. Kochar and Wiens(1987) defined new partial orderings of life distributions and studied the relationships of some partial orderings. These extend the concepts of DMRL, NBUE, HNBUE to compare ageing properties of two distributions. Kochar(1989) studied some further consequences of these orderings in terms of their mean residual life function, total time on test transforms and Lorenz curves. Singh(1989) defined two new partial orderings and discussed relevance of these partial orderings for comparing life of a new unit with residual life of a used unit. Deshpande(1990) investigated partial orders relations with existing partial orders of probability distributions for describing the phenomenon of ageing. Singh and Vijayasree(1991) studied preservation of partial orderings(LR, FR, ST, MR HAMR etc.) under the formation of k-out-of-n systems of i.i.d. components.

So, we shall convey preservation of some partial orderings and closures of some positive ageing classes under k-out-of-n systems of like components. That is, if the life time of a component A is larger than that of a component B in any of the NBU, DMRL, NBUE, HNBUE, NBUFR and NBUFRA orderings, then a k-out-of-n system formed by i.i.d. components of type A has larger life time, in that ordering, than that of a similar system consisting of n i.i.d. components type of B. And using these partial orderings, closures of positive aging classes(NBU, DMRL,NBUE, HNBUE, NBUFR and NBUFRA) under the coherent system like components.

2. Preservation of partial orderings and ageing classes under k-out-of-n systems like components

We study the effects on a partial orderings of forming a k-out-of-n systems of the components or, equivalently, taking order statistics. A useful identity is given in the following lemmas.

Lemma 1. Let F_{jn} be the distribution of the j-th order statistics in a sample of n from F. Then $F_{jn} = B_{jn}[F(x)]$ for $0 < x < \infty$, where

$$B_{jn}(p) = \frac{n!}{(j-1)!(n-j)!}\int_0^p \mu^{j-1}(1-\mu)^{n-j}\,d\mu, \quad \text{for } 0 \le p \le 1.$$

Definition 1. We say that F is more IFR than G and write $F \overset{IFR}{\prec} G$ (or $F \overset{C}{\prec} G$) if $G^{-1} \circ F(x)$ is convex.

Definition 2. We say that F is more NBU than G and write $F \overset{NBU}{\prec} G$ (or

$F \overset{SU}{\prec} G$) if $G^{-1} \circ F(x)$ is super-additive.

Proposition 1. If $F \overset{NBU}{\prec} G$, then $F_{jn} \overset{NBU}{\prec} G_{jn}$.

Proof. $G_{jn}{}^{-1} \circ F_{jn}(x) = (B_{jn}G)^{-1} B_{jn}G(x) = G^{-1} \circ F(x)$, so that superadditivity is preserved.

Proposition 2. If F be NBU, then F_{jn} is NBU.

Proof. Let F be NBU and $G(t) = 1 - e^{-t}$. Using Proposition 1 and $G_{jn} \overset{C}{\prec} G$, we show that $F_{jn} \overset{NBU}{\prec} G$. Hence F_{jn} is NBU.

We have a unit in operation whose life distribution is F. As soon as this unit fails, another new unit, which acts independently of the first and has the same life distribution, is activated. This renewal of the system is continued indefinitely. Then the residual life of the unit under operation at time t as $t \to \infty$ is given by $F_e(\cdot)$, the equilibrium distribution. Hence, obviously, stochastic comparisons between the equilibrium distributions corresponding to life distributions, F and G, can be considered. The equilibrium distribution has been used to describe the concepts of positive ageing in Deshpande, Kochar and Singh(1986).

Let equilibrium survival function $\overline{F}_e(x) = \int_x^\infty \overline{F}(t)dt, \quad 0 \le x < \infty.$

Define $W_F(\mu) = \overline{F} \circ \overline{F}_e^{-1}(\mu), \ W_G(\mu) = \overline{G} \circ \overline{G}_e^{-1}(\mu), \ 0 \le \mu \le 1.$ Note that W_F are W_G proper distributions on [0,1]. They are related to the scaled total time on test transform $H_F^{-1}(\mu) = F_e \circ F^{-1}(\mu)$ studied by Klefsjo(1986) through the relationship $\overline{W}_F^{-1} = H_F(1-\mu)$. Define also $\alpha(x) = G^{-1} \circ F(x) = \overline{G}^{-1} \circ \overline{F}(x), \beta(x) = G_e^{-1} \circ F_e(x)$ $= \overline{G}_e^{-1} \circ \overline{F}_e(x)$

Definition 3. We say that F is more decreasing in mean residual life than G and write $F \overset{DMRL}{\prec} G$ if $W_F^{-1} \circ W_G(\mu)$ is star-shaped $\mu \in [0,1]$.

Theorem 1. If $F \overset{DMRL}{\prec} G$, then $F_{jn} \overset{DMRL}{\prec} G_{jn}$.

Proof. $W_{F_{jn}}^{-1} \circ W_{G_{jn}}(\mu) = (B_{jn}W_F)^{-1}(B_{jn}W_G(\mu)) = W_F^{-1} \circ W_G(\mu)$, so that starshapedness is preserved.

Theorem 2. If F be DMRL, then F_{jn} is DMRL.

Proof. Let F be DMRL and $G(t)=1-e^{-t}$. Using Theorem 1 and $G_{jn} \overset{C}{\prec} G$, and Theorem 2.1 of Kochar and Wiens(1987), we show that $F_{jn} \overset{DMRL}{\prec} G$. Hence F_{jn} is DMRL.

Definition 4. We say that F is more new better than used in expectation than G and write $F \overset{NBUE}{\prec} G$ if $\alpha(x) \geq \beta(x),\ x \geq 0.$

Lemma 2. Let $\alpha(x) = G^{-1} \circ F(x) = \overline{G}^{-1} \circ \overline{F}(x),\quad \beta(x) = G_e^{-1} \circ F_e(x) = \overline{G_e}^{-1} \circ \overline{F_e}(x)$. Then $\alpha_{jn}(x) = \alpha(x)$ and $\beta_{jn}(x) = \beta(x)$, where $\alpha_{jn}(x) = G_{jn}^{-1} \circ F_{jn}(x)$ and $\beta_{jn}(x) = G_{e_{jn}}^{-1} \circ F_{e_{jn}}(x)$.

Proof. Since $F_{jn}(x) = B_{jn}(F(x))$, and $G_{jn}(x) = B_{jn}(G(x))$. Then $\alpha_{jn}(x) = G_{jn}^{-1} \circ F_{jn}(x) = (B_{jn}G(x))^{-1} B_{jn}F(x) = G^{-1} \circ F(x)$.
Similarly $\beta_{jn}(x) = \beta(x)$.

Theorem 3. If $F \overset{NBUE}{\prec} G$, then $F_{jn} \overset{NBUE}{\prec} G_{jn}$.

Proof. Using Lemma 2, $\alpha(x) \geq \beta(x)$, then $\alpha_{jn}(x) \geq \beta_{jn}(x)$.

Theorem 4. If F be NBUE, then F_{jn} is NBUE.

Proof. Let F be NBUE and $G(t)=1-e^{-t}$. Using Theorem 3 and $G_{jn} \overset{C}{\prec} G$, and Theorem 3.1 of Kochar and Wiens(1987), we show that $F_{jn} \overset{NBUE}{\prec} G$. Hence F_{jn} is NBUE.

Definition 5. We say that F is more harmonic new better than used in expectation than G and write $F \overset{HNBUE}{\prec} G$ if $\dfrac{G_e^{-1} \circ F_e(x)}{x} \geq \dfrac{\mu_F}{\mu_G}, \quad x \geq 0.$

Theorem 5. If $F \overset{HNBUE}{\prec} G$, then $F_{jn} \overset{HNBUE}{\prec} G_{jn}$.

Proof. Since $\beta_{jn}(x) = \beta\ (x)$, $\dfrac{G_{e_{jn}}^{-1} \circ F_{e_{jn}}(x)}{x} = \dfrac{G_e^{-1} \circ F_e(x)}{x}$. So, this inequality satisfied.

Theorem 6. If F be HNBUE, then F_{jn} is HNBUE.

Proof. Let F be HNBUE and $G(t)=1-e^{-t}$. Using Theorem 3 and $G_{jn} \overset{C}{\prec} G$, and

Theorem 3.2 of Kochar and Wiens(1987), we show that $F_{jn} \overset{HNBUE}{\prec} G$. Hence F_{jn} is HNBUE.

Assume that F and G are absolutely continuous, so that $\alpha(x) = G^{-1} \circ F(x)$ is differentiable.

Definition 6. We say that F is more new better than used in failure rate (in failure rate average) than G and write $F \overset{NBUFR}{\prec} G$ ($F \overset{NBUFRA}{\prec} G$) if $\alpha'(x) \geq \alpha'(0)$ [$\alpha(x) \geq x\alpha'(0)$], $x \geq 0$.

Theorem 7. If $F \overset{NBUFR}{\prec} G$ ($F \overset{NBUFRA}{\prec} G$), then $F_{jn} \overset{NBUFR}{\prec} G_{jn}$ ($F_{jn} \overset{NBUFRA}{\prec} G_{jn}$).
Proof. Using the fact that $\alpha_{jn}(x) = \alpha(x)$.

Theorem 8. If F be NBUFR(NBUFRA), then F_{jn} is NBUFR(NBUFRA).
Proof. Trivial, so this theorem's proof is omitted.

Using the fact that $\beta(x)$ of original distribution is equal to $\alpha(x)$ of their equilibrium distributions, we obtain next two results.

Proposition 3. $F \overset{NBUE}{\prec} G$ if and only if $F_e \overset{NBUFR}{\prec} G_e$

Proposition 4. $F \overset{HNBUE}{\prec} G$ if and only if $F_e \overset{NBUFRA}{\prec} G_e$

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