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Optimal Designs for Constant Stress Partially Accelerated Life Tests under Type I Censoring

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Abstract The inferences on a series system under the usual condition using data from constant stress partially accelerated life tests and type I censoring is studied. Two optimal designs to determine the sample proportion allocated each stress level model are also presented, which minimize the sum of the generalized asymptotic variances of maximum likelihood estimators of the failure rate and the acceleration factors and the sum of the asymptotic variances of maximum likelihood estimators of the acceleration factors for each component. Each component of a system is assumed to follow an exponenial distribution.

1. Introduction

As the life testing at the usual conditions requires a long time to get the test data, accelerated life tests(ALTs) and the partially accelerated life tests(PALTs) are often used to be shorten the lifetimes of test units. Test units are run only at the conditions of greater stress than the usual conditions in ALTs, and in PALTs, test units are subjected to the usual and accelerated conditions.

Meeker(1975,1984) studied the design for constant stress accelerated life tests and gave the optimal test conditions and sample allocations under type I ceneoring. Bai and Chung(1992) studied two optimal designs and compared the performances of the step stress and the constant stress partially accelerated life tests(PALTs) under tampered random variable model.

In this paper, the inferences on the estimation of a series system with m independent components is considered under three level constant stress partially accelerated life tests, assuming that the lifetime of each component follows an exponenetial distribution.

Under constant stress accelerated life tests, Klein and Basu(1981) discussed the problem on analysis of the reliability function of a series system when the lifetime

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of each component follows a Weibull distribution with common shape parameters. They(1982) also considered the situation in which the lifetime of each component follows a Weibull distribution with different shape parameters under Type I, Type II or progressively censorings.

For three level constant stress PALTs, $n\phi_1(=n_1)$ units randomly chosen among n test units are allocated to the use stress s_1, $n\phi_2(=n_2)$ units to the stress s_2 and the remaining $n-n\phi_1-n\phi_2(=n-n_1-n_2=n_3)$ units to the stress s_3. Each unit placed on each stress level is examined until the specified censoring time T, and the test condition is not changed. It is assumed that the lifetime of the component j of a series system at the use stress s_1 follows an exponential distribution with failure rate λ_{1j}, the lifetime of a component j at the stress s_2 an exponential distribution with failure rate $\alpha_{1j}\lambda_{ij}$, and the lifetime of a component j at the stress s_3 an exponential distribution with failure rate $\alpha_{1j}\alpha_{2j}\lambda_{1j}$.

Some useful notations are introduced as follows.

(1) r_{ij}, $i=1,2,3$, $j=1,2,\cdots,m$ is the number of failed systems due to the component j at the stress s_i before the censoring time.

(2) r_i, $i=1,2,3$ is the number of failed systems at te stress s_i regardless of the cause of failure before the censoring time.

(3) c_{ip_i}, $i=1,2,3$, $p_i=1,2,\cdots,n_i$ is the component which causes the failure of system at the stress s_i.

If X_p, $p=1,2,\cdots,n$ are lifetimes of systems from the constant stress PALTs, then by the assumption, the likelihood function of observations is given as

$$
\begin{aligned}
L &= \prod_{p_1=1}^{n_1}\prod_{j=1}^{m}\left[\lambda_{1j}\exp(-\lambda_1 x_{p_1})\right]^{\delta_j(c_{1p_1})}\left[\exp(-\lambda_1 T)\right]^{1-\delta_j(c_{1p_1})} \\
&\times \prod_{p_2=1}^{n_2}\prod_{j=1}^{m}\left[\alpha_{1j}\lambda_{1j}\exp(-\lambda_2 x_{p_2})\right]^{\eta_j(c_{2p_2})}\left[\exp(-\lambda_2 T)\right]^{1-\eta_j(c_{2p_2})} \\
&\times \prod_{p_3=1}^{n_3}\prod_{j=1}^{m}\left[\alpha_{1j}\alpha_{2j}\lambda_{1j}\exp(-\lambda_3 x_{p_3})\right]^{\varepsilon_j(c_{3p_3})}\left[\exp(-\lambda_3 T)\right]^{1-\varepsilon_j(c_{3p_3})} \qquad (1) \\
&= \prod_{j=1}^{m}\Bigg\{\lambda_{1j}^{r_{1j}}\exp(-\lambda_{1j}\sum_{p_1=1}^{r_1}x_{p_1})\exp(-(n_1-r_1)\lambda_{1j}T) \\
&\times (\alpha_{1j}\lambda_{1j})^{r_{2j}}\exp(-\alpha_{1j}\lambda_{1j}\sum_{p_2=1}^{r_2}x_{p_2})\exp(-(n_2-r_2)\alpha_{1j}\lambda_{1j}T)
\end{aligned}
$$

$$\times (\alpha_{1j}\alpha_{2j}\lambda_{1j})^{r_{3j}} \exp(-\alpha_{1j}\alpha_{2j}\lambda_{1j}\sum_{p_3=1}^{r_3} x_{p_3})\exp(-(n_3-r_3)\alpha_{1j}\alpha_{2j}\lambda_{1j}T)\Bigg\},$$

where $\lambda_1 = \sum_{j=1}^{m} \lambda_{1j}$, $\lambda_2 = \sum_{j=1}^{m} \alpha_{1j}\lambda_{1j}$, $\lambda_3 = \sum_{j=1}^{m} \alpha_{1j}\alpha_{2j}\lambda_{1j}$ and

$$\delta_j(c_{ip_i}) = \begin{cases} 1 & if \quad c_{ip_i} = j \\ 0 & if \quad c_{ip_i} \neq j \end{cases}, \ i = 1,2,3, \ p_i = 1,2,\cdots,n_i$$

The log-likelihood function function of observations is then given by

$$\log L = \sum_{j=1}^{m} \log L_j ,$$

with

$$\begin{aligned} \log L_j &= r_{1j} \log \lambda_{1j} - \lambda_{1j} \sum_{p_1=1}^{r_1} x_{p_1} - (n_1 - r_1)\lambda_{1j} T \\ &+ r_{2j} \log \alpha_{1j}\lambda_{1j} - \alpha_{1j}\lambda_{1j} \sum_{p_2=1}^{r_2} x_{p_2} - (n_2 - r_2)\alpha_{1j}\lambda_{1j} T \\ &+ r_{3j} \log \alpha_{1j}\alpha_{2j}\lambda_{1j} - \alpha_{1j}\alpha_{2j}\lambda_{1j} \sum_{p_3=1}^{r_3} x_{p_3} - (n_3 - r_3)\alpha_{1j}\alpha_{2j}\lambda_{1j} T . \end{aligned} \tag{2}$$

From (2), maximum likelihood estimators of the failure rate and the acceleration factors can be obtained as

$$\hat{\lambda}_{1j} = \frac{r_{1j}}{\sum_{p_1=1}^{r_1} x_{p_1} + (n_1 - r_1)T}, \quad \hat{\alpha}_{1j} = \frac{r_{2j}[\sum_{p_1=1}^{r_1} x_{p_2} + (n_1 - r_1)T]}{r_{1j}[\sum_{p_2=1}^{r_2} x_{p_2} + (n_2 - r_2)T]}, \quad \hat{\alpha}_{2j} = \frac{r_{3j}[\sum_{p_2=1}^{r_2} x_{p_2} + (n_2 - r_2)T]}{r_{2j}[\sum_{p_3=1}^{r_3} x_{p_3} + (n_3 - r_3)T]}$$

Now, the optimal designs to determine optiaml sample proportion allocated to each stress level among n test units minimizing the sum of the generalized asymptotic variances of MLEs for the failure rate and acceleration factors and the sum of the asymptotic variances of acceleration factors of component j is considered.

2. Optimal Designs for Constant Stress PALTs

Let I_j, $j = 1,2,\cdots,m$, be the Fisher information matrices for the failure rate and acceleration factors of component j. Then the Fisher information matrix for the failure rate and acceleration factors of a system is given by

$$I = Diag(I_1, I_2, \cdots, I_m)$$

whose diagonal submatrices I_j, $j = 1,2,\cdots,m$, are 3×3 matrices.

The Fisher information matrix I_j, $j = 1,2,\cdots,m$, is given as

$$I_j = \begin{pmatrix} \frac{n}{\lambda_{1j}}\left(\frac{1}{\lambda_1}\phi_1 p_1 + \frac{\alpha_{1j}}{\lambda_2}\phi_2 p_2 + \frac{\alpha_{1j}\alpha_{2j}}{\lambda_3}\phi_3 p_3\right) & n\left(\frac{1}{\lambda_2}\phi_2 p_2 + \frac{\alpha_{2j}}{\lambda_3}\phi_3 p_3\right) & \frac{n\alpha_{1j}}{\lambda_3}\phi_3 p_3 \\ & \frac{n}{\alpha_{1j}}\left(\frac{\lambda_{1j}}{\lambda_2}\phi_2 p_2 + \frac{\alpha_{2j}\lambda_{1j}}{\lambda_3}\phi_3 p_3\right) & \frac{n\lambda_{1j}}{\lambda_3}\phi_3 p_3 \\ & & \frac{n\alpha_{1j}\lambda_{1j}}{\alpha_{2j}\lambda_3}\phi_3 p_3 \end{pmatrix}$$

where $\phi_3 = 1 - \phi_1 - \phi_2$ and p_i, $i = 1,2,3$ represents the probability that a system fails at each stress s_i before the censoring time T with $p_1 = 1 - \exp(-\lambda_1 T)$, $p_2 = 1 - \exp(-\lambda_2 T)$, $p_3 = 1 - \exp(-\lambda_3 T)$

By the Fisher information matrix, the sum of the generalized asymptotic variances of MLEs of the failure rate and the acceleration factors can be obtained, which is given as

$$\text{SGAVR} = \frac{\lambda^* \lambda_1 \lambda_2 \lambda_3}{n^3} \frac{1}{p_1 p_2 p_3 \phi_1 \phi_2 (1 - \phi_1 - \phi_2)}, \qquad (3)$$

where $\lambda^* = \sum_{j=1}^{m} \alpha_{2j} / \lambda_{1j}$.

By (3), each optimal sample proportion ϕ_i allocated to each stress level under constant stress PALTs is always given as 1/3 regardless of sample size, which can be seen that equating the first derivatives of SGAVR with respect to ϕ_1 and ϕ_2 to zero.

From the above facts, we can observe that for the constant stress PALTs, in order to determine optimal sample proportions allocated three level stresses minimizing the sum of the generalized asymptotic variances, pre-estimates of p_1, p_2 and p_3 are not needed and therefore, the sum of the generalized asymptotic variances is not affected by the wrong pre-estimates of p_1, p_2 and p_3.

The sum of the asymptotic variances of MLEs of acceleration factors is also considered as another criterion to search optimal sample proportion allocated to each stress level. It is useful where acceleration factors are used to extrapolate the lifetime at accelerated conditions into the usual condition and therefore, it is important to estimate acceleration factors more precisely than the failure rate at the usual condition.

From the Fisher information matrix, the sum of asymptotic variances of MLEs for acceleration factors is given as

$$\text{SAVR} = \frac{\lambda_1 d_1}{n}\frac{1}{\phi_1 p_1} + \frac{\lambda_2 d_2}{n}\frac{1}{\phi_2 p_2} + \frac{\lambda_3 d_3}{n}\frac{1}{(1-\phi_1-\phi_2)p_3}, \tag{4}$$

where $d_1 = \sum_{j=1}^{m}\frac{\alpha_{1j}^2}{\lambda_{1j}}$, $d_2 = \sum_{j=1}^{m}\frac{\alpha_{2j}}{\lambda_{1j}}\left(\frac{\alpha_{1j}}{\alpha_{2j}}+\frac{\alpha_{2j}}{\alpha_{1j}}\right)$, $d_3 = \sum_{j=1}^{m}\frac{\alpha_{2j}}{\alpha_{1j}\lambda_{1j}}$

Optimal sample proportions ϕ_1 and ϕ_2 minimizing the sum of asymptotic variances of MLEs for acceleration factors are unique solutions of

$$\frac{\lambda_1 d_1}{n}\frac{1}{\phi_1^2 p_1} - \frac{\lambda_3 d_3}{n}\frac{1}{(1-\phi_1-\phi_2)^2 p_3} = 0,$$

and

$$\frac{\lambda_2 d_2}{n}\frac{1}{\phi_2^2 p_2} - \frac{\lambda_3 d_3}{n}\frac{1}{(1-\phi_1-\phi_2)^2 p_3} = 0. \tag{5}$$

In order to use this criterion for optimality, unknown parameters p_1, p_2 a nd p_3 must be approximated by the past data set while the sum of the generalized asymptotic variances of MLEs for the faliure rate and acceleration factors is influenced by the pre-estimated values of parameters. However, the wrong pre-estimated values of parameters may not lead to optimal sample proportions and result in the poor estimators of parameters at the usual condition. Thus the effects of the pre-estimated values of parameters are investigated.

It is assumed that $\alpha_{11} = \alpha_{12} = \cdots = \alpha_{1m} = \alpha_1$ and $\alpha_{21} = \alpha_{22} = \cdots = \alpha_{2m} = \alpha_2$ for the sake of simplicity. The true values of p_1 and p_2 are assumed to be 0.3 and 0.6, respectively and $p_3 = 0.9$ is fixed. The behaviors of SAVR relative to optimal SAVR due to wrong pre-estimated values of p_1 and p_2 are shown in Figure 1. If the pre-estimated values of p_1 and p_2 are not too far from true values, the behaviors of SAVR tend to be stable.

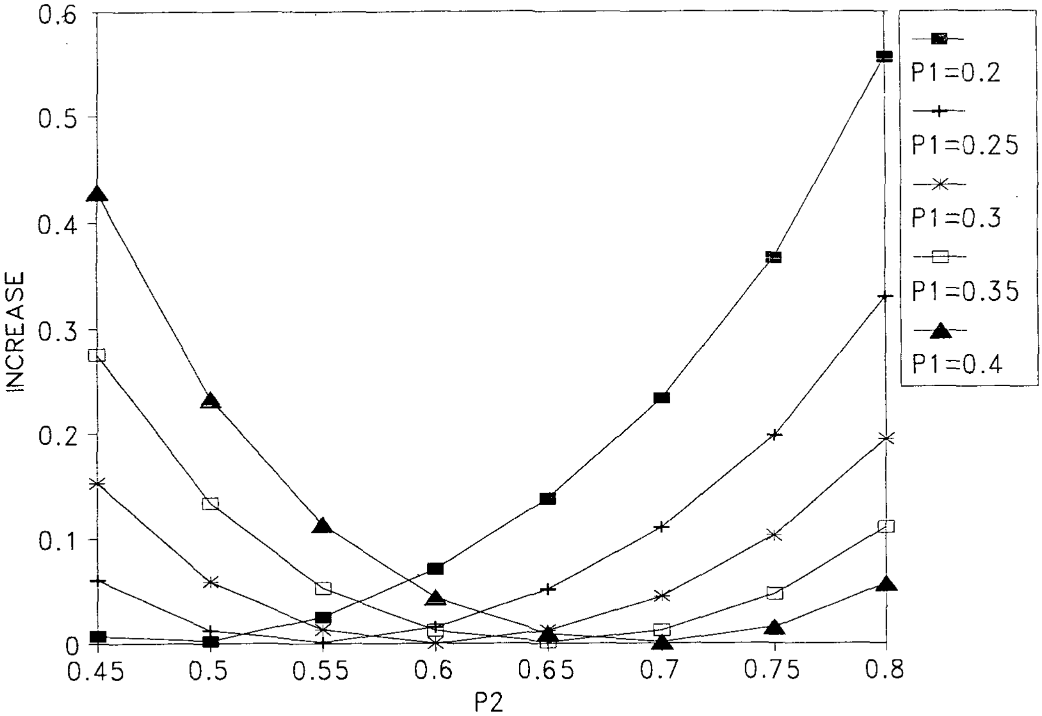

Figure 1. The Behavior of SAVR

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