Journal of Statistical
Theory & Methods
1995, Vol. 6, pp 65~76

The Estimation of Mean Residual Life Function under Left Truncation and Right Censoring Model

Gyoung Ae Moon[1], Im Hee Shin and Hyeon Suk Chae[2]

Abstract The importance of left truncated and right censoring cases has considered for better information in medical follow-up and engineering life testing studies. We propose some estimation procedure for the mean residual life function with consistency and asymptotic normality on the left truncated and right censoring model. And then, the comparision with Kaplan-Meier estimator ignoring the left truncated effect and the small sample properities are investigated by asymptotic biases and M.S.E.'s throgh Monte Carlo study.

1. Introduction

In medical follow-up, or in engineering life testing studies, many authors have been interested in the lifetime and considered right censoring cases for almost statistical analysis of survival data although additional restrictions on the observation of failure times may occur in many situations. Individuals who come observation only some known time after the time origin may be further subject to the usual right censoring during the follow-up period. Thus left truncation and right censoring(LTRC) case may arise and be common one among the different form in which incomplete data appear.

Some authors have been interested in the estimation problem using LTRC model. Wang, Jewell and Tsai(1987) considered the product limit estimator of survival function under LTRC model and studied the asymptotic behavior of this estimator, and also constructed of large sample simultaneous confidence bands for survival function. They showed the Kaplan-Meier estimator ignoring left truncation effects overestimates the survival function than the product limit

1 Instructor of Donghae Junior College
2 Department of Statistics, Kyungpook National University

estimator. Lai and Ying(1991) proposed a minor modification of the product limit estimator for LTRC model and showed the uniform strong consistency and the weak convergency. The estimation problem of the mean residual life function has been investigated by many researchers under parametric and nonparametric model since Cox(1961) and Swartz(1973) represented the mean residual life function at age t , $e(t)$ in terms of $E(X-t|X>t)$.

For a long time, many researchers also have interested in the nonparametric estimation of the mean residual life function for a long time. In the case of no censoring, Yang(1978) proposed the empirical estimator of mean residual life function and showed the strong consistency and asymptotic normality of it. Ghorai and Rejto(1987) presented the estimator of the mean residual life function based on the maximum likelihood estimator for reliability function and obtained the strong consistency and weak convergence of the estimator under the proportional hazard model.

In random censoring, Yang(1977) proposed the truncated version of the estimator for the mean residual life function based on the reliability estimator of Kaplan-Meier and Nelson-Aalen types under the competing risk model and also showed the asymptotic properties on the bounded interval.

In this paper, we propose the estimator for the mean residual life function using the product limit estimator on the LTRC model and show this estimator estimates the mean residual life function better than the estimator using Kaplan-Meier estimator in terms of the M.S.E.'s for the estimators.

2. Preliminaries and Proposed Estimator.

In order to construct the estiamtor of the survival function, we use the following notations.

Let U_i, $i=1,2,\cdots$ be the lifetime following the distribution F. Let T_i, C_i, $i=1,2,\cdots$ be the random truncation time and the censoring time with distribution function G and H, respectively to be independent of U_i, $i=1,2,\cdots$. Suppose that (X_i, δ_i, T_i) is observable only when $X_i \geq T_i$, where $X_i = \min(U_i, C_i)$ and

$$\delta_i = \begin{cases} 1 & \text{if} \quad U_i \leq C_i \\ 0 & \text{if} \quad U_i \geq C_i \end{cases}$$

Thus, the observed data is given by a set of n independent identically distributed observations (X_i, δ_i, T_i), $i=1,2,\cdots,n$.

Tsai, Jewell and Wang(1987) suggested the product-limit estimator for the survival function $S(t)$ given by

$$\hat{S}(t) = \sum_{x_j \le t} \frac{n_j - d_j}{n_j}$$

with d_j is the number of failures at time X_j and $n_j = \sum I(T_i \le X_j \le Y_i)$ is the number in the risk set at time X_j, where I is the usual indicator function. Note that $\hat{S}(t)$ reduces to the Kaplan and Meier estimator(1958) for right censored data if $T_1 = T_2 = \cdots = T_n = 0$. In addition to, if there is no censoring, $\hat{S}(t)$ reduces to the product-limit estimator based on truncated data studied by Woodroofe(1985) and Wang, Tewell and Tsai(1986).

However, it is not possible over the support of lifetime distribution F without further restriction on the underlying distribution functions. For any distribution R on $[0,\infty)$, let $a_R = \inf\{t > 0: R(t) > 0\} \ge 0$ and $b_R = \sup\{t > 0: R(t) < 1\} \le \infty$. If $a_F < a_G$, then we can not expect to estimate $S(t)$ without parametric assumption of $S(t)$. Similarly, if $b_H < b_F$, then we can not estimate $S(t)$ for $t > b_H$ without parametric assumption. But, If $a_F < a_G$, then we seek to estimate $S^*(t)$ for any fixed $a_G \le T^* \le t$ given by $S^*(t) = S(t)/S(T^*) = \Pr(U \ge t | U \ge T^*)$. Note that if $a_G < a_F$, then we can choose T^* so that $S^*(t) = S(t)$. In either case, if $b_H < b_F$, we seek to estimate $S(t)$ or $S^*(t)$ only for $t \le K \le b_H$. For reasons of identifiability, we usually consider the estimation of $S^*(t)$ for $t \in \lfloor T^*, K \rfloor$, where $a_G \le T^*$ and $K \le b_H$.

First, we consider the estimation $S^*(t)$ for $t \in \lfloor T^*, K \rfloor$. Let $X_{(1)}, X_{(2)}, \cdots X_{(n)}$ be the order statistics of $X_1, X_2, \cdots, X_n$ and $s_{(1)}, s_{(2)}, \cdots, s_{(r)}$ among $X_1, X_2, \cdots, X_n$ be the distinct observed lifetimes which are greater than or equal to T^*. For $t \ge T^*$, let

$$N(t) = \sum_{i=1}^{n} I(X_i \le t, \delta_i = 1)$$

$$Y(t) = \sum_{i=1}^{n} I(T_i \le t \le X_i).$$

It can be easily seen that the estimator of $S^*(t)$ is the step function with possible jumps at $s_{(1)}, s_{(2)}, \cdots, s_{(r)}$. Then Tsai, Jewell and Wang(1987) suggested the nonparametric conditional maximum likelihood estimator for the survival function $S^*(t)$ given by

$$\hat{S}_{PL}(t) = \prod_{i: T^* \le s_{(i)} \le t} \left(1 - \frac{\Delta N(s_{(i)})}{Y(s_{(i)})}\right).$$

Tsai, Jewell and Wang(1987) showed the consistency and the weak convergency of the product-limit estimator $\hat{S}_{PL}(t)$.

Lemma 2.1.(Tsai, Jewell and Wang(1987)) Let F, G and H be continuous with $a_G \le T^* \le K \le b_F$. Then for $T^* \le t \le K$,

$$|\hat{S}_{PL}(t) - S^*(t)| \rightarrow 0$$

with probability 1 as $n \rightarrow \infty$.

Lemma 2.2.(Tsai, Jewell and Wang(1987)) Let F, G and H be continuous with $a_G \le T^* \le K \le b_F$. Then for $T^* \le t \le K$,

$$\sqrt{n}(\hat{S}_{PL}(t) - S^*(t)) \rightarrow W(t)$$

weakly in $D\lfloor T^*, K \rfloor$, where $W(t)$ is a mean zero Gaussian process with covariance structure given by

$$Cov(W(s), W(t)) = S^*(s)S^*(t)\int_{T^*}^{\min(s,t)} \frac{dF^*(u)du}{(1-H^*(u))^2},$$

where $-H^*(t) = \Pr(T \le t \le X | X \ge T)$ and $F^*(t) = \Pr(X \le t, \delta = 1 | X \ge T)$.

In this section, we consider the estimation of the mean residual life function based on the product-limit estimator.

The mean residual life function $e^*_K(t)$ at age $t \ge T^*$ *is defined to be the expected* value remaining lifetime given reliability to age t as follows:

$$e^*_K(t) = E(X - t | X > t) = (S^*(t))^{-1} \int_t^K S^*(u)du.$$

Then the estimator $\hat{e}_K^{PL}(t)$ is proposed using the product-limit estimator $\hat{S}_{PL}(t)$ as follows ;

$$\hat{e}_K^{PL}(t) = (\hat{S}_{PL}(t))^{-1} \int_t^K \hat{S}_{PL}(u)du.$$

3. Asymptotic Properties of the Proposed Estimators.

We shall show that $\hat{e}_K^{PL}(t)$ converges weakly to a Gaussian process as $n \rightarrow \infty$. The proof utilizes Lemma 2.1, 2.2 and 3.1 to be established below.

The next lemma is essential for the application of the continuity theorem in Billingsley(1968, Theorem 5.1).

Lemma 3.1 (Yang(1977)). Let $D[0, K]$ be the space of functions on the interval $[0, K]$ that are right continuous and have left limits. Let d be the

Skorohod metric on $D[0,K]$. Define a map $H: D[0,K] \to D[0,K]$ by having

$$H(W)(x) = S^*(x)\int_x^K W(u)du - W(x)\int_K^x S^*(u)du,$$

for $W \in D[0,K]$. Then H is a continuous map and with respect to d.

Theorem 3.1. Let F, G and H be continuous with $a_G \leq T^* \leq K \leq b_F$. Then for $T^* \leq t \leq K$,

$$|\hat{e}_K{}^{PL}(t) - e^*{}_K(t)| \to 0$$

with probability 1 as $n \to \infty$.

Proof. For fixed $t \in [T^*, K]$ and $s \leq t$,

$$|\hat{e}_K{}^{PL}(t) - e^*(t)| = \left| \frac{\int^K \hat{S}_{PL}(u)du}{\hat{S}_{PL}(t)} - \frac{\int^K S^*(u)du}{S^*(t)} \right|$$

$$= (\hat{S}_{PL}(t)S^*(t))^{-1} |S^*(t)\int^K \hat{S}_{PL}(u)du - \hat{S}_{PL}(t)\int^K S^*(u)du|$$

$$= (\hat{S}_{PL}(t)S^*(t))^{-1} |S^*(t)\int^K (\hat{S}_{PL}(u)du - S^*(u)du)$$

$$= (\hat{S}_{PL}(t)S^*(t))^{-1} |S^*(t)\int^K (\hat{S}_{PL}(u)du - S^*(u)du)$$

$$+(S^*(t) - \hat{S}_{PL}(t))\int_t^K S^*(u)du|$$

$$\leq (\hat{S}_{PL}(t)S^*(t))^{-1}(S^*(t)\ {}_t^K |\hat{S}_{PL}(u) - S^*(u)|du$$

$$+|S^*(t) - \hat{S}_{PL}(t)|\ {}_t^K S^*(u)du)$$

$$\leq (\hat{S}_{PL}(t)S^*(t))^{-1}(S^*(t)K \| \hat{S}_{PL}(t) - S^*(t)\|_K$$

$$+|S^*(t) - \hat{S}_{PL}(t)|\ {}_t^K S^*(u)du),$$

where $\|f\|_K = \sup\{|f(u)| \,|\, t \leq u \leq K\}$. So the result follows by lemma 2.1.

Now, we have the weak convergency of $\hat{e}_K{}^{PL}(t)$ of the mean residual life function $e^*{}_K(t)$. The following theorem implies the asymptotic normality of the estimator $\hat{e}_K{}^{PL}(t)$ for each $t \in [T^*, K]$.

Theorem 3.2. Let F, G and H be continuous with $a_G \leq T^* \leq K \leq b_F$. Then for $T^* \leq t \leq K$,

$$\sqrt{n}(\hat{e}_K{}^{PL}(t) - e^*(t)) \to Z(t)$$

weakly in $D[T^*, K]$, where $Z(t)$ is a mean zero Gaussian process given by

$$Z(t) = (S^*(t))^{-2}\left(S^*(t)\int_t^K W(u)du - W(t)\int_t^K S^*(u)du\right)$$

and the covariance structure of Z is given by

$$Cov(Z(s),Z(t)) = (S^*(s)S^*(t))^{-2}\Big(S^*(s)S^*(t)E(\int_s^K\int_t^K W(u)W(v)dudv)$$
$$+E(W(s)W(t))\int_s^K S^*(v)dv\int_t^K S^*(u)du$$
$$-S^*(s)\int_t^K S^*(u)duE(W(t)\int_s^K W(v)dv)$$
$$-S^*(t)\int_s^K S^*(v)dvE(W(s)\int_t^K W(u)du)\Big)$$

for $T^* \le s \le t \le K$.

Proof. $n(\hat{e}_K^{PL}(t) - e^*(t)) = n(\hat{S}_{PL}(t)S^*(t))^{-1}\Big(S^*(t)\int_t^K(\hat{S}_{PL}(u)du - S^*(u))du$
$$-(\hat{S}_{PL}(t) - S^*(t))\int_t^K S^*(u)du\Big).$$

By lemma 2.1, the asymptotic distribution of $n(\hat{e}_K^{PL}(t) - e^*(t))$ is the same as that of

$$n(S^*(t))^{-2}\left(S^*(t)\int_t^K(\hat{S}_{PL}(u)du - S^*(u))du - (\hat{S}_{PL}(t) - S^*(t))\int_t^K S^*(u)du\right).$$

Thus, the result follows from lemma 2.2 and continuity theorem in Billingsley(1968) that can be applied by lemma 3.1. And the covariance function is evaluated by the interchange of expectation with integral signs.

4. Simulation Study

In this section, we investigate the small sample performances of the proposed estimator $\hat{e}_K^{PL}(t)$ for the mean residual life function $e_K^*(t)$ based on the biases and mean squared errors(MSE's) via Monte Carlo simulation.

We performed a simulation to compare the small sample properties of $\hat{e}_K^{PL}(t)$ with those of the Kaplan-Meier type estimator $\hat{e}_K^{KM}(t)$ based on the Kaplan-Meier estimator $\hat{S}_{KM}(t)$ for the survival function $S^*(t)$ obtained by ignoring the truncation effects.

The lifetime model considered in the simulation is Weibull distribution with constant failure rate, decreasing failure rate and increasing failure rate according to the scale and shape parameters. The censoring distributions are supposed to be exponential and uniform with the censoring rate about 10% and 30%.

A Monte Carlo simulation is performed for the lifetime distribution types in Table 4.1 for sample size n equal to 20, 50. This procedure is repeated 500 times in order to get estimates of biases and MSE's of the estimators $\hat{e}_K^{PL}(t)$ and $\hat{e}_K^{KM}(t)$ for each value of $t \geq T^*$ such that $t = S^{-1}(p), \quad p = 0.9, 0.8, \cdots, 0.1$.

The results are listed in Table 1 through Table 8. From these tables, we can draw the following facts.

(1) When the lifetime distribution is $Weib(1.0,1.0)$ and the censoring distribution is exponential , the product-limit type estimator $\hat{e}_K^{PL}(t)$ tends to have smaller bias and MSE than the Kaplan-Meier type estimator $\hat{e}_K^{KM}(t)$ except in the region of upper tail of the distribution for the case of the censoring rate 10% with $n = 20$. But, $\hat{e}_K^{PL}(t)$ is more efficient than $\hat{e}_K^{KM}(t)$ for the case of the censoring rate 30% regardless of sample size. When the censoring distribution is given as uniform, $\hat{e}_K^{PL}(t)$ always seems to be more efficient than $\hat{e}_K^{KM}(t)$ with respect to bias and MSE.

(2) When the lifetime distribution is $Weib(1.0,1.5)$ and the censoring distribution is exponential or uniform with the sample size 20 and the censoring rate 10%, the product-limit type estimator $\hat{e}_K^{PL}(t)$ has smaller bias than the Kaplan-Meier type estimator $\hat{e}_K^{KM}(t)$. But with respect to MSE, $\hat{e}_K^{KM}(t)$ seems to be a little more efficient than $\hat{e}_K^{PL}(t)$ in the upper tail region of the distribution. While when the sample size is 50, $\hat{e}_K^{PL}(t)$ has tendency to be more efficient than $\hat{e}_K^{KM}(t)$ regardless of the censoring rate.

From above facts, we can conclude that with data having the truncation effect, theproduct-limit type estimator seems to be more reasonable than Kaplan-Meier type estimator for the mean residual life function in the aspect of biases and MSE's.

Table 1. The Estimated Biases and MSE's (n=20, T^* =.4705)

Lifetime : Wei(1.0,1.0) Truncation : Exp(5.0)

Cesoring : Exp(.111) Censoring : Exp(.429)

$e^*(t)$		K-M	P-L	K-M	P-L
	EST	.2025	.7439	.0220	.2484
1.000	BIAS	-.7975	-.2561	-.9780	-.7516
	MSE	.8060	.8649	.9696	.8678
	EST	.4362	1.1094	.2407	.8060
1.000	BIAS	-.5638	.1094	-.7593	-.1940
	MSE	.4724	.5727	.6920	.6541
	EST	.4997	1.1211	.3849	.9937
1.000	BIAS	-.5003	.1211	-.6151	-.0063
	MSE	.3472	.3261	.4649	.4260
	EST	.5069	1.1021	.4404	1.0248
1.000	BIAS	-.4931	.1021	-.5596	.0248
	MSE	.2967	.2261	.3663	.2867
	EST	.5590	1.0782	.4929	1.0166
1.000	BIAS	-.4410	.0782	-.5071	.0166
	MSE	.2425	.1398	.3019	.2031
	EST	.6144	1.0640	.5606	1.0815
1.000	BIAS	-.3856	.0640	-.4394	.0185
	MSE	.1895	.1098	.2373	.1620

Table 2. The Estimated Biases and MSE's (n=50, T^* =.2735)

Lifetime : Wei(1.0,1.0) Truncation : Exp(5.0)

Cesoring : Exp(.111) Censoring : Exp(.429)

$e^*(t)$		K-M	P-L	K-M	P-L
	EST	.4320	1.1132	.2285	.6932
1.000	BIAS	-.5680	.1132	-.7715	-.3068
	MSE	.5467	.4788	.7252	.7553
	EST	.3116	1.0796	.2896	1.0027
1.000	BIAS	-.6884	.0796	-.7104	.0027
	MSE	.5014	.1731	.5445	.2565
	EST	.3334	1.0568	.2826	1.0092
1.000	BIAS	-.6666	.0568	-.7174	.0092
	MSE	.4658	.1219	.5288	.1605
	EST	.3803	1.0460	.3305	1.0053
1.000	BIAS	-.6197	.0460	-.6695	.0053
	MSE	.4001	.0968	.4632	.1157
	EST	.4385	1.0355	.3928	1.0015
1.000	BIAS	-.5615	.0355	-.6072	.0015
	MSE	.3272	.0605	.3824	.0782
	EST	.4954	1.0278	.4609	1.0002
1.000	BIAS	-.5046	.0278	-.5391	.0002
	MSE	.2652	.0448	.3035	.0587
	EST	.5583	1.0304	.5279	.9989
1.000	BIAS	-.4417	.0304	-.4721	-.0011
	MSE	.2049	.0388	.2356	.0492

Table 3. The Estimated Biases and MSE's (n=20, T^*=.7305)
Lifetime : Wei(1.0,1.5) Truncation : Exp(2.5)
Cesoring : Exp(.119) Censoring : Exp(.432)

$e^*(t)$		K-M	P-L	K-M	P-L
.4550	EST	.1235	.4030	.0366	.2299
	BIAS	-.3315	-.0520	-.4184	-.2251
	MSE	.1552	.1925	.1891	.1749
.4960	EST	.2518	.5813	.1912	.4870
	BIAS	-.2442	.0853	-.3048	-.0090
	MSE	.1056	.1167	.1387	.1362
.5300	EST	.2711	.6077	.2422	.5818
	BIAS	-.2589	.0777	-.2878	.0515
	MSE	.0924	.0686	.1128	.0819
.5620	EST	.2095	.6245	.2650	.6096
	BIAS	-.2715	.0625	-.2970	.0476
	MSE	.0906	.0535	.1058	.0581
.5930	EST	.3324	.6463	.2983	.6338
	BIAS	-.2606	.0533	-.2947	.0408
	MSE	.0833	.0378	.0998	.0423

Table 4. The Estimated Biases and MSE's (n=50, T^* =.4214)
Lifetime : Wei(1.0,1.5) Truncation : Exp(2.5)
Cesoring : Exp(.119) Censoring : Exp(.432)

$e^*(t)$		K-M	P-L	K-M	P-L
.4550	EST	.1942	.5144	.1551	.4768
	BIAS	-.2608	.0594	-.2999	.0218
	MSE	.1022	.0733	.1265	.1312
.4960	EST	.1694	.5339	.1758	.5397
	BIAS	-.3266	.0379	-.3202	.0437
	MSE	.1142	.0291	.1128	.0498
.5300	EST	.2002	.5596	.1762	.5556
	BIAS	-.3298	.0296	-.3538	.0256
	MSE	.1147	.0210	.1302	.0265
.5620	EST	.2462	.5873	.2144	.5824
	BIAS	-.3158	.0253	-.3476	.0204
	MSE	.1058	.0172	.1260	.0214
.5930	EST	.3009	.6158	.2652	.6092
	BIAS	-.2921	.0228	-.3278	.0162
	MSE	.0902	.0123	.1126	.0151
.6260	EST	.3601	.6464	.3239	.6400
	BIAS	-.2659	.0204	-.3021	.0140
	MSE	.0758	.0106	.0965	.0124
.6630	EST	.4280	.6836	.3919	.6785
	BIAS	-.2350	.0206	-.2711	.0155
	MSE	.0608	.0104	.0786	.0123

Table 5. The Estimated Biases and MSE's (n=20, T^*=.4705)

Lifetime : Wei(1.0,1.0) Truncation : Exp(5.0)
Cesoring : Uni(10.0) Censoring : Uni(3.197)

$e^*(t)$		K-M	P-L	K-M	P-L
1.000	EST	.2269	.7186	.0061	.0722
	BIAS	-.7731	-.2814	-.9939	-.9278
	MSE	.8229	.8200	.9897	.8961
1.000	EST	.4241	1.0684	.1768	.5432
	BIAS	-.5759	.0684	-.8232	-.4568
	MSE	.5013	.3952	.7436	.4056
1.000	EST	.4325	1.0803	.3256	.8111
	BIAS	-.5675	.0803	-.6744	-.1889
	MSE	.3951	.2420	.5138	.1772
1.000	EST	.4432	1.0710	.3576	.8871
	BIAS	-.5568	.0710	-.6424	-.1129
	MSE	.3575	.1883	.4525	.1306
1.000	EST	.4698	1.0604	.3848	.9116
	BIAS	-.5302	.0604	-.6152	-.0884
	MSE	.3216	.1440	.4022	.1013
1.000	EST	.5071	1.0455	.4296	.9242
	BIAS	-.4929	.0455	-.5704	-.0758
	MSE	.2726	.1079	.3473	.0851

Table 6. The Estimated Biases and MSE's (n=50, T^* =.2735)

Lifetime : Wei(1.0,1.0) Truncation : Exp(5.0)
Cesoring : Uni(10.0) Censoring : Uni(3.197)

$e^*(t)$		K-M	P-L	K-M	P-L
1.000	EST	.3952	1.0575	.0725	.2349
	BIAS	-.6048	.0575	-.9275	-.7651
	MSE	.5118	.3553	.8841	.6661
1.000	EST	.2964	1.0497	.2527	.7478
	BIAS	-.7036	.0497	-.7473	-.2522
	MSE	.5242	.1277	.5932	.1448
1.000	EST	.3103	1.0307	.2369	.8370
	BIAS	-.6897	.0307	-.7631	-.1630
	MSE	.4905	.0835	.5911	.0857
1.000	EST	.3563	1.0231	.2854	.8818
	BIAS	-.6437	.0231	-.7146	-.1182
	MSE	.4260	.0614	.5203	.0636
1.000	EST	.4180	1.0230	.3447	.9049
	BIAS	-.5820	.0230	-.6553	-.0951
	MSE	.3509	.0477	.4384	.0500
1.000	EST	.4741	1.0202	.4051	.9157
	BIAS	-.5259	.0202	-.5949	-.0843
	MSE	.2883	.0405	.3628	.0440
1.000	EST	.5341	1.0202	.4706	.9298
	BIAS	-.4659	.0202	-.5294	-.0702
	MSE	.2285	.0337	.2883	.0340

Table 7. The Estimated Biases and MSE's (n=20, T^*=.6430)
Lifetime : Wei(1.0,1.5) Truncation : Exp(2.5)
Cesoring : Uni(9.024) Censoring : Uni(3.002)

$e^*(t)$		K-M	P-L	K-M	P-L
.4550	EST	.1255	.3959	.0264	.1493
	BIAS	-.3295	-.0591	-.4286	-.3057
	MSE	.1644	.1698	.1921	.1578
.4960	EST	.2466	.5767	.1480	.4337
	BIAS	-.2494	.0807	-.3480	-.0623
	MSE	.1004	.1043	.1533	.1014
.5300	EST	.2688	.6020	.2306	.5412
	BIAS	-.2612	.0270	-.2994	-.0112
	MSE	.0927	.0610	.1145	.0622
.5620	EST	.2935	.6230	.2528	.5744
	BIAS	-.2685	.0610	-.3092	.0214
	MSE	.0899	.0451	.1125	.0429
.5930	EST	.3327	.6463	.2852	.6043
	BIAS	-.2603	.0533	-.3078	.0113
	MSE	.0835	.0360	.1066	.0350

Table 8. The Estimated Biases and MSE's (n=50, T^* =.4214)
Lifetime : Wei(1.0,1.5) Truncation : Exp(2.5)
Cesoring : Uni(9.024) Censoring : Uni(3.002)

$e^*(t)$		K-M	P-L	K-M	P-L
.4550	EST	.1969	.5067	.1107	.3220
	BIAS	-.2581	.0517	-.3443	-.1330
	MSE	.0949	.0588	.1359	.0848
.4960	EST	.1695	.5341	.1645	.4893
	BIAS	-.3265	.0381	-.3315	-.0067
	MSE	.1137	.0241	.1208	.0292
.5300	EST	.1973	.5575	.1661	.5220
	BIAS	-.3327	.0275	-.3639	-.0080
	MSE	.1160	.0182	.1371	.0209
.5620	EST	.2458	.5875	.2048	.5610
	BIAS	-.3162	.0255	-.3572	-.0010
	MSE	.1052	.0135	.1318	.0174
.5930	EST	.3003	.6156	.2581	.5924
	BIAS	-.2927	.0226	-.3349	-.0006
	MSE	.0907	.0113	.1168	.1040
.6260	EST	.3609	.6477	.3208	.6309
	BIAS	-.2651	.0217	-.3052	.0049
	MSE	.0755	.0097	.0983	.0112
.6630	EST	.4261	.6802	.3866	.6636
	BIAS	-.2369	.0172	-.2764	.0006
	MSE	.0617	.0087	.0816	.0089

References

Billingsley, P. (1968). *Convergence of Probability Measures*, John Wiley & Sons,Cox, D. R. (1961). *Renewal Theory*, Methuen, London.

Ghorai, J. K. and Rejto, L. (1987). Estimation of mean residual life with censored data under the proportional hazard model, *Communication and Statistics-Theory. and Method.* 16. 2097- 2114.

Kaplan, E. and Meier, P.(1958). Nonparametric estimation from Incomplete observations, *Journal of American Statistical Association,* 53, 457-481.

Lai, T. Z. and Ying, Z. (1991). Estimating a distribution function with truncate and censored data, *the Annals of Statistics,* 19, 417-442.

Swartz, G. B. (1973). The Mean Residual Life function, *IEEE transaction reliability R22*, 108-109.

Wang, M. C., Jewell, N. P. and Tsai, W. Y. (1986). Asymptotic properties of the product limit estimate under radom Truncation, *The Annals of Statistics,* 14, 1597-1605.

Woodroofe, M. (1985). Estimating a distribution function with truncated data. *The Annals of Statististics,* 13, 163-177.

Tsai, W. Y., Jewell, N. P. and Wang, M. C. (1987). A note on the product-limit estimator under right censoring and left truncation, *Biometrika*. 74, 883-886.

Yang, G. (1977). Life expectancy under reandom censorship, *Stochastic Processes and their Applications*, 6, 33-38.

Yang, G. (1978). Estimation of a Biometric Function, *The Annals of Statistics*,6, 112-116.